Generalized Spectral Decomposition Approach to a Stochastic Finite Integration Technique Electrokinetic Formulation

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Abstract — In order to efficiently solve the stochastic finite integration technique formulation for electrokinetics, a recently proposed generalized spectral decomposition approach is applied. Compared to the standard approach, the proposed method drastically reduces the computational burden. The results are validated by comparison with those obtained with high order polynomial chaos expansion, taken as the reference solution.

Index Terms — Electrokinetics, finite integration technique, polynomial chaos expansion, resistance welding, uncertainty cuantification.

I. INTRODUCTION

The spectral stochastic finite element method (FEM) [1] based on the polynomial chaos expansion (PCE) is a well-established method and has been applied to various electromagnetic problems, e.g., [2]. In order to take advantages of the finite integration technique (FIT), a well-known alternative to the FEM, in [3] the authors presented its spectral stochastic formulation in the case of a prototype problem of electrokinetics type.

The approach in [3] exhibits large computational costs both in the storage requirements and in execution time, due to the large dimensions of the linear systems to be constructed and solved. In order to alleviate these drawbacks, in this paper the benefits of a generalized spectral decomposition approach to the stochastic FIT formulation are shown. Such approach, based on [4], allows to reduce the storage requirement to that of about one deterministic problem and the computational complexity of about one order of magnitude.

These results are validated considering a simplified geometry of a typical system for resistance welding [2, 3]. Here, three aluminium electrodes over a conductive aluminium substrate are considered. The three contact resistances are modelled as sheets of depth equal to d and conductivities σ_1, σ_2, and σ_3. The conductivity of aluminium is indicated with σ_4. Only the conductive region is discretized. Voltages V_1, V_2, and V_3 are imposed as Dirichlet boundary conditions and the three currents I_1, I_2, and I_3 are computed from the field solution.

II. DETERMINISTIC FIT FORMULATION

The stochastic FIT formulation starts from a deterministic FIT formulation, derived discretizing the spatial region of the problem Ω by a pair of three-dimensional oriented dual grids, $\mathcal{G}$ and $\tilde{\mathcal{G}}$. In order to manage boundary conditions, following [5] *unlike* standard FIT, the pair of two-dimensional oriented dual grids, $\mathcal{G}_b$ and $\tilde{\mathcal{G}}_b$ are also introduced, where $\mathcal{G}_b$ is the trace of the primal grid $\mathcal{G}$ onto the boundary $\partial\Omega$ of Ω and $\tilde{\mathcal{G}}_b$ is its dual. The continuity law for currents is discretized, in exact form, as:

$$\widetilde{\mathbf{D}}\tilde{\boldsymbol{\imath}} + \widetilde{\mathbf{D}}_b\tilde{\boldsymbol{\imath}}_b = \mathbf{0}, \tag{1}$$

in which $\tilde{\boldsymbol{\imath}}$ and $\tilde{\boldsymbol{\imath}}_b$ are the vectors of the fluxes of the electric current through the faces of $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{G}}_b$ respectively, $\widetilde{\mathbf{D}}$ is the volume-face incidence matrix of $\tilde{\mathcal{G}}$ and $\widetilde{\mathbf{D}}_b$ is the incidence matrix between the volumes of $\tilde{\mathcal{G}}$ and the faces of $\tilde{\mathcal{G}}_b$.

The irrotationality of the electric field is expressed, in exact form, as:

$$\boldsymbol{v} = -\mathbf{G}\boldsymbol{\varphi}, \tag{2}$$

in which $\boldsymbol{v}$ is the vector with the circulations of the electric field along the edges of $\mathcal{G}$, $\boldsymbol{\varphi}$ is the vector of the electric potentials at the nodes of $\mathcal{G}$ and $\mathbf{G}$ is the edge-node incidence matrix of $\mathcal{G}$. As it is well known [5-7], for any pair of dual grids $-\mathbf{G}^T = \widetilde{\mathbf{D}}$.

Ohm's law, relating the electric field to the electric current density by means of the electric conductivity σ, assumed to be strictly positive, is discretized in approximate form by means of a discrete material matrix $\mathbf{M}_\sigma$, as:

$$\tilde{\boldsymbol{\imath}} = \mathbf{M}_\sigma \boldsymbol{v}. \tag{3}$$

The discrete material matrix $\mathbf{M}_\sigma$ is assumed to be derived using the energetic approach introduced in [6] for a tetrahedral grid and extended in [7] for generic polyhedral grids. As detailed in [6], in this way this matrix is *symmetric* and *positive definite*. In usual electrokinetics problems it can be assumed that the electric conductivity is uniform and equal to σ_k in each subregion Ω_k composed of a distinct material. As a result, the discrete constitutive matrix takes the form:

$$\mathbf{M}_\sigma = \sum_{k=1}^{K} \sigma_k \, \mathbf{M}_k. \tag{4}$$

Submitted On: February 24, 2015
Accepted On: November 8, 2015

Lastly boundary conditions are introduced. For a Dirichlet problem, here considered for the sake of simplicity, such conditions can be written in exact form in terms of the incidence matrix $\widetilde{\mathbf{D}}_b$ as follows [5]:

$$\widetilde{\mathbf{D}}_b^T \boldsymbol{\varphi} = \boldsymbol{\varphi}_b, \tag{5}$$

in which $\boldsymbol{\varphi}_b$ is the vector of the electric potential on the boundary $\partial\Omega$, hence assumed known.

After grouping the unknown electric potentials at the nodes of $\mathcal{G}$, *not* belonging to $\mathcal{G}_b$, in the $N \times 1$ vector $\boldsymbol{u}$, it can be written:

$$\boldsymbol{\varphi} = \widetilde{\mathbf{D}}_{\overline{b}} \boldsymbol{u} + \widetilde{\mathbf{D}}_b \boldsymbol{\varphi}_b,$$

in which $\widetilde{\mathbf{D}}_{\overline{b}}$ is the matrix mapping the electric potentials of $\boldsymbol{u}$ onto the electric potentials of $\boldsymbol{\varphi}$. Then eliminating all variables different from $\boldsymbol{u}$, (1)-(3), (5) can be reduced to the form:

$$\mathbf{A}\boldsymbol{u} = \boldsymbol{b}, \tag{6}$$

in the unknown $\boldsymbol{u}$, in which

$$\mathbf{A} = \widetilde{\mathbf{D}}_{\overline{b}}^T \mathbf{G}^T \mathbf{M}_\sigma \mathbf{G} \widetilde{\mathbf{D}}_{\overline{b}},$$
$$\boldsymbol{b} = -\widetilde{\mathbf{D}}_{\overline{b}}^T \mathbf{G}^T \mathbf{M}_\sigma \mathbf{G} \widetilde{\mathbf{D}}_b \boldsymbol{\varphi}_b.$$

In case (4) holds, (6) takes the simplified form:

$$\textstyle\sum_{k=1}^K \sigma_k \, \mathbf{A}_k \boldsymbol{u} = \sum_{k=1}^K \sigma_k \, \boldsymbol{b}_k,$$

in which

$$\mathbf{A}_k = \widetilde{\mathbf{D}}_{\overline{b}}^T \mathbf{G}^T \mathbf{M}_k \mathbf{G} \widetilde{\mathbf{D}}_{\overline{b}},$$
$$\boldsymbol{b}_k = -\widetilde{\mathbf{D}}_{\overline{b}}^T \mathbf{G}^T \mathbf{M}_k \mathbf{G} \widetilde{\mathbf{D}}_b \boldsymbol{\varphi}_b.$$

As a consequence of the symmetric, positive definite properties of the discrete material matrix $\mathbf{M}_\sigma$, also matrix $\mathbf{A}$ is symmetric, positive definite. Thus, robust sparse numerical methods for symmetric positive definite matrices, both direct and iterative, can be used for solving (6).

III. STOCHASTIC FIT FORMULATION

The electric conductivity is now assumed to depend on a small number Q of random variables $\xi_1, \cdots, \xi_Q$, that can be assumed to be statistically independent and to form a vector $\boldsymbol{\xi} = [\xi_q]$. As a result the constitutive matrix (4) depends on $\boldsymbol{\xi}$ and is indicated by $\mathbf{M}_\sigma(\boldsymbol{\xi})$. Hence, also the coefficient matrix, the right hand vector and unknowns in (6) depend on $\boldsymbol{\xi}$ so that it can be written:

$$\mathbf{A}(\boldsymbol{\xi})\boldsymbol{u}(\boldsymbol{\xi}) = \boldsymbol{b}(\boldsymbol{\xi}). \tag{7}$$

A PCE can now be used to approximate the unknown vector $\boldsymbol{u}(\boldsymbol{\xi})$ in the form:

$$\boldsymbol{u}(\boldsymbol{\xi}) = \textstyle\sum_{|\boldsymbol{\alpha}| \le P} \boldsymbol{u}_{\boldsymbol{\alpha}} \, \psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}), \tag{8}$$

in which $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_Q)$ are multi-indices of q elements, $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots \alpha_Q$ and

$$\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \psi_{\alpha_1}^1(\xi_1)\psi_{\alpha_1}^2(\xi_2) \cdots \psi_{\alpha_Q}^Q(\xi_Q),$$

where $\psi_j^q(\xi_q)$, with $j = 0,1, \dots, P$, are polynomials of degree not greater than P, forming an orthonormal basis in the probability space of random variable ξ_q, with $q = 1, \dots, Q$. Thus, the functions $\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})$ are polynomials of degrees not greater than P, forming a basis of dimension:

$$M = \binom{P+Q}{P}, \tag{9}$$

which is orthonormal in the probability space of random vector $\boldsymbol{\xi}$. Equivalently, by defining the column vector $\boldsymbol{\psi}(\boldsymbol{\xi}) = [\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})]$ and the $N \times M$ matrix $\boldsymbol{\mathcal{U}} = [\boldsymbol{u}_{\boldsymbol{\alpha}}]$, obtained by juxtaposing $\boldsymbol{u}_{\boldsymbol{\alpha}}$ in lexicographic order, it can be written:

$$\boldsymbol{u}(\boldsymbol{\xi}) = \boldsymbol{\mathcal{U}}\boldsymbol{\psi}(\boldsymbol{\xi}). \tag{10}$$

The spectral stochastic FIT equations are achieved by substituting (10) into (7), multiplying both members on the right by $\boldsymbol{\psi}^T(\boldsymbol{\xi})$ and taking the expected value $E[\cdot]$, so that it results in:

$$E[(\mathbf{A}(\boldsymbol{\xi})\boldsymbol{\mathcal{U}}\boldsymbol{\psi}(\boldsymbol{\xi}) - \boldsymbol{b}(\boldsymbol{\xi}))\boldsymbol{\psi}^T(\boldsymbol{\xi})] = \mathbf{0}. \tag{11}$$

Using the vec$(\cdot)$ operator [8], (11) can be written as:

$$\boldsymbol{\mathcal{A}}\boldsymbol{U} = \boldsymbol{\mathcal{B}}, \tag{12}$$

a linear system of MN equations in the MN unknowns forming column vector $\boldsymbol{U} = \text{vec}(\boldsymbol{\mathcal{U}})$. The coefficient matrix $\boldsymbol{\mathcal{A}}$ and right-hand vector $\boldsymbol{\mathcal{B}}$ of this system of equations can be written respectively in the form:

$$\boldsymbol{\mathcal{A}} = E[\boldsymbol{\psi}(\boldsymbol{\xi})\boldsymbol{\psi}^T(\boldsymbol{\xi}) \otimes \mathbf{A}(\boldsymbol{\xi})] =$$
$$(\mathbf{1}_n \otimes \widetilde{\mathbf{D}}_{\overline{b}}^T \mathbf{G}^T) \, E[\boldsymbol{\psi}(\boldsymbol{\xi})\boldsymbol{\psi}^T(\boldsymbol{\xi}) \otimes \mathbf{M}_\sigma(\boldsymbol{\xi})] \, (\mathbf{1}_n \otimes \mathbf{G}\widetilde{\mathbf{D}}_{\overline{b}}),$$
$$\boldsymbol{\mathcal{B}} = E[\boldsymbol{\psi}(\boldsymbol{\xi}) \otimes \boldsymbol{b}(\boldsymbol{\xi})] =$$
$$-(\mathbf{1}_n \otimes \widetilde{\mathbf{D}}_{\overline{b}}^T \mathbf{G}^T) \, E[\boldsymbol{\psi}(\boldsymbol{\xi}) \otimes \mathbf{M}_\sigma(\boldsymbol{\xi})] \, (\mathbf{1}_n \otimes \mathbf{G}\widetilde{\mathbf{D}}_b \boldsymbol{\varphi}_b).$$

being $\otimes$ the tensor product.

In case (4) holds, (11) and (12) can be respectively written in the simplified forms:

$$\textstyle\sum_{k=1}^K \mathbf{A}_k \boldsymbol{\mathcal{U}} \mathbf{S}_k = \sum_{k=1}^K \boldsymbol{b}_k \boldsymbol{e}_1^T \mathbf{S}_k,$$
$$(\textstyle\sum_{k=1}^K \mathbf{S}_k \otimes \mathbf{A}_k)\boldsymbol{U} = \sum_{k=1}^K \mathbf{S}_k \, \boldsymbol{e}_1 \otimes \boldsymbol{b}_k,$$

being

$$\mathbf{S}_k = E[\sigma_k(\boldsymbol{\xi})\boldsymbol{\psi}(\boldsymbol{\xi})\boldsymbol{\psi}^T(\boldsymbol{\xi})].$$

a symmetric positive definite matrix and being $\boldsymbol{e}_1$ the $M \times 1$ vector with all zeros except the first element equal to one.

As a consequence of the symmetric, positive definite properties of the discrete material matrix $\mathbf{M}_\sigma$, the coefficient matrix of (12) is symmetric, positive definite. Thus, also for problem (12), the robust sparse numerical methods specific for symmetric positive definite matrices, both direct and iterative can be used. However, as with stochastic FEM, difficulties arise with the solution of this system of equations when the number of independent random variables Q and PCE order P increase, due to the increased dimensionality MN of the problem, as a consequence of (9).

IV. GENERALIZED SPECTRAL DECOMPOSITION

An iterative approach, recently proposed in literature [4], is here adapted to approximate the solution of (12). In this approach, given at step $j-1$ an approximation for $\boldsymbol{u}(\boldsymbol{\xi})$ in a form analogous to (10),

$$\boldsymbol{u}_{j-1}(\boldsymbol{\xi}) = \boldsymbol{\mathcal{U}}_{j-1}\boldsymbol{\psi}(\boldsymbol{\xi}), \tag{13}$$

a new approximation $\boldsymbol{u}_j(\boldsymbol{\xi})$ is obtained as:

$$\boldsymbol{u}_j(\boldsymbol{\xi}) = \boldsymbol{u}_{j-1}(\boldsymbol{\xi}) + \textstyle\sum_{r=1}^R \boldsymbol{w}_r \, \lambda_r(\boldsymbol{\xi}), \tag{14}$$

in which a number $R \ll M$ of $N \times 1$ vectors $\boldsymbol{w}_r$ are searched for together with a number R of functions $\lambda_r(\boldsymbol{\xi})$ belonging to the space spanned by the $\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi})$ functions expressed as:

$$\lambda_r(\boldsymbol{\xi}) = \sum_{|\boldsymbol{\alpha}| \le P} l_{r\boldsymbol{\alpha}}\, \psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}), \tag{15}$$

in which $l_{r\boldsymbol{\alpha}}$ are coefficients. Let $\boldsymbol{\lambda}(\boldsymbol{\xi}) = [\lambda_r(\boldsymbol{\xi})]$ be an $R \times 1$ vector. Also let $\boldsymbol{\mathcal{W}} = [\boldsymbol{w}_r]$ be an $N \times R$ matrix and let it be $\boldsymbol{\mathcal{L}} = [l_{r\boldsymbol{\alpha}}]$ an $R \times M$ matrix so that $\boldsymbol{\lambda}(\boldsymbol{\xi}) = \boldsymbol{\mathcal{L}}\boldsymbol{\psi}(\boldsymbol{\xi})$ and (13) can be rewritten in the form:

$$\boldsymbol{\mathcal{U}}_j = \boldsymbol{\mathcal{U}}_{j-1} + \boldsymbol{\mathcal{W}}\boldsymbol{\mathcal{L}}, \tag{16}$$

$$\boldsymbol{u}_j(\boldsymbol{\xi}) = \boldsymbol{\mathcal{U}}_j \boldsymbol{\psi}(\boldsymbol{\xi}), \tag{17}$$

in which $\boldsymbol{\mathcal{W}}$ and $\boldsymbol{\mathcal{L}}$ have to be determined. In order to compute $\boldsymbol{\mathcal{W}}$ and $\boldsymbol{\mathcal{L}}$, an iterative procedure is adopted. Precisely, assuming that $\boldsymbol{\mathcal{W}}$ is known, $\boldsymbol{\mathcal{L}}$ is computed by solving the equations obtained substituting (16) into (11) and multiplying on the left by $\boldsymbol{\mathcal{W}}^T$,

$$\boldsymbol{\mathcal{W}}^T E[(\mathbf{A}(\boldsymbol{\xi})(\boldsymbol{\mathcal{U}}_{j-1} + \boldsymbol{\mathcal{W}}\boldsymbol{\mathcal{L}})\boldsymbol{\psi}(\boldsymbol{\xi}) - \boldsymbol{b}(\boldsymbol{\xi}))\boldsymbol{\psi}^T(\boldsymbol{\xi})] = 0. \tag{18}$$

After applying the vec(·) operator, this equation is written in the form of the system of RM equations:

$$E[\boldsymbol{\psi}(\boldsymbol{\xi})\boldsymbol{\psi}^T(\boldsymbol{\xi}) \otimes \boldsymbol{\mathcal{W}}^T\mathbf{A}(\boldsymbol{\xi})\boldsymbol{\mathcal{W}}]\,\boldsymbol{L} = E[\boldsymbol{\psi}(\boldsymbol{\xi}) \otimes \boldsymbol{\mathcal{W}}^T\boldsymbol{b}(\boldsymbol{\xi})] - E[\boldsymbol{\psi}(\boldsymbol{\xi})\boldsymbol{\psi}^T(\boldsymbol{\xi}) \otimes \boldsymbol{\mathcal{W}}^T\mathbf{A}(\boldsymbol{\xi})]\boldsymbol{U}_{j-1}, \tag{19}$$

in the RM unknowns forming the column vector $\boldsymbol{L} = \mathrm{vec}(\mathcal{L})$. In case (4) holds, Equations (18) and (19) can be respectively rewritten in the simplified forms:

$$\sum_{k=1}^{K} \boldsymbol{\mathcal{W}}^T\mathbf{A}_k\boldsymbol{\mathcal{W}}\boldsymbol{\mathcal{L}}\mathbf{S}_k = \sum_{k=1}^{K} \boldsymbol{\mathcal{W}}^T\boldsymbol{b}_k\boldsymbol{e}_1^T\mathbf{S}_k - \sum_{k=1}^{K} \boldsymbol{\mathcal{W}}^T\mathbf{A}_k\boldsymbol{\mathcal{U}}_{j-1}\mathbf{S}_k, \tag{20}$$

$$(\sum_{k=1}^{K} \mathbf{S}_k \otimes \boldsymbol{\mathcal{W}}^T\mathbf{A}_k\boldsymbol{\mathcal{W}})\,\boldsymbol{L} = \sum_{k=1}^{K} \mathbf{S}_k\,\boldsymbol{e}_1 \otimes \boldsymbol{\mathcal{W}}^T\boldsymbol{b}_k + -(\sum_{k=1}^{K} \mathbf{S}_k \otimes \boldsymbol{\mathcal{W}}^T\mathbf{A}_k)\boldsymbol{U}_{j-1}. \tag{21}$$

Then, assuming $\boldsymbol{\mathcal{L}}$ to be known, $\boldsymbol{\mathcal{W}}$ is computed by solving the equations obtained substituting (16) into (11), multiplying on the right by $\boldsymbol{\mathcal{L}}^T$ and using that $\boldsymbol{\lambda}^T(\boldsymbol{\xi}) = \boldsymbol{\psi}^T(\boldsymbol{\xi})\boldsymbol{\mathcal{L}}^T$, so that

$$E[(\mathbf{A}(\boldsymbol{\xi})(\boldsymbol{\mathcal{U}}_{j-1} + \boldsymbol{\mathcal{W}}\boldsymbol{\mathcal{L}})\boldsymbol{\psi}(\boldsymbol{\xi}) - \boldsymbol{b}(\boldsymbol{\xi}))\boldsymbol{\lambda}^T(\boldsymbol{\xi})] = 0. \tag{22}$$

After applying the vec(·) operator, this equation is written in the form of the system of RN equations:

$$E[\boldsymbol{\mathcal{L}}\boldsymbol{\psi}(\boldsymbol{\xi})\boldsymbol{\psi}^T(\boldsymbol{\xi})\boldsymbol{\mathcal{L}}^T \otimes \mathbf{A}(\boldsymbol{\xi})]\,\boldsymbol{W} = E[\boldsymbol{\mathcal{L}}\boldsymbol{\psi}(\boldsymbol{\xi}) \otimes \boldsymbol{b}(\boldsymbol{\xi})] + -E[\boldsymbol{\psi}(\boldsymbol{\xi})\boldsymbol{\psi}^T(\boldsymbol{\xi})\boldsymbol{\mathcal{L}}^T \otimes \mathbf{A}(\boldsymbol{\xi})]\boldsymbol{U}_{j-1}, \tag{23}$$

in the RN variables forming the column vector $\boldsymbol{W} = \mathrm{vec}(\boldsymbol{\mathcal{W}})$. In case (4) holds, Equations (22) and (23) can be respectively rewritten in the forms:

$$\sum_{k=1}^{K} \mathbf{A}_k\boldsymbol{\mathcal{W}}\boldsymbol{\mathcal{L}}\mathbf{S}_k\boldsymbol{\mathcal{L}}^T = \sum_{k=1}^{K} \boldsymbol{b}_k\boldsymbol{e}_1^T\mathbf{S}_k\boldsymbol{\mathcal{L}}^T - \sum_{k=1}^{K} \mathbf{A}_k\boldsymbol{\mathcal{U}}_{j-1}\mathbf{S}_k\,\boldsymbol{\mathcal{L}}^T, \tag{24}$$

$$(\sum_{k=1}^{K} \boldsymbol{\mathcal{L}}\,\mathbf{S}_k\boldsymbol{\mathcal{L}}^T \otimes \boldsymbol{A}_k)\,\boldsymbol{W} = \sum_{k=1}^{K} \boldsymbol{\mathcal{L}}\,\mathbf{S}_k\boldsymbol{e}_1 \otimes \boldsymbol{b}_k + -\big(\sum_{k=1}^{K} \boldsymbol{\mathcal{L}}\,\mathbf{S}_k\boldsymbol{\mathcal{L}}^T \otimes \mathbf{A}_k\big)\boldsymbol{U}_{j-1}. \tag{25}$$

Equations (21), (23) are iteratively solved until convergence of $\boldsymbol{\mathcal{W}}\boldsymbol{\mathcal{L}}$ in the energy norm. This procedure can be stopped after S iterations, typically in the range $1 \div 4$ [4]. The whole procedure is then repeated, increasing j until the residual of (12) when $\boldsymbol{U} = \boldsymbol{\mathcal{U}}_j$ is less than a tolerance, as detailed in Algorithm 1.

Algorithm 1: Approximation of **U**

```
Set i :=0
Set j :=0
Set 𝒰_j :=0
repeat
  Set j:=j+1
  Choose ℒ randomly
  repeat S times
    Set i:=i+1 (total number of iterations)
    Orthonormalize rows of ℒ
    Solve (25) for W = vec(𝒲).
    Orthonormalize columns of 𝒲
    Solve (21) for L = vec(ℒ)
  end
  Update 𝒰_j by (16)
until convergence
```

For each step i Algorithm 1 requires the solutions of the systems of RM Equation (24) and the solutions of the systems of RN Equation (20). Since the coefficient matrices of these problems are ensured to be symmetric, positive definite, the same robust numerical methods for solving the deterministic problems, both direct and iterative, can be used.

Using iterative methods the coefficient matrix is not directly constructed, but the product of the coefficient matrix with column vectors are computed. Since (20) and (21) are equivalent, the left-hand-side of (21) can be more efficiently computed as:

$$vec(\sum_{k=1}^{K} \boldsymbol{\mathcal{W}}^T\mathbf{A}_k\boldsymbol{\mathcal{W}}\boldsymbol{\mathcal{L}}\mathbf{S}_k).$$

Similarly since (24) and (25) are equivalent, the left-hand-side of (25) can be more efficiently computed as:

$$vec(\sum_{k=1}^{K} \mathbf{A}_k\boldsymbol{\mathcal{W}}\boldsymbol{\mathcal{L}}\mathbf{S}_k\boldsymbol{\mathcal{L}}^T).$$

In order to reduce computational complexity, while maintaining satisfactory convergence properties, small values of R are chosen. While in [4] only the case R=1 is considered, here it is shown that small values of R greater than 1 can be more effective choices. It is noted that for small numbers Q of random variables and relatively low PCE orders P, as in the cases considered here, M is usually much smaller than N, so that the only computationally demanding problem is problem (25) in RN variables. The orthonormalization of the rows of $\boldsymbol{\mathcal{L}}$ and of the columns of $\boldsymbol{\mathcal{W}}$ is performed in order to ensure robustness, as detailed in [4].

V. NUMERICAL RESULTS

The 3D problem shown in Fig. 1 is chosen as a test case. Conductivities σ_1, σ_2, and σ_3 are modelled as uniformly distributed random variables, in the range 1.41 MS/m – 11.3 MS/m. Voltages are imposed with values V_1=V_3= 1 V and V_2=0. Only the conductive region, without the surrounding air region, is discretized.

Homogeneous Neumann's boundary conditions are used everywhere, with the exception of the three electrodes at which Dirichlet's boundary conditions are used, as indicated in Fig. 1.

A set of primal tetrahedral grids is generated. Among these a tetrahedral grid with n=6441 nodes is chosen whose further refining introduces variations in the potential distribution lesser that 1% in the energy norm. The energetic approach to FIT is used for constructing discrete material matrices. In particular, the basis functions introduced in [6] for tetrahedral grids are adopted.

The PCE of the electric current I_1 is estimated in the case P=7. In the PCE intrusive approach the computational time for solving (12) by the Conjugate Gradient (CG) algorithm is about 152 s on a 2.3 GHz Intel Core i7 with tolerance equal to 10^{-9}. Memory storage requirement is about 120 MB.

In Fig. 2, the probability density function (pdf) of the current I_1 computed following the GSD approach (Algorithm 1) and the one obtained with the standard PCE intrusive approach are compared. In [3] the computation of the same pdf is carried out in about 10 hours by means of 10^5 Monte Carlo simulations using the same computer, validating previous results. A good agreement among the three methods is observed.

The convergence of the approximations to the solution **U** provided by Algorithm 1 for various choices of R and S is shown in Figs. 3-8, when the relative error is defined as:

$$error = \|\boldsymbol{U}_{j-1} - \boldsymbol{U}\|/\|\boldsymbol{U}\|. \qquad (22)$$

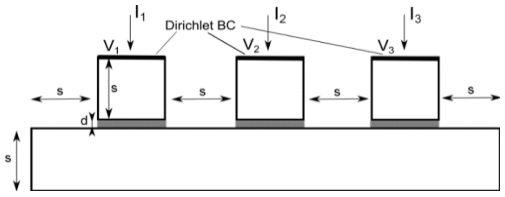

Fig. 1. 2D section of the test case. Depth is equal to s, with s =10 mm and d = 1 mm.

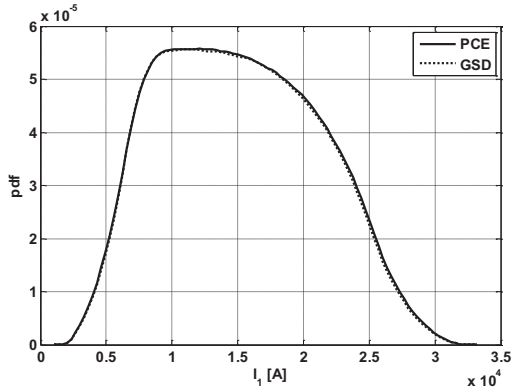

Fig. 2. Probability density function of the current I_1.

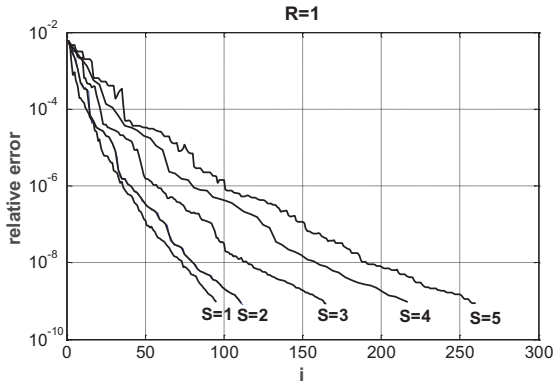

Fig. 3. Relative error (22) vs. number of iterations i of Algorithm 1 when R=1, for different values of S.

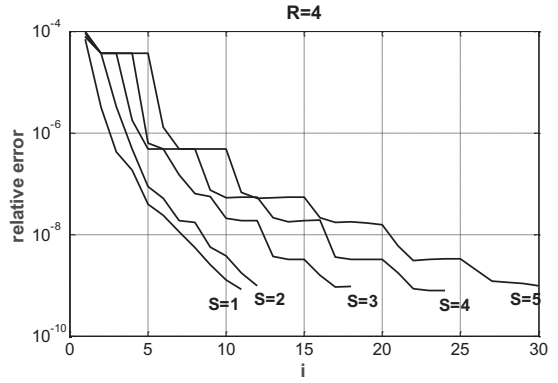

Fig. 4. Relative error vs. number of iterations i when R=4, for different values of S.

From these results it follows that for each choice of R the best results in terms of efficiency are obtained for S=1; furthermore R=4 appears as the optimal value. The computational times for this case (S=1 and R=4) are reported in Figs. 6-7. The corresponding memory storage requirement is about 1.5 MB, comparable to the 1.3 MB memory storage requirement of a single deterministic electrokinetic problem.

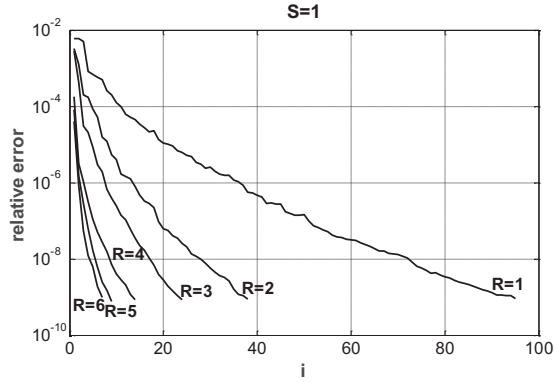

Fig. 5. Relative error vs. number of iterations i when S=1, for different values of R.

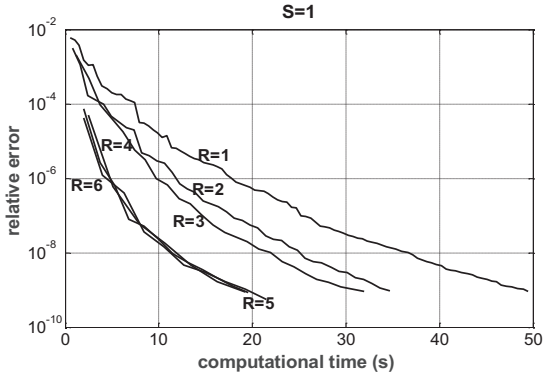

Fig. 6. Relative error vs. computational time of Algorithm 1 when S=1, for different values of R.

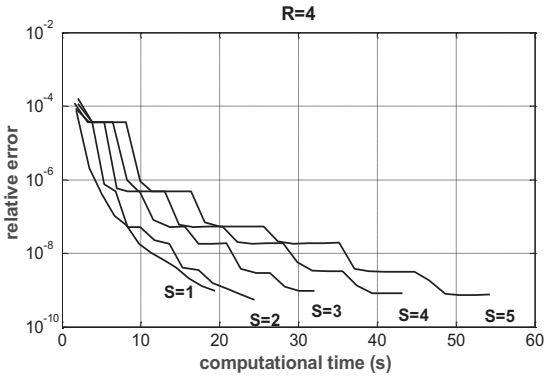

Fig. 7. Relative error vs. computational time of Algorithm 1 when R=4, for different values of S.

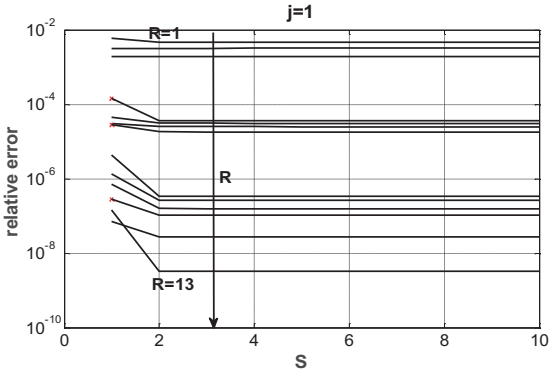

Fig. 8. Relative error for increasing values of R when j=1.

VI. CONCLUSION

In this paper it is shown how a generalized spectral decomposition approach can be applied to the spectral stochastic formulation of FIT based on PCE. As an application example, a typical system for resistance welding is analyzed. The results show that the proposed approach allows to drastically reduce the storage requirement and the computational time of the spectral stochastic formulation of FIT based on PCE.

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