Electromagnetic Scattering by Multiple Cavities Embedded in the Infinite 2D Ground Plane

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Abstract **—** This paper is concerned with the mathematical analysis and numerical computation of the electromagnetic wave scattering by multiple open cavities, which are embedded in an infinite two-dimensional ground plane. By introducing a new transparent boundary condition on the cavity apertures, the scattering problem is reduced to a boundary value problem on the two-dimensional Helmholtz equation imposed in the separated interior domains of the cavities. The existence and uniqueness of the weak solution for the model problem is studied by using a variational approach. A block Gauss-Seidel iterative method is introduced to solve the coupled system. Numerical examples are presented to show the efficiency and accuracy of the proposed method.

Index Terms **-** Electromagnetic cavity, finite element method, Helmholtz equation, variational formulation.

I. INTRODUCTION

A cavity is referred to as a local perturbation of the infinite ground plane. Given the cavity structure and an incident wave, the scattering problem is to predict the electromagnetic field scattered by the cavity. It has been extensively examined by researchers for the time-harmonic analysis of cavity-backed apertures with penetrable material filling the cavity interior [14-16, 18, 28]. Mathematical analysis of the problem including overfilled cavities, where the aperture is not planar and may protrude the ground plane, can be found in [1-4, 17, 19-24, 27]. A lot of work has been devoted to solve the problem by various numerical methods including finite element, finite difference, boundary element, and hybrid methods [5, 7, 8, 11, 12, 25, 26, 29, 30]. All the model problems have been focused on a single cavity, which may limit the application of the problem in industry and military. This paper aims to extend the single cavity model to a more general multiple cavity model, and analyze and develop numerical methods for the associated boundary value problem.

In this paper we focus on the Transverse Magnetic polarization (TM), where the modeling equation is the two-dimensional Helmholtz equation. Based on Fourier transform, a nonlocal transparent condition is introduced on the aperture, which connects the electric field in each individual cavity. By using the boundary condition, we reduce the multiple cavity problem into a boundary value problem imposed in the interiors of the cavities. The existence and uniqueness of the weak solution for the model problem is studied by using a variational approach. A block Gauss-Seidel iterative method is introduced to solve the coupled system, where only a single cavity problem needs to be solved at each iteration. Thus, it is applicable of any efficient single cavity solver to the multiple cavity problem. Numerical examples are presented to show the efficiency and accuracy of the proposed method. We refer to [9, 10, 13] for numerical methods to solve a related multiple obstacle scattering problem.

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The paper is outlined as follows. In Section 2, a mathematical model for the single cavity problem is introduced, the variational formulation is presented, and the uniqueness and existence of the solution are examined. Section 3 is devoted to the study of the solution for the multiple cavity problem. The major new ingredient is the introduction of a transparent boundary condition. Section 4 addresses the numerical implementation and examples are shown to illustrate the method. The paper is concluded with some general remarks and directions for future research in Section 5.

II. SINGLE CAVITY SCATTERING

In this section, we study a single cavity problem, which is intended to serve as a basis for the multiple cavity problem.

A. Model problem

We focus on a two-dimensional geometry. The medium is assumed to be non-magnetic and has a constant magnetic permeability; i.e., $\mu = \mu_0$, where μ_0 is the magnetic permeability of vacuum. The medium is characterized by the dielectric permittivity ε.

As shown in Fig. 1, an open cavity Ω enclosed by the aperture Γ and the wall S, is placed on a perfectly conducting ground plane Γ^c. Above the flat surface $\{y=0\} = \Gamma \cup \Gamma^c$, the medium is assumed to be homogeneous with a positive dielectric permittivity ε_0. The medium inside the cavity is inhomogeneous with a variable dielectric permittivity $\varepsilon(x,y)$. Assume further that:

$$\varepsilon \in L^\infty(\Omega),\ \mathrm{Re}\,\varepsilon > 0,\ \mathrm{Im}\,\varepsilon \ge 0 .$$

For the TM polarization, the magnetic field is transverse to the invariant direction. The time-harmonic Maxwell equations can be reduced to the two-dimensional Helmholtz equation:

$$\Delta u + \kappa^2 u = 0 \text{ in } \Omega \cup R_+^2 . \tag{1}$$

The total field satisfies the boundary condition:

$$u = 0 \text{ on } \Gamma^c \cup S , \tag{2}$$

where $\kappa^2 = \omega^2 \varepsilon \mu_0$ is the wavenumber and ω is the angular frequency.

Let an incoming plane wave $u^i = e^{i\kappa_0(x\sin\theta - y\cos\theta)}$ be incident on the cavity from above, where θ is the incident angle with respect to the positive y axis, and $\kappa_0 = \omega\sqrt{\varepsilon_0\mu_0}$ is the wavenumber of the free space.

Denote the reference field u^{ref} as the solution of the homogeneous Helmholtz equation in the upper half space:

$$\Delta u^{ref} + \kappa_0^{\,2} u^{ref} = 0 \text{ in } R_+^2 , \tag{3}$$

together with boundary condition:

$$u^{ref} = 0 \text{ on } \Gamma^c \cup \Gamma . \tag{4}$$

It can be shown from (3) and (4) that the reference field consists of the incident field and the reflected field:

$$u^{ref} = u^i + u^r ,$$

where $u^r = -e^{i\kappa_0(x\sin\theta + y\cos\theta)}$.

The total field is composed of the reference field and the scattered field:

$$u = u^{ref} + u^s .$$

It can be verified from (1) and (3) that the scattered field satisfies:

$$\Delta u^s + \kappa_0^2 u^s = 0 \text{ in } R_+^2 . \tag{5}$$

In addition, the scattered field is required to satisfy the radiation condition:

$$\lim_{\rho\to\infty} \sqrt{\rho}\left(\frac{\partial u^s}{\partial \rho} - i\kappa_0 u^s\right) = 0, \rho = |(x,y)| \cdot \tag{6}$$

To describe the boundary value problem, we need to introduce some functional spaces. For $u \in L^2(\Gamma^c \cup \Gamma)$, which is identified with $L^2(R)$, we denote by $\hat{u}$ the Fourier transform of u defined as:

$$\hat{u}(\xi) = \int_R u(x) e^{ix\xi} dx.$$

Using Fourier modes, the norm on the space $L^2(R)$ can be characterized by:

$$\| u \|_{L^2(R)} = \left[\int_R |u|^2 \, dx \right]^{\frac{1}{2}} = \left[\int_R |\hat{u}|^2 \, d\xi \right]^{\frac{1}{2}} .$$

Denote the Sobolev space:

$$H^1(\Omega) = \{u : D^s u \in L^2(\Omega), \text{ for } |s| \le 1\} ,$$

and the trace functional space:

$$H^s(R) = \{u \in L^2(R) : \int_R (1+\xi^2)^s \, |\hat{u}|^2 \, d\xi < \infty\} ,$$

whose norm is defined by:

$$\| u \|_{H^s(R)} = \left[\int_R (1+\xi^2)^s \, |\hat{u}|^2 \, d\xi \right]^{\frac{1}{2}} \cdot$$

By taking the Fourier transform of (5) with respect to x, we obtain:

$$\frac{\partial^2 \hat{u}^s(\xi,y)}{\partial y^2} + (\kappa_0^2 - \xi^2)\hat{u}(\xi,y) = 0, y > 0. \tag{7}$$

Since the solution of (7) satisfies the radiation condition (6), we deduce that:

$$\hat{u}^s(\xi,y) = \hat{u}^s(\xi,0) e^{i\beta(\xi)y} , \tag{8}$$

where

$$\beta(\xi)=\begin{cases}(\kappa_0^2-\xi^2)^{\frac{1}{2}} \text{ for } |\xi|<\kappa_0,\\ i(\xi^2-\kappa_0^2)^{\frac{1}{2}} \text{ for } |\xi|>\kappa_0.\end{cases}$$

Taking the inverse Fourier transform of (8), we find that:

$$u^s(x,y)=\int_R \hat{u}^s(\xi,0)e^{i\beta(\xi)y}e^{-i\xi x}d\xi \text{ in } R_+^2.$$

Taking the normal derivative on $\Gamma^c\cup\Gamma$, which is the partial derivative with respect to y, and evaluating at $y=0$ yield:

$$\partial_y u^s(x,y)|_{y=0}=\int_R i\beta(\xi)\hat{u}(\xi,0)e^{-i\xi x}d\xi. \quad (9)$$

For given u on $\Gamma^c\cup\Gamma$, define the boundary operator T:

$$Tu=\int_R i\beta(\xi)\hat{u}(\xi,0)e^{-i\xi x}d\xi, \quad (10)$$

which leads to the transparent boundary condition for the scattered field on $\Gamma^c\cup\Gamma$:

$$\partial_y(u-u^{ref})=T(u-u^{ref}).$$

Equivalently we have a transparent boundary condition for the total field:

$$\partial_y u=Tu+g \text{ on } \Gamma^c\cup\Gamma, \quad (11)$$

where

$$g=\partial_y u^{ref}-Tu^{ref}=-2i\kappa_0\cos\theta e^{i\kappa_0 x\sin\theta}.$$

It can be shown that the boundary operator is continuous from $H^{\frac{1}{2}}(R)$ to $H^{-\frac{1}{2}}(R)$. Furthermore, it has the following properties which ensures the uniqueness of the solution of the single cavity problem.

Lemma 1. *Let* $u\in H^{\frac{1}{2}}(R)$. *It holds that* $\mathrm{Re}\langle Tu,u\rangle\le 0$ *and* $\mathrm{Im}\langle Tu,u\rangle\ge 0$. *Furthermore, if* $\hat{u}$ *is an analytical function with respect to* ξ, $\mathrm{Re}\langle Tu,u\rangle=0$ *or* $\mathrm{Im}\langle Tu,u\rangle=0$ *implies* $u=0$.

To derive a transparent boundary condition for the total field on the aperture Γ, we need to make the zero extension as follows: for any given u on Γ, define

$$\tilde{u}(x)=\begin{cases}u, x\in\Gamma,\\ 0, x\in\Gamma^c.\end{cases}$$

The zero extension is consistent with the problem since the ground plane is a perfectly electrical conductor. Based on the extension and the transparent boundary condition (11), we have the transparent boundary condition for the total field on the aperture:

$$\partial_y u=T\tilde{u}+g \text{ on } \Gamma. \quad (12)$$

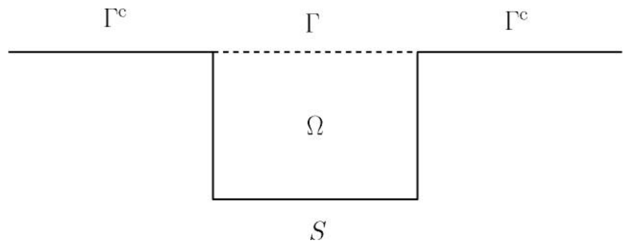

Fig. 1. The problem geometry of a single cavity.

B. Well-posedness

Define a trace functional space:

$$\tilde{H}^{\frac{1}{2}}(\Gamma)=\{u:\tilde{u}\in H^{\frac{1}{2}}(R)\},$$

whose norm is defined as the $H^{\frac{1}{2}}(R)$ norm for its extension; i.e.,

$$\|u\|_{\tilde{H}^{\frac{1}{2}}(\Gamma)}=\|\tilde{u}\|_{H^{\frac{1}{2}}(R)}.$$

Define a dual paring:

$$\langle u,v\rangle_\Gamma=\int_\Gamma u\bar{v}.$$

This dual paring for u and v is equivalent to the scalar product in $L^2(R)$ for their extensions; i.e.,

$$\langle u,v\rangle_\Gamma=\langle\tilde{u},\tilde{v}\rangle.$$

Denote by $H^{-\frac{1}{2}}(\Gamma)$ the dual space of $\tilde{H}^{\frac{1}{2}}(\Gamma)$; i.e., $H^{-\frac{1}{2}}(\Gamma)=(\tilde{H}^{\frac{1}{2}}(\Gamma))'$. The norm on this space is characterized by:

$$\|u\|_{H^{-\frac{1}{2}}(\Gamma)}=\sup_{\tilde{v}\in H^{\frac{1}{2}}(R)}\frac{\langle\tilde{u},\tilde{v}\rangle}{\|\tilde{v}\|_{H^{\frac{1}{2}}(R)}}.$$

Introduce a space:

$$H_S^1(\Omega)=\{u\in H^1(\Omega):u=0 \text{ on } S\},$$

which is a Hilbert space with the usual norm.

Multiplying a test function v on both sides of (1) and using the boundary conditions (2) and (12), we may deduce a variational problem: find u such that

$$a(u,v)=\langle g,v\rangle_\Gamma \text{ for all } v\in H_S^1(\Omega), \quad (13)$$

where the sesquilinear form is:

$$a(u,v)=\int_\Omega(\nabla u\cdot\nabla\bar{v}-\kappa^2u\bar{v})-\langle T\tilde{u},v\rangle_\Gamma. \quad (14)$$

Theorem 1. *The variational problem (13) has a unique weak solution in* $H_S^1(\Omega)$ *and the solution* *satisfies the estimate:*

$$\|u\|_{H^1(\Omega)} \le C\|g\|_{H^{-\frac{1}{2}}(\Gamma)},$$

where C is a positive constant.

Proof: Decompose the sesquilinear form (14) into $a = a_1 - a_2$, where

$$a_1(u,v) = \int_\Omega \nabla u \cdot \nabla \bar{v} - \langle T\tilde{u}, \tilde{v}\rangle,$$

and

$$a_2(u,v) = \int_\Omega \kappa^2 u\bar{v}.$$

We conclude that from Lemma 1 and Poincare inequality that a_1 is coercive from:

$$\operatorname{Re} a_1(u,u) = \int_\Omega |\nabla u|^2 - \operatorname{Re}\langle T\tilde{u}, \tilde{v}\rangle \ge \int_\Omega |\nabla u|^2$$
$$\ge C\|u\|^2_{H^1(\Omega)} \text{ for all } u \in H^1_S(\Omega).$$

Next we prove the compactness of a_2. Define an operator $K: L^2(\Omega) \to H^1(\Omega)$ by:

$$a_1(Ku,v) = a_2(u,v) \text{ for all } v \in H^1_S(\Omega),$$

which explicitly gives that for all $v \in H^1_S(\Omega)$,

$$\int_\Omega \nabla Ku \cdot \nabla \bar{v} - \langle TK\tilde{u}, \tilde{v}\rangle = \int_\Omega \kappa^2 u\bar{v}.$$

Using the coercivity of a_1 and the Lax-Milgram lemma, it follows that:

$$\|Ku\|_{H^1(\Omega)} \le C\|u\|_{L^2(\Omega)}. \tag{15}$$

Thus, K is bounded from $L^2(\Omega)$ to $H^1(\Omega)$ and $H^1(\Omega)$ is compactly imbedded into $L^2(\Omega)$. Hence, $K: L^2(\Omega) \to L^2(\Omega)$ is a compact operator.

Define a function $w \in L^2(\Omega)$ by requiring $w \in H^1_S(\Omega)$ and satisfying:

$$a_1(w,v) = \langle g, v\rangle_\Gamma \text{ for all } v \in H^1_S(\Omega).$$

It follows from the Lax-Milgram lemma again that:

$$\|w\|_{H^1(\Omega)} \le C\|g\|_{H^{-\frac{1}{2}}(\Gamma)}. \tag{16}$$

Using the operator K, we can see that the variational problem (13) is equivalent to find $u \in L^2(\Omega)$ such that:

$$(I - K)u = w. \tag{17}$$

It follows from the uniqueness result and the Fredholm alternative that the operator $I - K$ has a bounded inverse. We then have the estimate:

$$\|u\|_{L^2(\Omega)} \le C\|w\|_{L^2(\Omega)}.$$

Combining (15)-(17), we deduce that:

$$\|u\|_{H^1(\Omega)} \le \|Ku\|_{H^1(\Omega)} + \|w\|_{H^1(\Omega)}$$
$$\le C\|u\|_{L^2(\Omega)} + \|w\|_{H^1(\Omega)}$$
$$\le C\|w\|_{H^1(\Omega)} \le C\|g\|_{H^{-\frac{1}{2}}(\Gamma)},$$

which completes the proof.

III. MULTIPLE CAVITY SCATTERING

As shown in Fig. 2, we consider a situation of n cavities, where the multiple open cavities $\Omega_1,\ldots,\Omega_n$ enclosed by the apertures $\Gamma_1,\ldots,\Gamma_n$ and the walls $S_1,\ldots,S_n$ are placed on Γ^c. Above the flat surface $\{y=0\} = \Gamma_1 \cup \cdots \cup \Gamma_n \cup \Gamma^c$, the medium is assumed to be homogeneous with a positive dielectric permittivity ε_0. The medium inside the cavity Ω_j is inhomogeneous with a variable dielectric permittivity $\varepsilon_j(x,y)$, which satisfies $\varepsilon_j \in L^\infty(\Omega)$, $\operatorname{Re}\varepsilon_j > 0$, $\operatorname{Im}\varepsilon_j \ge 0$ for $j = 1,\ldots,n.$

We consider the two-dimensional Helmholtz equation for the total field:

$$\Delta u + \kappa^2 u = 0, \text{ in } \Omega_1 \cup \cdots \cup \Omega_n \cup R^2_+, \tag{18}$$

together with the boundary condition:

$$u = 0, \text{ on } S_1 \cup \cdots \cup S_n \cup \Gamma^c. \tag{19}$$

Let the plane wave u^i be incident on the cavities from above. The total field u is consisted of the incident field u^i, the reflected field u^r, and the scattered field u^s, where the scattered field is required to satisfy the radiation condition (6).

To reduce the problem into the bounded domains $\Omega_j, j = 1,\ldots,n,$ we need to derive a transparent boundary condition on Γ_j. Rewrite (18)-(19) into n single cavity scattering problem:

$$\begin{aligned} \Delta u_j + \kappa_j^2 u_j &= 0 \text{ in } \Omega_j, \\ u_j &= 0 \text{ on } S_j, \end{aligned} \tag{20}$$

where $\kappa_j^2 = \omega^2 \varepsilon_j \mu_0$. If u is the solution of (18)-(19) and u_j is the solution of (20), respectively, then we have $u_j = u|_{\Omega_j}$ for $j = 1,\ldots,n.$

For $u_j(x,0)$, define its zero extension:

$$\tilde{u}_j(x,0) = \begin{cases} u_j(x,0) \text{ for } x \in \Gamma_j, \\ 0 \text{ for } x \in R \setminus \Gamma_j. \end{cases}$$

For the total field u, define its extension:

$$\tilde{u}(x,0) = \begin{cases} u_j(x,0) \text{ for } x \in \Gamma_j, \\ 0 \text{ for } x \in \Gamma^c. \end{cases}$$

It follows from the definition of the extensions that we have:

$$\tilde{u} = \sum_{j=1}^n \tilde{u}_j \text{ on } \Gamma_1 \cup \cdots \cup \Gamma_n \cup \Gamma^c.$$

Repeating the same steps as those for the single cavity problem, we have the following transparent boundary condition for the extended field:

$$\partial_y \tilde{u} = T\tilde{u} + g \text{ on } \Gamma_1 \cup \cdots \cup \Gamma_n \cup \Gamma^c, \tag{21}$$

which gives the transparent boundary for u_j:

$$\partial_y u_j = T\tilde{u}_j + \sum_{i=1, i\neq j}^{n} T\tilde{u}_i + g \text{ on } \Gamma_j. \tag{22}$$

As we can see from (22), the boundary condition for $u_j, j = 1,\ldots,n$ is coupled with each other, which is the major difference between the single cavity problem and the multiple cavity problem.

Next we present a variational formulation for the multiple cavity problem. Denote $\Omega = \Omega_1 \cup \cdots \cup \Omega_n$, $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_n$ and $S = S_1 \cup \cdots \cup S_n$. Define a trace functional space:

$$\tilde{H}^{\frac{1}{2}}(\Gamma) = \tilde{H}^{\frac{1}{2}}(\Gamma_1) \times \cdots \times \tilde{H}^{\frac{1}{2}}(\Gamma_n).$$

Its norm is characterized by:

$$\|u\|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma)} = \sum_{j=1}^{n} \|u_j\|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma_j)}.$$

Denote $H^{-\frac{1}{2}}(\Gamma) = H^{-\frac{1}{2}}(\Gamma_1) \times \cdots \times H^{-\frac{1}{2}}(\Gamma_n)$, which is the dual space of $\tilde{H}^{\frac{1}{2}}(\Gamma)$. The norm on the space is characterized by:

$$\|u\|^2_{H^{-\frac{1}{2}}(\Gamma)} = \sum_{j=1}^{n} \|u_j\|^2_{H^{-\frac{1}{2}}(\Gamma_j)}.$$

Introduce the space:

$$H^1_S(\Omega) = H^1_{S_1}(\Omega_1) \times \cdots \times H^1_{S_n}(\Omega_n),$$

which is a Hilbert space with norm characterized by:

$$\|u\|^2_{H^1(\Omega)} = \sum_{j=1}^{n} \|u_j\|^2_{H^1(\Omega_j)}.$$

Similarly, we may obtain the variational formulation for the multiple cavity problem: find $u \in H^1_S(\Omega)$ with $u_j = u|_{\Omega_j}$ such that

$$a(u,v) = \sum_{j=1}^{n} \left\langle g, v_j \right\rangle_{\Gamma_j} \text{ for all } v \in H^1_S(\Omega), \tag{23}$$

where the sesquilinear form is:

$$a(u,v) = \sum_{j=1}^{n} \int_{\Omega_j} (\nabla u_j \cdot \nabla \bar{v}_j - \kappa_j^2 u_j \bar{v}_j) - \sum_{j=1}^{n} \sum_{i=1}^{n} \left\langle T\tilde{u}_i, \tilde{v}_j \right\rangle.$$

We have the following well-posedness result. The proof is similar in nature as that of the single cavity problem and is omitted here for brevity.

Theorem 2. *The variational problem (23) has a unique weak solution in and the solution satisfies the estimate*:

$$\|u\|_{H^1(\Omega)} \le C \|g\|_{H^{-\frac{1}{2}}(\Gamma)},$$

where C is a positive constant.

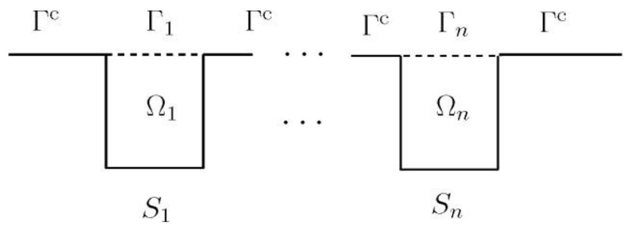

Fig. 2. The problem geometry for multiple cavities.

IV. NUMERICAL EXPERIMENTS

In this section, we discuss the computational aspects and present some examples for the multiple cavity problem.

A. Finite element formulation

Let M_j be a regular conforming triangulation of Ω_j and $V_j \subset H^1_{S_j}(\Omega_j)$ be the conforming linear finite element space over M_j. Denote $V = V_1 \times \cdots \times V_n$. The finite element approximation to the multiple cavity problem is to find u^h with $u^h \in V_j$ such that

$$a(u^h, v^h) = \sum_{j=1}^{n} \left\langle g, v_j^h \right\rangle_{\Gamma_j} \text{ for all } v^h \in V, \tag{24}$$

where the sesquilinear form

$$a(u^h, v^h) = \sum_{j=1}^{n} \int_{\Omega_j} (\nabla u_j^h \cdot \nabla \bar{v}_j^h - \kappa_j^2 u_j^h \bar{v}_j^h) - \sum_{j=1}^{n} \sum_{i=1}^{n} \left\langle T\tilde{u}_j^h, \tilde{v}_j^h \right\rangle.$$

For any $1 \le j \le n$, we denote by P_j the set of vertices of M_j, which are not on the cavity wall S_j, and let $\varphi_j(r) \in V_j$ be the nodal basis function belonging to vertex $r \in P_j$. Using the basis functions, the solution of (24) is represented as:

$$u_j^h = \sum_{r \in P_j} u_j(r) \varphi_j(r).$$

The discrete problem (24) is equivalent to the following system of algebraic equations:

$$AU = G, \tag{25}$$

where

$$A = \begin{bmatrix} A_1 - B_{1,1} & -B_{2,1} & \cdots & -B_{n,1} \\ -B_{1.2} & A_2 - B_{2,2} & \cdots & -B_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ -B_{1,n} & -B_{2,n} & \cdots & A_n - B_{n,n} \end{bmatrix},$$

$$U=(u_1,u_2,\ldots,u_n)^T,$$
$$G=(g_1,g_2,\ldots,g_n)^T.$$

Hence, each u_j is an unknown vector whose entries are $u_j(r)=u_j^h(r)$ for all $r\in P_j$, A_j is the stiffness matrix for the discrete problem and its entries are defined by:

$$A_j(r,r')=\int_{\Omega_j}[\nabla\varphi_j(r)\cdot\nabla\varphi_j(r')-\kappa_j^2\varphi_j(r)\varphi_j(r')],$$

for all $r,r'\in P_j$. The entries of $B_{j,i}$ are defined by:

$$B_{j,i}(r,r')=\left\langle T\tilde{\phi}_i,\tilde{\phi}_j\right\rangle \text{ for all } r,r'\in P_j\cap\Gamma_j,$$

and the entries of each vector g_j are given by:

$$g_j(r)=\left\langle g,\varphi_j(r)\right\rangle_{\Gamma_j} \text{ for all } r\in P_j\cap\Gamma_j.$$

A block Gauss-Seidel method is adopted to solve (25). Given $U^{(0)}$, define $U^{(k)},k\geq 1$ by the solution of the following system of equations:

$$(A_j-B_{j,j})u_j^{(k)}=g_j+\sum_{i=1}^{j-1}B_{j,i}u_i^{(k)} + \sum_{i=j+1}^{n}B_{j,i}u_i^{(k-1)},1\leq j\leq n. \tag{26}$$

The block Gauss-Seidel iteration (26) is equivalent to apply the finite element method to solve the following problem: let $u_j^{(0)}=0$, define $u_j^{(k)}$ for $k\geq 1$ by the solutions of the decoupled equations

$$\begin{aligned}\Delta u_j^{(k)}+\kappa_j^2u_j^{(k)}&=0 \text{ in } \Omega_j,\\ u_j^{(k)}&=0 \text{ on } S_j,\\ \partial_y u_j^{(k)}&=\sum_{i=1}^{j}T\tilde{u}_i^{(k)}+\sum_{i=j+1}^{n}T\tilde{u}_i^{(k-1)}+g \text{ on } \Gamma_j,\end{aligned} \tag{27}$$

for $j=1,\ldots,n$. Therefore, we only need to solve a single cavity problem (27) at each iteration.

B. Transparent boundary condition

The transparent boundary conditions (11) and (22) are not convenient to be implemented numerically. We take an alternative and equivalent transparent boundary condition [7].

Let

$$G(r,r')=\frac{i}{4}[H_0^{(1)}(\kappa_0\rho)-H_0^{(1)}(\kappa_0\bar{\rho})]$$

be the Green function of the two-dimensional Helmholtz equation in the upper half space, where $H_0^{(1)}$ is the Hankel function of the first kind with order zero; $r=(x,y),r'=(x',y'),\rho=|r-r'|,\bar{\rho}=|r-\bar{r}'|$, and $\bar{r}'=(x',-y')$ is the image of r' with respect to the real axis. By the Green's theorem and the radiation condition, we obtain:

$$\partial_y u^s(x,0)=\frac{i\kappa_0}{2}\int_\Gamma\frac{1}{|r-r'|}H_1^{(1)}(\kappa_0|r-r'|)u^s(r',0)dr',$$

where $H_1^{(1)}$ is the Hankel function of the first kind with order one. Hence, the alternative boundary condition is:

$$\partial_y u=Tu+g \text{ on } \Gamma, \tag{28}$$

where the boundary operator T is defined as:

$$Tu=\frac{i\kappa_0}{2}\int_\Gamma\frac{1}{|r-r'|}H_1^{(1)}(\kappa_0|r-r'|)u(r',0)dr'. \tag{29}$$

Here the integral is understood in the sense of Hadamard finite-part. For multiple cavities with apertures $\Gamma_1\cup\cdots\cup\Gamma_n$, the boundary operator is defined as:

$$Tu=\frac{i\kappa_0}{2}\sum_{j=1}^{n}\int_{\Gamma_j}\frac{1}{|r-r'|}H_1^{(1)}(\kappa_0|r-r'|)u(r',0)dr'.$$

The boundary operator (29) can be approximated by:

$$Tu(x_i)\approx\sum_{k=1}^{m}g_{ik}u(x_k,0),$$

where

$$\operatorname{Re} g_{ik}=-t_{ik}\frac{\kappa_0|x_i-x_j|}{2}Y_1(\kappa_0|x_i-x_j|),$$
$$\operatorname{Im} g_{ik}=\frac{\kappa_0 h_x}{2}\frac{J_1(\kappa_0|x_i-x_j|)}{|x_i-x_k|},$$

and

$$t_{ik}=\begin{cases}\frac{1}{h_x}(1-\ln 2) \text{ for } |i-k|=1,\\ -\frac{2}{h_x} \text{ for } |i-k|=0,\\ \frac{1}{h_x}\ln\frac{|i-k|^2}{|i-k|^2-1} \text{ for } |i-k|\geq 2,\end{cases}$$

where h_x is the step size of the partition for the cavity aperture Γ, Y_1 and J_1 are Bessel functions of the second and first kind with order one, respectively. Therefore, the boundary integral $\langle Tu,v\rangle$ in the weak formulation for the cavity problem can be approximated by any numerical quadratures.

C. Numerical examples

The physical parameter of interest is the Radar Cross Section (RCS), which is defined by:

$$\sigma=\frac{4}{\kappa_0}|P(\phi)|^2.$$

Here ϕ is the observation angle and P is the far-field coefficient given by:

$$P(\phi) = \frac{\kappa_0}{2} \sin\phi \int_{\Gamma} u(x,0) e^{i\kappa_0 x \cos\phi} dx.$$

When the incident and observation directions are the same, σ is called the backscatter RCS, which is defined by:

$$\text{Backscatter RCS } (\phi) = 10 \log_{10} \sigma(\phi) \text{dB}.$$

Example 1. Consider a plane wave scattering from a rectangular cavity with 1 meter wide and 0.25 meters deep at normal incidence; i.e., $\theta = 0$. Two different cases are considered: an empty cavity with $\kappa = \kappa_0$ and a cavity filled with a homogeneous medium with $\kappa^2 = \kappa_0^2(4+i)$. These two cases have been considered as standard test problems in [14]. The Rectangular domain $[-0.5, 0.5] \times [-0.25, 0.0]$ is first divided into 160×40 small equal rectangles and then each small rectangle is subdivided into two equal triangles. Numerical results are obtained by using a linear finite element over triangles at the wavenumber $\kappa_0 = 2\pi$. Figures 3 and 4 show the magnitude and the phase of the total field on the aperture at the normal incidence, the backscatter RCS for the empty cavity and the filled cavity, respectively. We observe the coincidence of the numerical results obtained in [19] (circled) and our numerical method (solid line).

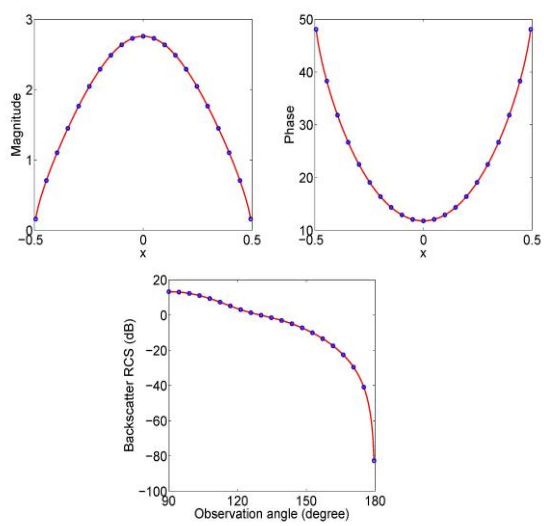

Fig. 3. The magnitude, phase, and backscatter RCS of the total field for Example 1 of the empty cavity.

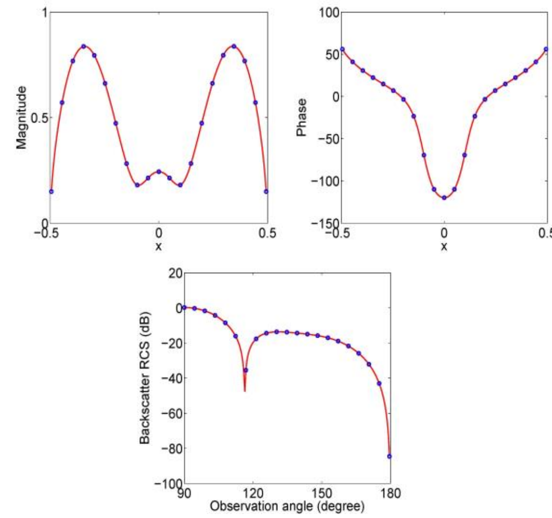

Fig. 4. The magnitude, phase, and backscatter RCS of the total field for Example 1 of the filled cavity.

Example 2. Consider the normal incidence of a plane wave onto two identical rectangular cavities. Each cavity is 1 meter wide and 0.25 meters deep; they are 1 meter distance away from each other. The two rectangular domains are given as follows:

cavity one: $[-1.5, -0.5] \times [-0.25, 0.0]$,

cavity two: $[0.5, 1.5] \times [-0.25, 0.0]$.

Each rectangular domain is divided into 160×40 small equal rectangles and then each small rectangle is subdivided into two equal triangles. Three types of cavities are considered: (type one) two empty cavities with $\kappa_1 = \kappa_2 = \kappa_0$; (type two) two filled cavities with $\kappa_1^2 = \kappa_2^2 = \kappa_0^2(4+i)$; (type three) one empty cavity with $\kappa_1 = \kappa_0$ and one filled cavity with $\kappa_2^2 = \kappa_0^2(4+i)$. Figures 5, 6 and 7 show the magnitude and the phase of the total field on the apertures at the normal incidence and the backscatter RCS for the type one, type two and type three cavities, respectively. These numerical results are obtained by the block Gauss-Seidel iterative method. To show the convergence of the iterative method, we define the error between two consecutive approximations:

$$e_k = \max_{1 \le j \le k} \left\| u_j^{(k)} - u_j^{(k-1)} \right\|_{L^2(\Gamma_j)},$$

where k is the number of iteration. Figure 8 shows the error e_k of two consecutive approximations against the number of iterations for all three types of cavities. It can be seen from Fig. 8, that more number of iterations are needed for the type one cavities to reach the same level accuracy as the other two types of cavities. The reason is that the cavity for either type two or type three is filled with complex medium, which accounts for the absorption of the energy, and thus, the damping of the amplitude of the field.

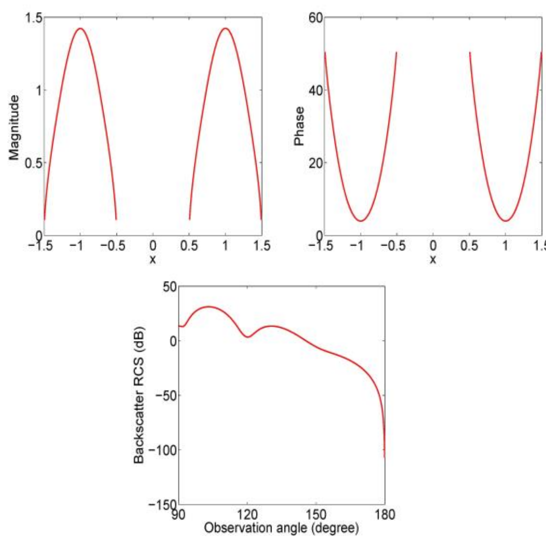

Fig. 5. The magnitude, phase, and backscatterer RCS for Example 2 of the type one cavity.

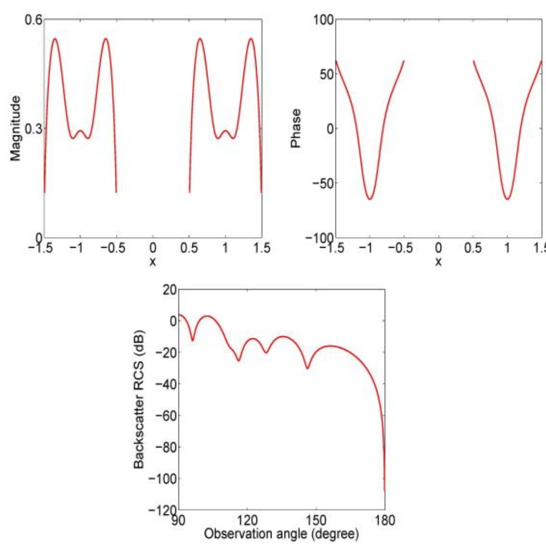

Fig. 6. The magnitude, phase, and backscatterer RCS for Example 2 of the type two cavity.

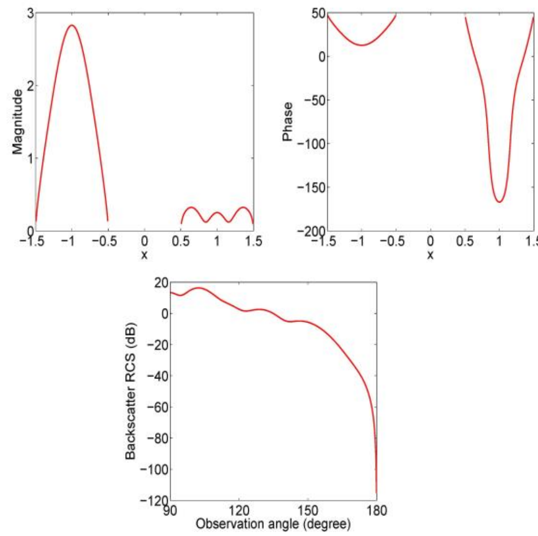

Fig. 7. The magnitude, phase, and backscatterer RCS for Example 2 of the type three cavity.

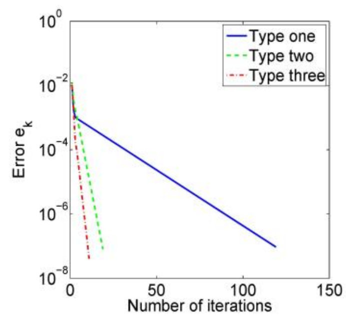

Fig 8. Convergence of the Gauss-Seidel iteration for Example 2.

Example 3. Consider the scattering of a triple cavity model. Let a plane wave be incident onto three identical rectangular cavities at the normal direction. Each cavity is 1 meter wide and 0.25 meters deep; there are 1 meter distance away from each other. The three rectangular domains are given as follows:

cavity one: $[-2.5,-1.5]\times[-0.25,0.0]$,

cavity two: $[-0.5,0.5]\times[-0.25,0.0]$,

cavity three: $[1.5,\ 2.5]\times[-0.25,\ 0.0]$.

Again, each rectangular domain is divided into 160×40 small equal rectangles and then each small rectangle is subdivided into two equal triangles.

Cavities one and three are filled with the same homogeneous medium with $\kappa_1^2 = \kappa_3^2 = \kappa_0^2(4+i)$ and cavity two is an empty cavity with $\kappa = \kappa_0$. Figure 9 shows the magnitude and the phase of the total field on the apertures at the normal incidence and the backscatter RCS.

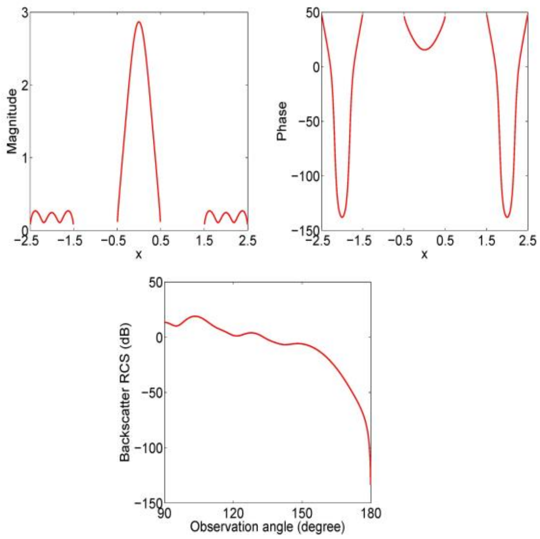

Fig 9. The magnitude, phase, and backscatter RCS of the total field for Example 3.

V. CONCLUSION

We studied the problem of electromagnetic scattering by multiple cavities embedded in the infinite two-dimensional ground plane. The scattering problem was reduced into a boundary value problem by introducing a transparent boundary condition. Based on the variational formulation, we proved the uniqueness and existence of the weak solution for the model problem. We employed a block Gauss-Seidel iterative method to decouple the coupled system arising from the multiple interaction among cavities. At each step of iteration, it required to solve only a single cavity problem. Three numerical examples were considered, a single cavity, two cavities and three cavities, with and/or without filling. The results show the convergence of the block Gauss-Seidel iterative method for the examples. We point out some future directions along the line of our present work. The first is to analyze the convergence of the Gauss-Seidel iterative method and investigate the parameters, such as separation distance among cavities, wavenumber and cavity size, which requires further mathematical analysis of the stability of the cavity scattering problem [6]. Another project is to study the multiple overfilled cavity problem and the model problem of three-dimensional Maxwell equations.

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