Analysis of Scattering Problems by MOM with Intervallic Wavelets and Operators

Sami Barmada, Marco Raugi
Dipartimento Sistemi Elettrici e Automazione,
Via Diotisalvi 2 56126 Pisa Italy.
Phone +39 050 565312 Fax +39 050 565333
E-mail: sami.barmada@dsea.unipi.it

Abstract— **In this paper a method for the solution of scattering problems is proposed. In particular the EFIE is solved by a collocation point procedure, where the unknown current density is expanded by Daubechies wavelets on the interval and the integration is performed in the wavelet domain by the integral operator, hence without the use of any quadrature formula. Comparison with induced currents calculated by a standard MoM and with fields calculated by a FEM code are reported.**

I. Introduction

Wavelet Expansion (WE) has become a widely used tool in electromagnetic analysis. The main reasons can be found in the strong interpolating properties of the wavelet functions and to the fact that WE and reconstruction can be performed by the use of fast algorithms (see for example [1] - [3]).

The analysis of scattering problems can be carried out expanding the unknown functions in terms of a chosen wavelet basis and performing a Galerkin procedure using the same wavelet functions as test functions (see [4] - [6]). The integration is then performed by the use of quadrature formulae.

In this paper the EFIE for the scattering on a conductive body is solved in a different way. By utilizing Daubechies wavelets on the interval the unknown current is expanded, then a collocation point method is used and the integration is performed by the use of the integral operator for Daubechies wavelets on the interval developed by the authors (see [8]). In this way the need of quadrature formulae is avoided; furthermore the well known capability of the wavelet functions of representing irregular signals with few coefficients, allows us to use bases of low dimension (if compared with the number of unknowns of a standard MoM); for these reasons both accuracy and CPU time saving are achieved.

II. Wavelets on the Interval and Operators

The concepts of scaling functions, wavelets, time-scale analysis, multiresolution analysis are here considered known [1]; there are many wavelet bases available in the literature, and we chose the Daubechies Wavelets on the interval [2] for their numerical properties. In particular the choice of wavelets that "survive" only on intervals is adopted because we are interested in the solution of a boundary value problem.

From the wavelet theory we know that WE must be performed starting from a signal known at a dyadic number of samples. This number is equal to the dimension of the basis on which we perform the WE; hence a signal represented by $n = 2^m$ samples when expanded in the wavelet domain leads to a number of 2^m coefficients. For the compact support wavelet there is an important relation that allows a straight computation of the coefficients at the higher resolution of a generic function: it is possible to obtain them from the samples of the functions itself according to the relation $\langle \phi_{J,k}, f \rangle = 2^{J/2} f(2^J k)$, where $\phi_{J,k}$ is the scaling function of order J, k of the adopted wavelet basis. Then, the vector of wavelet coefficients $\mathbf{g}$ representing the wavelet transform of a function $g(x)$ can be obtained by multiplying a matrix $\mathbf{W}$ related to the adopted wavelet basis and the samples $g(x_j)$ corresponding to 2^m equally spaced points in the interval $[0, 1]$. Further details about wavelet numerical computation can be found in [3].

When performing WE of a function, the notation that will be used throughout the paper is the following:

$$f(t) = \sum_i f_i b_i(x) = \mathbf{b}(x)\mathbf{f} \qquad (1)$$

where $\mathbf{b}(x) = [b_1(x), \cdots b_n(x)]$ is the wavelet basis

and $\mathbf{f} = [f_1, \cdots, f_n]^T$ is the vector of coefficients constituting the wavelet expansion of the signal.

Also operators can be represented in the wavelet domain, as described in [7]. The authors have obtained the representation of the integral operator for the Daubechies wavelets on the interval (see for example [8]). The convenience in using operators in the wavelet domain is that the integral of a function $f(x)$ can be calculated by the matrix-vector product $\mathbf{Tf}$ where $\mathbf{T}$ is the constant sparse matrix representing the operator in the wavelet domain and $\mathbf{f}$ is the wavelet expansion of the function $f(x)$. In the previous operation the result is the primitive function of f. In order to better understand the previous statement let us define $f(x) = x^3$; we can write that

$$\int_{[0,1]} f(x)dx = \int_{[0,1]} x^3 dx = \frac{1}{4}x^4 = g(x) \qquad (2)$$

where the integral limit $[0, 1]$ indicates that we are considering the interval $[0, 1]$. Expanding (2) according to (1) in the wavelet domain we can write that

$$\int_{[0,1]} \mathbf{b}(x)\mathbf{f}dx = \mathbf{b}(x)\mathbf{g} \qquad (3)$$

where $\mathbf{f}$ and $\mathbf{g}$ are the wavelet expansion of the two functions. Left multiplying by $\mathbf{b}(x)^T$ and taking into account the definition of the integral operator ($\mathbf{T} = \langle \mathbf{b}(x), \int \mathbf{b}(x) \rangle$) and the orthogonality properties of the wavelet basis ($\mathbf{I_d} = \langle \mathbf{b}(x), \mathbf{b}(x) \rangle$, whith $\mathbf{I_d}$ being the identity matrix) we can write that

$$\mathbf{Tf} = \mathbf{g} \qquad (4)$$

Hence as clearly stated in equation (4) the calculation of function $g(x)$ can be performed in the wavelet domain by multiplying matrix $\mathbf{T}$ by vector $\mathbf{f}$, the wavelet expansion of the function $f(x)$; then the result, vector $\mathbf{g}$, is inverse transformed, obtaining the function $g(x)$.

In order to compute a definite integration directly in the wavelet domain, two new (row) vectors must be introduced: vectors $\mathbf{b}_0 = \mathbf{b}(0)$ and $\mathbf{b}_1 = \mathbf{b}(1)$ that are the values of the basis functions on the left and right border, respectively, of the interval $[0, 1]$. The meaning of these two vectors is the following: given a function $f(x)$, its left border value $f(0)$ can be obtained by the coefficients of its WE simply by the use of the vector $\mathbf{b}_0$ as $f(0) = \mathbf{b}_0\mathbf{f}$.

Suppose that we want to compute

$$\int_0^1 f(x)dx = \int_0^1 x^3 dx = \frac{1}{4}x^4|_0^1 = g(x)|_0^1 = \frac{1}{4} \qquad (5)$$

Then starting from equation (4) it must be considered that the primitive function ($\mathbf{g}$) is already calculated, and we only need to evaluate it on the borders of the interval. Hence it yields

$$\int_0^1 f(x)dx = \mathbf{b}_1\mathbf{g} - \mathbf{b}_0\mathbf{g} \qquad (6)$$

which can be rewritten as

$$\int_0^1 f(x)dx = \mathbf{b}_1\mathbf{Tf} - \mathbf{b}_0\mathbf{Tf} \qquad (7)$$

As evidenced in equation (7) the calculation of the definite integral of a function $f(x)$ can be performed knowing its WE $\mathbf{f}$, the integral operator matrix $\mathbf{T}$ and the border vectors. The quantities $\mathbf{T}$, $\mathbf{b}_0$ and $\mathbf{b}_1$ are known once a wavelet basis has been chosen, so they need to be computed only once, and not at any analysis.

III. Method of Moments and Wavelet Expansion

A. General Considerations

In the study of scattering from conducting cylinders, an integral equation can be formulated, which in general has the form of

$$j\frac{\eta}{\beta}\left[\beta^2 \int\int_S \mathbf{J_s}(\mathbf{r}')\mathbf{G}(\mathbf{r_s}, \mathbf{r}')\mathbf{ds}' + \right.$$
$$\left. +\nabla \int\int_S \nabla' \cdot \mathbf{J_s}(\mathbf{r}')\mathbf{G}(\mathbf{r_s}, \mathbf{r}')\mathbf{ds}'\right] = \mathbf{E_t^i}(\mathbf{r} = \mathbf{r_s}) \qquad (8)$$

where $\eta = \sqrt{(\mu/\epsilon)}$ and $\beta^2 = \omega^2\mu\epsilon$; $\mathbf{J_s}$ is the current density induced on the scatterer, G is the green function for the three dimensional scatterer, $\mathbf{r}'$ and $\mathbf{r_s}$ are respectively the integration variable and the observation point, both on the surface scatterer; and $\mathbf{E_t^i}$ is the incident field.

In a simpler way and in one dimension equation (8) can be in general rewritten as

$$\int g(x')K(x, x')dx' + c(x)g(x) = h(x) \qquad (9)$$

where $g(x)$ is our unknown function. In the literature [5], function g is expanded in the wavelet

domain, and the expansion is substituted in (9). In order to obtain a linear system for the unknown coefficients, the resultant equation is tested with the same expansion functions (Galerkin's method). Quadrature formulae available in the literature for wavelets and scaling functions are used, and a square system is obtained, which solved gives the unknown coefficients.

Due to the high interpolating properties of the Daubechies wavelets on the interval, we have adopted this family of wavelets, for which the authors have developed the representation of the integral operator. Wavelets on the interval (and so operators in the wavelet domain) are defined on $[0,1]$, hence the contour of the scatterer must be mapped into the interval $[0,1]$. As it is suggested in [6] for an arbitrary contour of the scatterer two steps must be performed:

- The contour of the scatterer is discretized in boundary elements and then each boundary element (simply a first order element) is mapped into one dimensional standard element through shape functions or interpolation functions.
- The standard elements are mapped into corresponding portions of the interval $[0,1]$.

In this way the basis functions are defined in a standard way on the interval $[0,1]$, since the contour has been mapped on this interval.

B. Scattering from a Conductive Body

In case of a two dimensional problem with TM^z polarization the EFIE equation is the following:

$$\frac{\eta\beta}{4}\int_C J_z(\rho')H_0^{(2)}(\beta|\rho_m-\rho'|)dc' = E_z^i(\rho_m) \quad (10)$$

where ρ_m is any observation point on the scatterer, ρ' is any source point on the scatterer and C is the contour of the scatterer.

After the mapping on the elemental interval $[0,1]$ (described in the previous section) is performed, we perform a classical collocation point procedure: we evaluate equation (10) at a particular point $\bar{\rho}_m$ of the contour; hence equation (10) can be rewritten as

$$k\int_0^1 f(x)g(x)dx = h \quad (11)$$

where $k=\dfrac{\eta\beta}{4}$, $f(x)=H_0^{(2)}(\beta|\rho_m-\rho'|)$ and $g(x)=J_z(\rho')$.

Then the wavelet expansion in the space domain is performed. It has to be noticed that in equation (11) there is the product between two functions, in particular one of them is known ($f(x)$) while the other is our unknown. As explained in [8] it is possible to obtain the wavelet expansion of a the product between two functions $y(x)=f(x)g(x)$ as the product between a constant diagonal matrix $\mathbf{F}$ and a vector $\mathbf{g}$, where $\mathbf{g}$ is the wavelet expansion of the function $g(x)$ and $\mathbf{F}$ is a diagonal matrix whose entries are the samples at $n=2^m$ equally spaced points of the values of $f(x)$ in the interval. This approximation is as much accurate as the number of samples is high, hence as the resolution of the chosen basis increases. In case the two functions are know the above described procedure is useless, since the expansion of $y(x)$ can be performed directly. But in the cases when one of the two functions is unknown then the procedure is fundamental, since it allows to keep $\mathbf{g}$ as the unknown vector and anyway perform the wavelet expansion. Hence equation (11) can be expressed in the wavelet domain as

$$k\int_0^1 \mathbf{b}(x)\mathbf{F}\mathbf{g}dx = \mathbf{b}(x)\mathbf{h} \quad (12)$$

where $\mathbf{F}$ is the diagonal matrix with the samples of the Hankel function, $\mathbf{g}$ is the vector of unknown coefficients and $\mathbf{h}$ is the expansion of constant h on the interval $[0,1]$ i.e. considered as a constant function on the whole interval. By left multiplying equation (12) by $\mathbf{b}(x)^T$ we obtain

$$k\mathbf{b}(x)^T\int_0^1 \mathbf{b}(x)dx\mathbf{F}\mathbf{g} = \mathbf{b}(x)^T\mathbf{b}(x)\mathbf{h} \quad (13)$$

and taking into account the definition of the integral operator and the orthogonality properties of the chosen basis we obtain

$$k\mathbf{T}\mathbf{F}\mathbf{g} = \mathbf{h} \quad (14)$$

In this way equation (14) establishes a relation between the primitive of the product $f(x)g(x)$ and the constant function $h(x)=constant$. This is actually something different from what we want, hence the two vectors $\mathbf{b}_0$ and $\mathbf{b}_1$ (introduced in section II) must be employed. Hence we can write that

$$\mathbf{b}_1 k\mathbf{T}\mathbf{F}\mathbf{g} - \mathbf{b}_0 k\mathbf{T}\mathbf{F}\mathbf{g} = \mathbf{h} \quad (15)$$

Equation (15) is characterized by known matrices $\mathbf{T}$ and $\mathbf{F}$ and known vectors $\mathbf{b}_0$ and $\mathbf{b}_1$ and by the unknown vector of coefficients $\mathbf{g}$. Once the resolution of the wavelet basis is chosen the number of

basis functions is consequently fixed, hence equation (15) must be written for a number of n points on the interval itself. This leads to a sparse linear square system whose unknowns are the coefficients $\mathbf{g}$ and which can be solved in low CPU time.

IV. Numerical Results

The numerical results presented here are relative to the scattering of a square conductive object illuminated by a polarized TM_z field. The geometry of the system and the input signal are reported in figures 1 and 2.

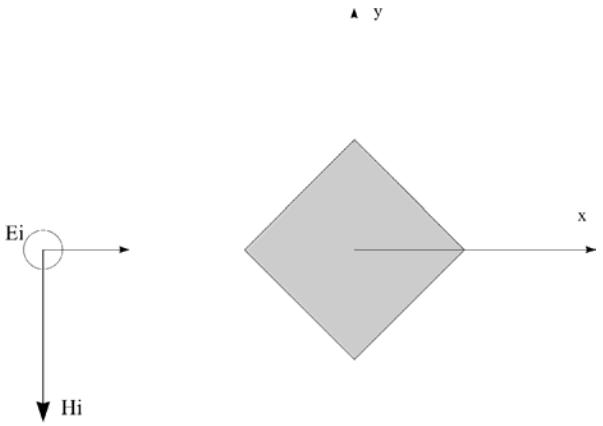

Fig. 1. Geometry of the system

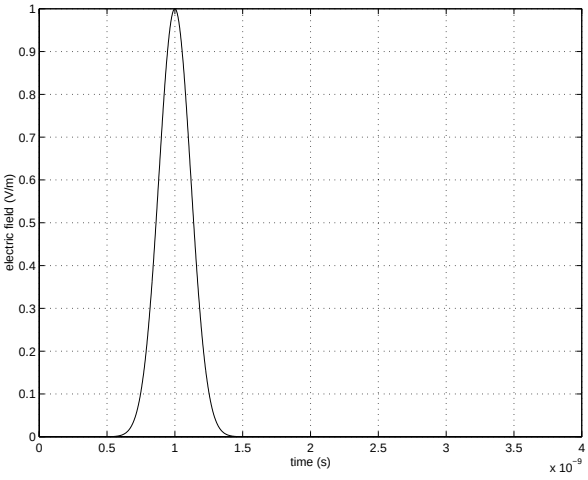

Fig. 2. Input signal

The diagonal of the scatterer is of 0.2m and the frequency content of the input signal is of the order of GHz. FFT has been used in order to obtain the behavior in the time domain. Figures 3 - 5 show the calculated current density on the surface scatterer for a number of 64, 128 and 256 wavelet functions for a frequency of 1 GHz.

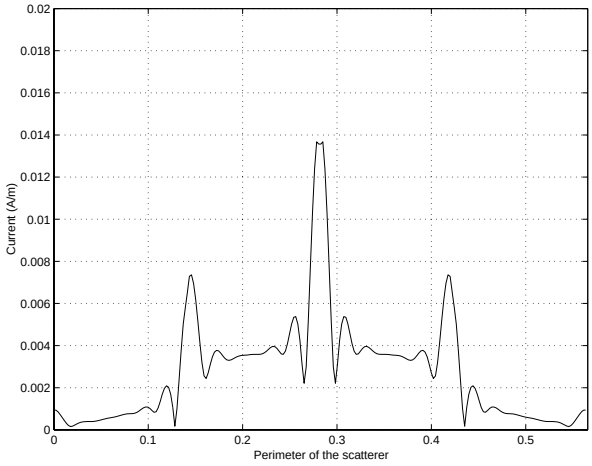

Fig. 3. Current density on the scatterer calculated by the use of 64 wavelets

It is evident that together with the increase of the resolution from 64 to 256, the accuracy of the results becomingh higher. Nevertheless even at lower resolutions the obtained current (at a very lo CPU time cost) is consistent witht the problem.

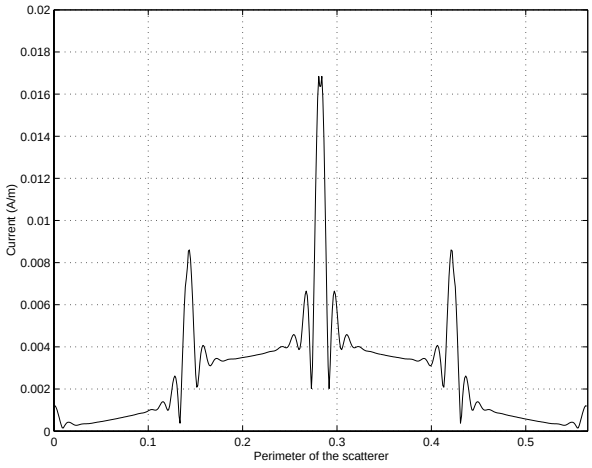

Fig. 4. Current density on the scatterer calculated by the use of 128 wavelets

Figure 6 shows the comparison between the calculations performed by the proposed method with a resolution of 256 wavelets and a standard MoM (with collocation point) technique with 500 points on the whole perimeter. It can be seen the very good agreement between the two different methods.

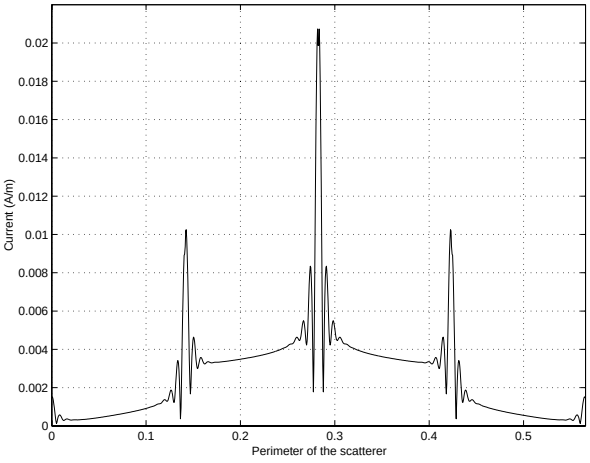

Fig. 5. Current density on the scatterer calculated by the use of 256

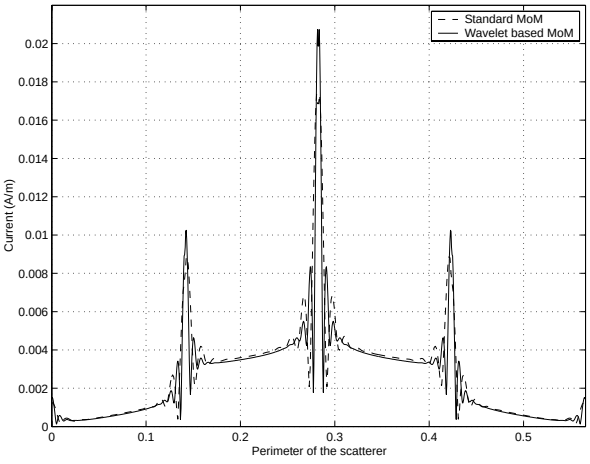

Fig. 6. Current density on the scatterer calculated by the use wavelets and MoM

As for the CPU time the method proposed is of the order of 2-4 times faster than the standard MoM, due to the high numerical efficiency of the wavelet expansion and integration, as described in the previous sections: the construction of the integral operator matrix and the border vectors is done only once and can be seen as a pre processing activity, while the integration is performed by a simple matrix - vector product, without the need of any quadrature formula. Furthermore the well known numerical properties of the wavelet functions (well addressed in the literature) allow the choice of bases of small dimensions in order to obtain accurate results.

Since the calculation of the current density is not the only important result from an engineering poin of view, a comparison with results obtained by the use of a FEM code of the total field are also reported. In figures 7 and 8 the electric field in point $P \equiv (-0.2, 0)$ and $P \equiv (0, -0.2)$ are reported, evaluated by the FEM code and by the proposed method, by using a wavelet basis of 256 functions.

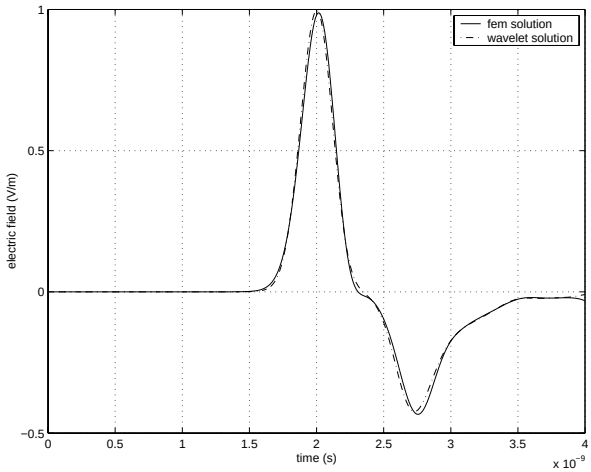

Fig. 7. Electric field evaluated at point $P \equiv (-0.2, 0)$

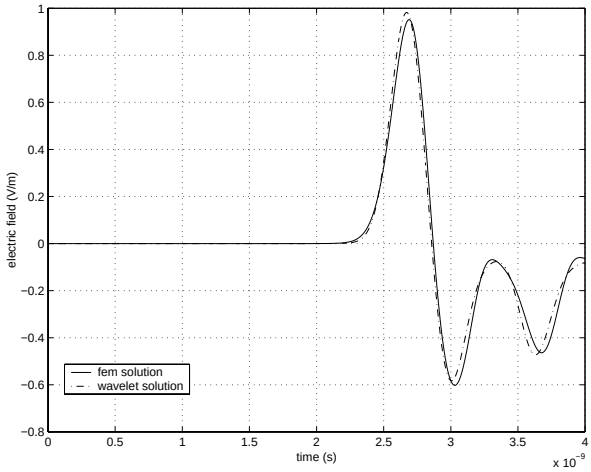

Fig. 8. Electric field evaluated at point $P \equiv (0, -0.2)$

Figure 9 reports the comparison between two solutions obtained by the proposed method at different resolutions, in particular with 64 and 256 wavelets, and shows the robustness of the method in terms of calculated fields

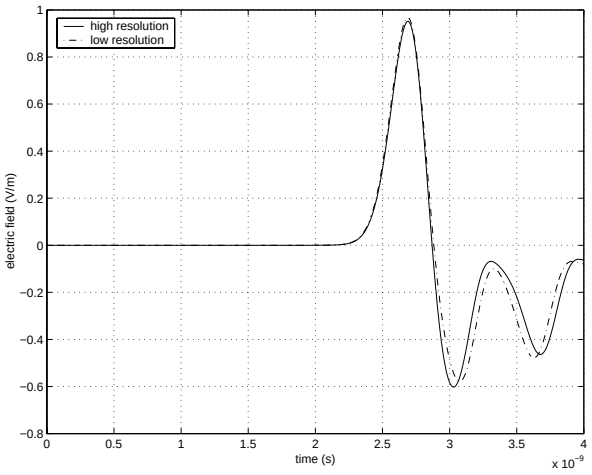

Fig. 9. Comparison at different resolutions of Electric field evaluated at point $P \equiv (0, -0.2)$

V. Conclusion

The method proposed here performs the analysis of scattering problems by the use of a MoM technique numerically implemented by wavelet expansion. In particular the use of the integral operator in the wavelet domain makes it possible to obtain the unknown current without the use of quadrature formulae, and the high interpolating properties of the chosen wavelet basis gives results with low CPU time. The good quality of the results is demostrated by comparisons with standard MoM and FEM computations.

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Sami Barmada was born in Livorno, Italy on November 18, 1970. He received his master degree in Electrical Engineering from the University of Pisa in 1995. He worked from 1995 to 1997 at ABB Teknologi (Oslo, Norway), on distribution network analysis and optimization. In 2001 he received the Ph. D. in Electrical Engineering at the department of Electrical System and Automation of the University of Pisa. He is currently assistant professor in the same department and he is working on numerical computation of electromagnetic fields, in particular on the modeling of MTL and to the application of the wavelet expansion to computational electromagnetics.

Marco Raugi received the degree in Electronic Engineering from the University of Pisa in 1985 and the Ph. D. in Electrical Engineering from the Department of Electrical Systems and Automation in 1990. He is currently a full professor of Electrical Engineering. His research activity is in the numerical methods for the analysis of electromagnetic fields in linear and nonlinear media. Main applications have been devoted to Non Destructive Testing, Electromagnetic compatibility in transmission lines, electromagnetic launchers. He is author of around 100 papers on international journals and conferences. He is the general chairman of the International conference PIERS (Progress In Electromagnetic Research Symposium) 28-31 March 2004 in Pisa. He has been chairman of the Editorial Boards and he served as chairman and session organizer of international conferences. He has been recipient of the IEEE-IndustryApplicationSociety 2002 Melcher Prize Paper Award.