On the Construction and Use of Two-Dimensional Wavelet-Like Basis Functions

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Abstract: An alternative method for generating higher dimensional wavelet-like basis functions is proposed in this paper. One method that has been used was to derive the two-dimensional wavelet-like basis from the two-dimensional traditional finite element basis. However, in this paper, products of one-dimensional wavelet-like functions are used as two-dimensional wavelet-like basis functions. The generation of linear wavelet-like functions is discussed in detail and the use of linear and higher order wavelet-like functions is also investigated. The advantages and disadvantages of this technique for deriving wavelet-like basis functions will be discussed.

Keywords: Wavelets, Iterative Techniques, Finite Element Methods

I. INTRODUCTION

Wavelets and wavelet analysis have recently become increasingly important in the computational sciences. Wavelets have many applications in areas such as signal analysis, image compression, and the numerical solution of partial differential equations and integral equations. Only rather recently, however, have wavelets begun being used in computational electromagnetics. The multiresolution time domain technique (MRTD), developed by Katehi et. al, has attracted abundant interest in the use of wavelets as basis functions [1]. Gordon has used wavelet-like basis functions in the numerical solution of elliptic partial differential equations [2]. The wavelet-like functions have also been used as the basis for a finite element time-domain algorithm [3]. Although the wavelet-like functions are not true wavelets, they do exhibit some of the benefits that have caused wavelets to receive attention. One advantage that will be discussed in detail in this paper is that wavelet-like basis functions have good stability and convergence properties.

II. GENERATION OF BASIS FUNCTIONS

The method for generating the wavelet-like basis was first discussed by Jaffard in [4]. Consider the generation of linear wavelet-like functions for which the domain, Ω, is the line segment $0 \le x \le 1$. Assume that the problem under consideration has Dirichlet boundary conditions so that the value of the solution is specified at both endpoints of the problem domain. This eliminates the necessity of nodes at the endpoints of the domain. To begin the multiresolution analysis (MRA) for the generation of the linear wavelet-like basis functions, an initial discretization is chosen such that there is a single node placed at the midpoint of the domain (Fig. 1). This corresponds to beginning with two segments in the initial discretization; one segment from $0 \le x \le 0.5$, and another segment from $0.5 \le x \le 1$. This particular discretization is not a requirement; a very simple or extremely complex segmentation may be used as the initial discretization of the problem domain.

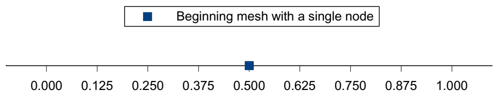

Fig. 1. Initial discretization with a single node in the mesh.

This beginning discretization is chosen for ease of illustration of the MRA. The traditional linear basis function associated with the node at $x = 0.5$ is normalized (Fig. 2).

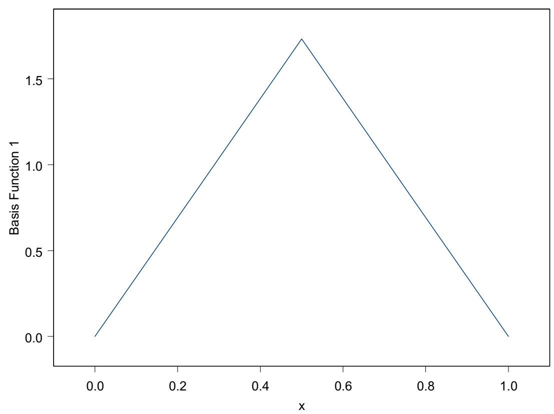

Fig. 2. Wavelet-like function from the first level of the MRA.

The first level of the MRA has now been completed and this function is considered to be the first wavelet-like basis function. To begin the second level of the MRA, each of the two segments from the first level is divided into two equal segments. After doing this, there are now four segments in the domain: one segment from $0 \le x \le 0.25$, another segment from $0.25 \le x \le 0.5$, another from $0.5 \le x \le 0.75$, and another from $0.75 \le x \le 1.0$. The node at $x = 0.5$ is not a new node, and the function associated with it is discarded. The nodes located at $x = 0.25$ and at $x = 0.75$ are new nodes (Fig. 3). Therefore, these two nodes need to be considered in the analysis. Next, the traditional basis functions that are associated with the two nodes are orthogonalized against the wavelet-like function from the first level; then, the resulting functions will be orthonormalized against each other.

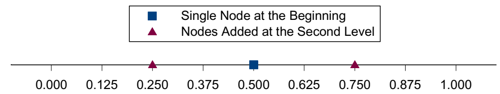

Fig. 3. Initial discretization with added nodes from second level of the MRA.

After the orthonormalization, the two functions can be added to the wavelet-like basis. Now the second level of the MRA is complete and there are three wavelet-like functions in the basis. The wavelet-like function associated with the node at $x = 0.25$ is shown in Fig. 4.

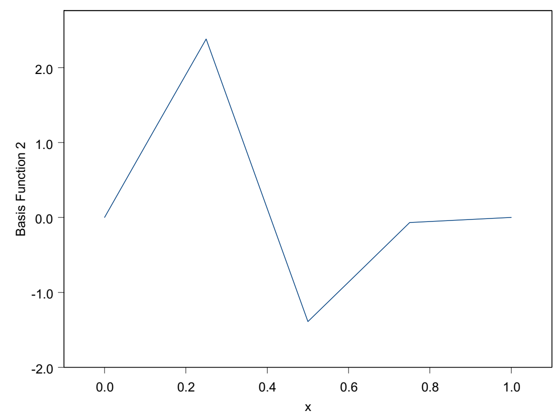

Fig. 4. Wavelet-like function added at the second level of the MRA.

The process of subdividing the segments and orthogonalizing traditional basis functions against previous wavelet-like basis functions and then orthonormalizing the resulting functions can be continued until the desired level of discretization is reached. Figures 5, 6, and 7 show the progression of the subdivision of the line segment from the third level to the fifth level of the analysis. Also, the linear wavelet-like basis function associated with the node at $x = 0.375$, which was added during the third level of the MRA, is shown in Fig. 8.

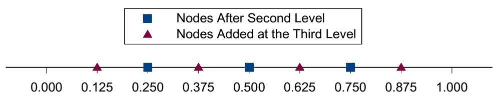

Fig. 5. Second Level discretization with added nodes from the third level.

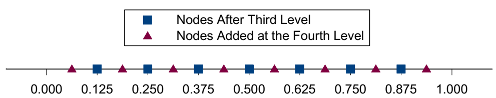

Fig. 6. Third level discretization with added nodes from the fourth level.

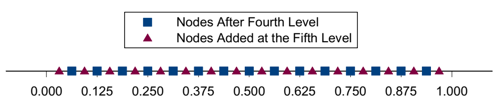

Fig. 7. Fourth level discretization with added nodes from the fifth level.

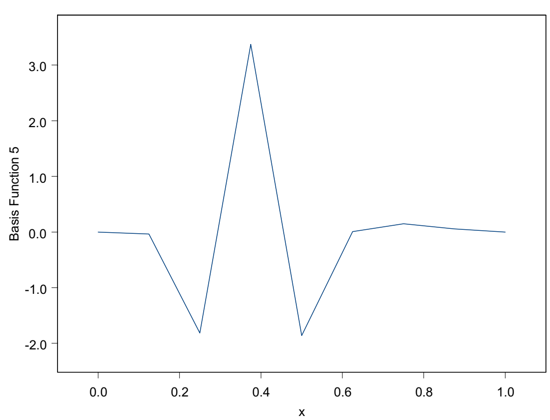

Fig. 8. Wavelet-like function added at the third level of the MRA.

This concludes the discussion of the generation of one-dimensional linear wavelet-like functions. Now there will be a brief discussion of the generation of higher dimensional and higher order wavelet-like functions.

There are two methods that have been used to generate higher dimensional wavelet-like functions. One possibility is to generate them from their higher dimensional traditional finite element counterparts. For example, a piecewise linear two-dimensional wavelet-like basis can be generated from the traditional two-dimensional tetrahedral basis. However, this is not how higher dimensional wavelets are typically created. Instead, they are generally formed from products of one-dimensional wavelets [5]. In two dimensions, this yields

$$\Psi_{m,n}(x,y) = \Psi_m(x)\Psi_n(y) \qquad (1)$$

Hutchcraft and Gordon have shown that this technique can also be employed using products of wavelet-like functions [6].

Just as higher dimensional wavelet-like functions can be generated using their traditional counterparts, so can higher order wavelet-like functions. These functions have been used by Hutchcraft and Gordon in the numerical solution of a one-dimensional problem in [7] in which the traditional piecewise cubic basis functions are used to generate piecewise cubic wavelet-like basis functions. Implementing both of these concepts, higher order, higher-dimensional wavelet-like functions can be generated by forming products of one-dimensional higher order wavelet-like functions.

III. EXAMPLES OF ONE AND TWO-DIMENSIONAL BASIS FUNCTIONS

Consider a rectangular region as the domain for a two-dimensional problem. To obtain a two-dimensional wavelet-like basis, one-dimensional wavelet-like functions need to be generated in both the x- and y-directions by the method outlined previously. For the two-dimensional wavelet-like basis, all products of a wavelet-like function in the x-direction with a wavelet-like function in the y-direction will be considered a two-dimensional wavelet-like basis function; thus, the total number of wavelet-like functions generated by this procedure will be the total number of wavelet-like functions in the x-direction multiplied by the total number of wavelet-like functions in the y-direction.

To aid in the visualization of these functions, Figs. 9-16 show several one- and two-dimensional linear and cubic wavelet-like functions. First, Figs. 9 and 10 illustrate one-dimensional cubic wavelet-like functions. In Fig. 10, the more slowly varying of the two functions is from the first level of the MRA. It has a single piecewise cubic representation over the entire domain. The other function in Fig. 10 is from the second level in the MRA. It is also piecewise cubic, but it has two different representations; one representation for the segment $0 \le x \le 0.5$ and another representation for the segment from $0.5 \le x \le 1.0$.

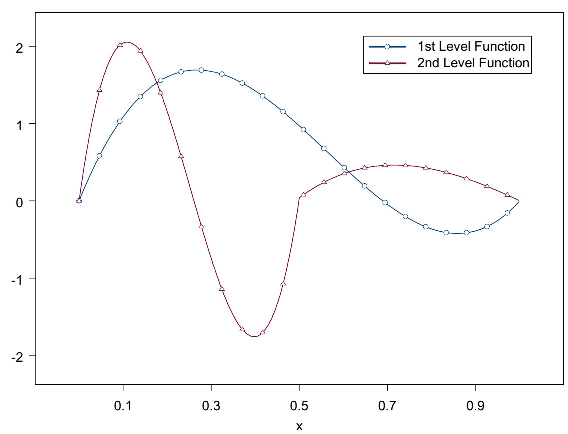

Fig. 9. Third order wavelet-like basis functions from the 1st and 2nd levels.

As discussed previously, two-dimensional wavelet-like functions are obtained by forming products of one-dimensional wavelet-like functions. Figure 11 shows a two-dimensional linear wavelet-like function. The linear wavelet-like function $B_6(x,y)$, which could also be written as $B_2(x)B_3(y)$ to denote that it is derived from the product of the 2nd basis function in the x-direction and the 3rd basis function in the y-direction, is formed from a function from the second level of the MRA in the x-direction and a function from the second level of the MRA in the y-direction.

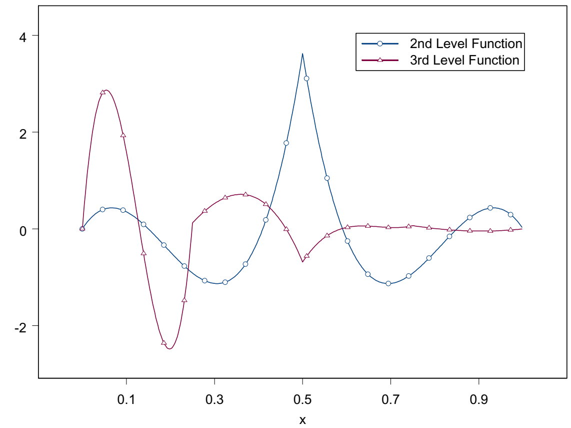

Fig. 10. Third order wavelet-like basis functions from the 2nd and 3rd levels.

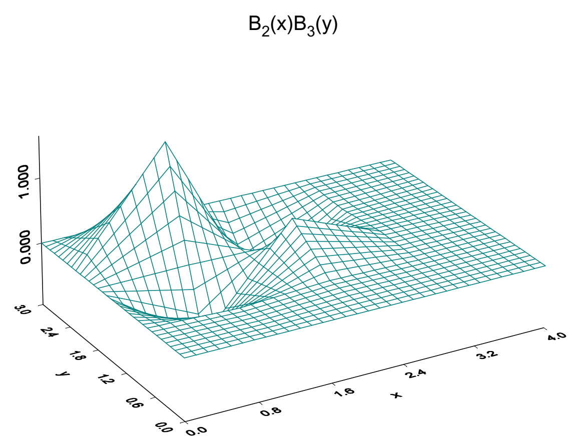

Fig. 11. Linear wavelet-like basis function obtained from a 2nd level x and 2nd level y function.

Plots of several two-dimensional cubic wavelet-like basis functions are shown in Figs. 12, 13, 14, and 15. $B_1(x,y)$ is a cubic wavelet-like basis function that is generated from the first level in both the x- and y-directions (Fig. 12). As can be seen from the figure, this function is nonzero over most of the domain. It is also a piecewise cubic polynomial in the x-direction and a piecewise cubic polynomial in the y-direction. $B_5(x,y)$ and $B_{10}(x,y)$ are both generated from the first level of the MRA in the y-direction and the second level of the MRA in the x-direction (Figs. 13 and 14). In the x-direction, each of these two functions has two different piecewise cubic representations; on the other hand, both of these functions have a single representation in the y-direction. Specifically, in the x-direction, there is one piecewise cubic representation for the segment $0 \le x \le 2.0$, and another piecewise cubic representation for the segment $2.0 \le x \le 4.0$. B_{15}(x,y) is a basis function that is obtained from a second level x-directed function and a second level y-directed function (Fig. 15). This function has two different piecewise cubic representations in both the x- and y-directions.

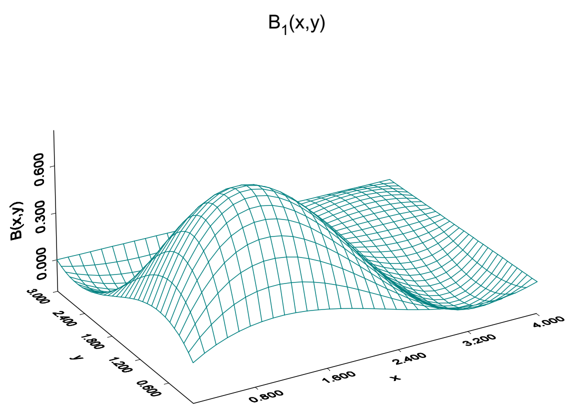

Fig. 12. Cubic wavelet-like basis function obtained from a 1st level x and 1st level y function.

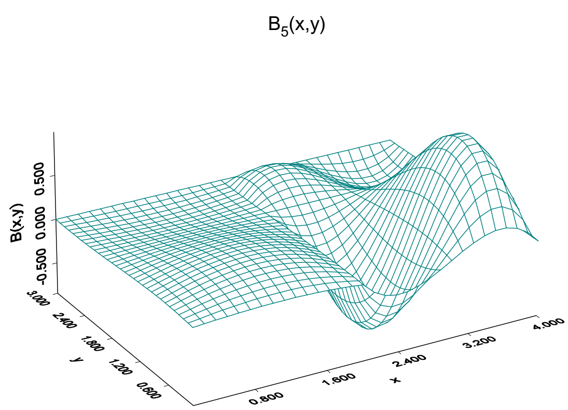

Fig. 13. Cubic wavelet-like basis function obtained from a 2nd level x and 1st level y function.

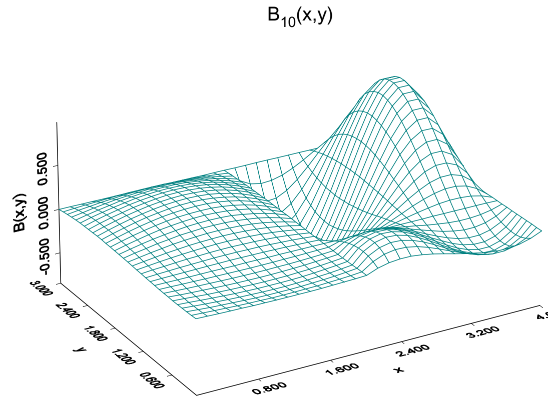

Fig. 14. Cubic wavelet-like basis function obtained from a 2nd level x and 1st level y function.

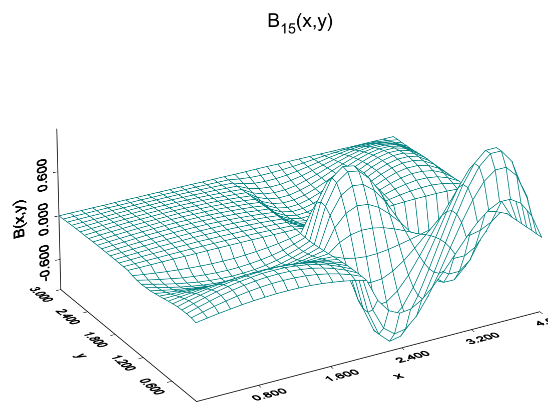

Fig. 15. Cubic wavelet-like basis function obtained from a 2nd level x and 2nd level y function.

With wavelet analysis, as levels in the MRA are added, the wavelets become more localized. As can be seen from these figures, the wavelet-like basis functions also possess this property; they have a large magnitude in a smaller portion of the domain as the level in the MRA for either (or both) the x- or y-directions increases. B_1(x,y) has a rather large magnitude over the entire domain. Again, B_5(x,y) and B_{10}(x,y) are from the second level in the x-direction and the first level in the y-direction; notice that these two functions have a large value only in half of the region. B_{15}(x,y) is a function from the second level in both the x- and y-directions and its value is large only in one-quarter of the domain.

IV. EXAMPLE PROBLEM

As an example of the use of the wavelet-like basis functions, consider the following differential equation

$$-\nabla\cdot\left(a\left(x,y\right)\nabla u\left(x,y\right)\right)+b\left(x,y\right)u\left(x,y\right)=g\left(x,y\right) \quad (2)$$

in which the domain is the rectangular region from $x=0.0$ to $x=4.0$ and from $y=0.0$ to $y=3.0$. Laplace's equation can be obtained by choosing the following: $a(x,y)=-1.0$, $b(x,y)=0.0$, and $g(x,y)=0.0$. An illustration of the problem domain along with the boundary conditions is shown in Fig. 16.

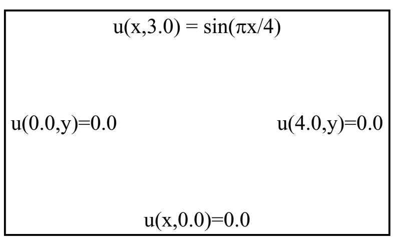

Fig. 16. Problem domain.

Solutions were obtained using the traditional two-dimensional basis functions, two-dimensional basis functions that were products of linear wavelet-like basis functions, and two-dimensional basis functions that were products of cubic wavelet-like basis functions. A comparison of the analytic solution and the numerical solution, which was found using 961 linear wavelet-like basis functions, is made along the line $x=1.5$ (Fig. 17). The numerical solution in this case corresponds to 31 wavelet-like functions in each direction (31*31=961 total basis functions). To illustrate the accuracy when the linear wavelet-like basis is used, the curves for the numerical solution and the analytic solution lie on top of each other. To illustrate that an accurate solution is also obtained when the cubic wavelet-like basis functions is used, the analytic solution and the numerical solution, which was obtained with 55 cubic wavelet-like basis functions, are compared along the line y=1.5 (Fig. 18). Again, these two curves are indistinguishable on the graph. The numerical solution obtained with only 25 cubic wavelet-like basis functions is plotted in figure 19. For this graph, five cubic wavelet-like functions in each direction were used as the two-dimensional basis. As expected, very few cubic basis functions are necessary to obtain an accurate solution. From these figures, it is seen that the solutions obtained are accurate when either cubic or linear wavelet-like basis functions are used.

Although the ability of any basis function to accurately model an arbitrary function is quite important, the wavelet-like basis also has other advantages. Previously, wavelet-like functions have been shown to have extremely good convergence and stability properties. After diagonal preconditioning, the condition number of the system matrix was calculated. Figure 20 illustrates how the condition number varies as the number of basis functions is increased. Because the condition number of the system matrix is much smaller for the wavelet-like bases in comparison with the rapidly rising condition number when the traditional basis is used, the condition numbers when the linear and cubic wavelet-like basis functions are used are shown separate in Fig. 21. The benefits of this low condition number are especially evident when looking at the number of steps required for convergence of the conjugate gradient method. In Fig. 22, the number of steps required for convergence of the conjugate gradient method is plotted as the number of basis functions is increased. With approximately 225 basis functions, the traditional basis requires 78 steps for convergence; this is in contrast to only 34 for 225 linear wavelet-like basis functions and only 18 for 253 cubic wavelet-like basis functions.

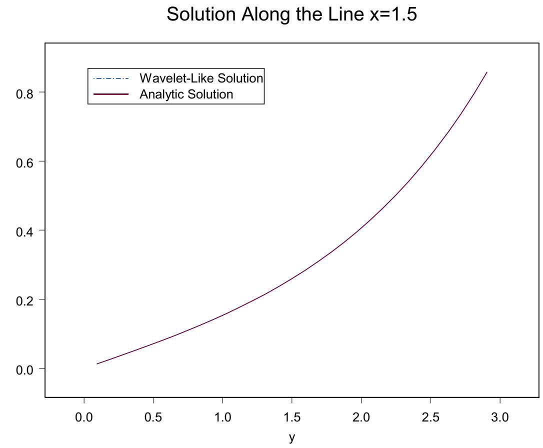

Fig. 17. Numerical (with 961 linear 2D wavelet-like basis) and analytic solutions along the line x = 1.5.

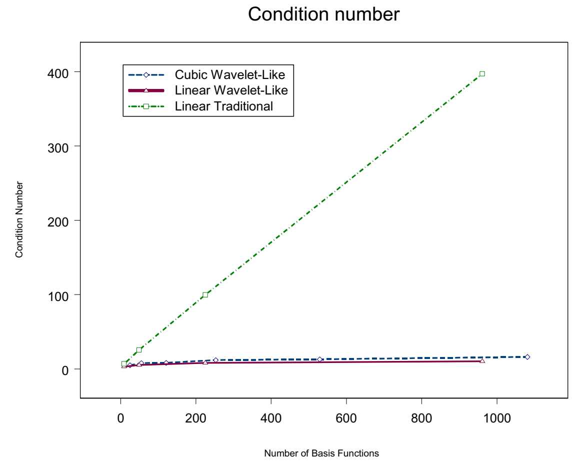

Fig. 20. Condition number comparison of wavelet-like and traditional basis.

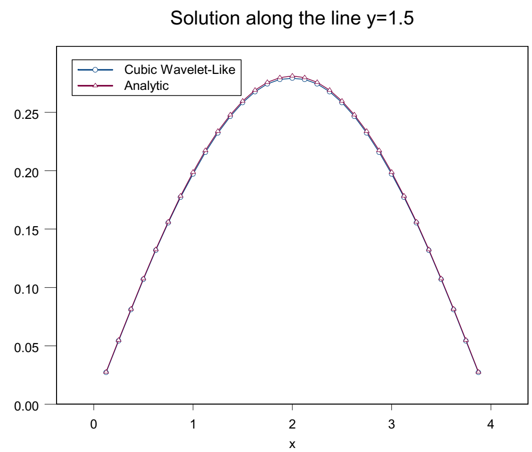

Fig. 18. Numerical and analytic solution along the line y=1.5.

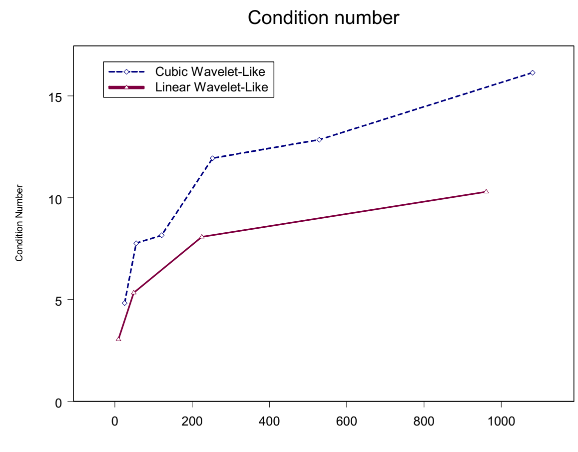

Fig. 21. Condition number when wavelet-like bases Are used.

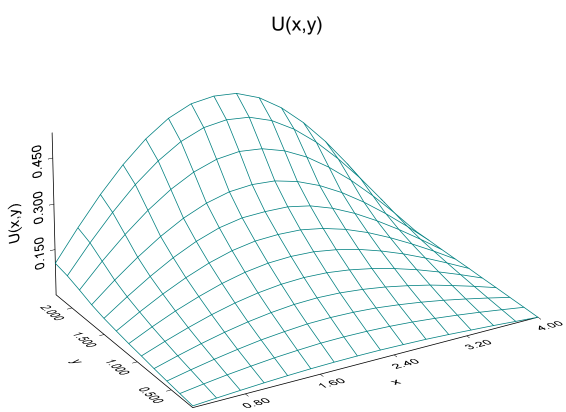

Fig.19. Numerical solution when 25 cubic wavelet-like functions are used.

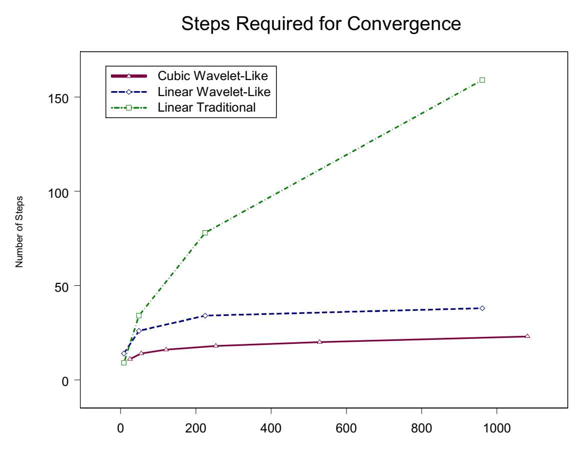

Fig. 22. Steps required for convergence.

V. CONCLUSION

It has been shown that one-dimensional wavelet-like basis functions can be multiplied to obtain two-dimensional basis functions that give accurate results when they are used to obtain numerical solutions. The stability of the condition number and the rapid convergence when the conjugate gradient method is used have also been shown to be two advantages of using either linear or higher order wavelet-like rather than traditional basis functions. As is the case with higher order traditional basis functions, fewer cubic wavelet-like basis functions are required for high accuracy. One disadvantage of this method is that the mesh would resemble more of a finite difference mesh rather than the triangular patches that are typically associated with the finite element method; however, non-uniform spacing is still rather easily accomplished with the wavelet-like method.

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