A HYBRID FINITE DIFFERENCES & CHARGE SIMULATION METHOD FOR THE COMPUTATION OF HIGH VOLTAGE ELECTRIC FIELDS

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Abstract : A new treatment is proposed to the hybrid method of Finite Differences and Charge Simulation for the computation os electric fields, entirely applicable to the similar hybrid method of Finite Element and Charge Simulation. The resulting system of linear equations is solved by using the fixed point theory, the QR decomposition and the Conjugate Gradients Squared method with a preconditioning technique. New procedures are suggested for the discretization of the boundary conditions, which lead to results with higher precision. Case studies are included.

Key words : Electric fields, numerical methods, high voltage engineering, electrical engineering computing, digital simulation.

1. INTRODUCTION

The Finite Difference Method (FDM), the Finite Element Method (FEM) and the Charge Simulation Method (CSM) are very commonly used for field analysis of high voltage insulation systems. The CSM is suitable for unbounded problems, but becomes complicated for problems including dielectrics. On the other hand,the FDM and FEM are suitable for multi-dielectric cases, but require truncation of the domain for unbounded problems. Thus, a hybrid method may be a promising tool for unbounded problems including dielectrics, where the FDM or the FEM is applied inside a limited arbitrary region, and the CSM is applied to the unbounded exterior. Along the coupling surface continuity conditions are imposed.

Early research work [1,2,3] presented practical examples of this hybrid method, pointing out its advantages and disadvantages. Either the hybrid FDM & CSM or the FEM & CSM require the solution of a system of linear equations, whose matrix of coefficients (square in principle) is composed by full submatrices (resulting from the CSM) and sparse submatrices (resulting from the FDM or FEM). The present work deals with the formulation of adequate procedures for the solution of this linear system, which results in a significant smaller computer storage, smaller CPU computer time and results with higher precision if compared to [1,2,3]. In addition, the convergence of the proposed iterative method is not dependent on the arbitrary initial conditions as [1] does.

The application of the CSM to the unlimited exterior region as it was previously considered by [1,2,3], leads to full square submatrices and demands an excessive amount of computer memory. It is shown in section 3 that it is possible to significantly reduce the required computer memory, by choosing a number of simulated charges which is smaller than the number of contour points. This leads to a least squares problem, which is solved by means of the QR decomposition [4,5] of the resulting full rectangular submatrix, using the Modified Gram-Schmidt method. This procedure only demands from 30% to 50% of the computer memory initially required by the original problem, without loss of precision.

The application of the FDM or the FEM to the limited region leads to a sparse linear system of the $\mathbf{A}\,\mathbf{x} = \mathbf{b}$ type. It is possible to use the Conjugate Gradients method (CG) [4], a Krylov Subspace method (KS) [6] or the Conjugate Gradients Squared method (CGS) [7] for the solution of this linear system. These methods have advantages over the traditional Gauss-Seidel, SOR and others (used by [1,2,3]), since they reach the exact solution in a number of steps at most equal to the matrix dimension. The CGS with a preconditioning technique is suggested for the solution of the problem, since it demands less CPU computer time and less computer memory than other methods (see section 4).

The hybrid method requires a discretization for the normal component of the electric flux density vector $\mathbf{D}$ (i. e. $\mathbf{n}\cdot\mathbf{D}$) all over the arbitrary rectangular surface of discontinuity on the electrical permittivity ε (coupling boundary). This leads to a discretization of the normal derivative of the scalar electric potential function (since $\mathbf{D} = -\varepsilon \nabla\Phi$). The error of such discretization as computed by [1,2,3] for rectangular surfaces is of the order of h (i.e. O(h)), where h is the grid step (distance between two consecutive nodes). It is shown (section 5.1) by the use of the Taylor series expansion, that it is possible to obtain a simple discretization with an error of $O(h^2)$. A discretization for curvilinear surfaces is also presented (section 5.2), being of interest when a discontinuity on ε exists. These methodologies for the discretization of $\mathbf{n}\cdot\nabla\Phi$ lead to results of higher precision if compared to those used by [1,2,3].

Finally, adequate procedures for the solution of the combined system of linear equations that results from the hybrid method are stated. A direct solution is not recommended due to the irregular structure of the coefficient matrix. Reference [1] proposes an iterative scheme which depends on a good initial estimate. In addition, it may not reach the solution for certain cases. The approach used here (section 6) is based on the fixed point theory of linear systems [4,5]. The utilization of an arbitrary but rather predictable parameter (θ) leads to convergence in a significantly greater class of problems (if not all), not depending on the initial estimates for the electric potentials. Some examples are included and the results are analysed (section 7).

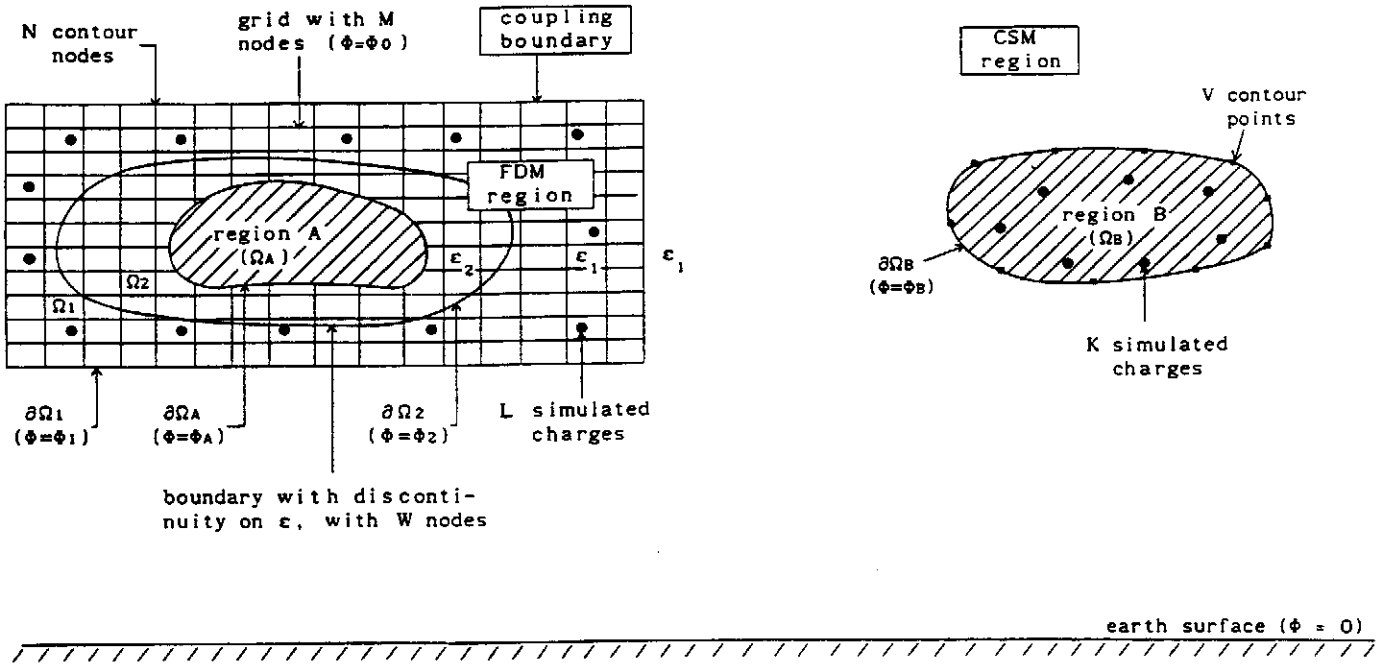

Figure 1. General problem treated with the FDM & CSM

Considering the points above, it is supposed that the present work may represent a new motivation for the utilization of the hybrid FDM & CSM or FEM & CSM on the computation of two-dimensional or three-dimensional with axial symmetry electric fields, as frequently found in high voltage engineering.

2. DEFINITION OF THE COMBINED SYSTEM OF LINEAR EQUATIONS

The hybrid FDM & CSM applied to the general two-dimensional problem of Figure 1 will be considered in the following sections.

Suppose that it is needed to calculate the electric potential distribution near electrode A, sorrounded by a region (Ω_2) with electrical permittivity ε_2. An arbitrary rectangular boundary $\partial\Omega_1$ involving A and Ω_2 is defined, inside which a finite difference grid with M nodes is placed. This also will define N nodes on $\partial\Omega_1$ and W nodes on $\partial\Omega_2$. A discretization with V contour points is made on the boundary $\partial\Omega_B$ of the external electrode B, and K+L charges are simulated as indicated in Figure 1.

It was already shown by [1,2,3] that, for two-dimensional or three-dimensional problems with axial symmetry, the application of the FDM & CSM (or the FEM & CSM) to Figure 1 leads to a system of linear equations, which is represented here by the following equivalent system with submatrices :

$$\begin{bmatrix} P_1 & -I & 0 \\ P_2 & 0 & 0 \\ F & S_0 & S_1 \\ 0 & S_2 & D \end{bmatrix} \begin{bmatrix} q \\ \Phi_1 \\ \Phi_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \Phi_B \\ 0 \\ c \end{bmatrix} \qquad (1)$$

where

a) The matriz equations $\mathbf{P_1}\,\mathbf{q} - \mathbf{I}\,\Phi_1 = \mathbf{0}$ and $\mathbf{P_2}\,\mathbf{q} = \Phi_B$ result from the CSM, and

$\mathbf{P_1}$ = full matrix, dimension N x (K+L)
$\mathbf{P_2}$ = full matrix, dimension V x (K+L)
$\mathbf{I}$ = Identity matrix, dimension N x N
$\mathbf{q}$ = vector of simulated charges, dimension K+L
Φ_1 = vector of electric potentials on $\partial\Omega_1$, dimension N
Φ_B = vector of electric potentials on $\partial\Omega_B$, dimension V
$\mathbf{P_1}$ and $\mathbf{P_2}$ are the Maxwell electric potential coefficient matrices

b) The matrix equation $\mathbf{F}\,\mathbf{q} + \mathbf{S_0}\,\Phi_1 + \mathbf{S_1}\,\Phi_0 = \mathbf{0}$ results from the continuity condition $\mathbf{n}\cdot\mathbf{D}$ on $\partial\Omega_1$, and

$\mathbf{F}$ = full matrix, dimension N x (K+L)
$\mathbf{S_0}$ = sparse matrix, dimension NxN
$\mathbf{S_1}$ = sparse matrix, dimension NxM
Φ_0 = vector of electric potentials on Ω_1 , Ω_2 and Ω_A, dimension M
$\mathbf{F}$ is the Maxwell electric field coefficient matrix

c) The matrix equation $\mathbf{S_2}\,\Phi_1 + \mathbf{D}\,\Phi_0 = \mathbf{c}$ results from the FDM (discretization of the Laplacian operator [3,9]), and

$\mathbf{S_2}$ = sparse matrix, dimension M x N
$\mathbf{D}$ = sparse matrix, dimension M x M
$\mathbf{c}$ = sparse vector which depends on the given electric potential vector Φ_A of the electrode A.

The form (1) is adequate for a problem with only one dielectric. A small modification is suggested for the general problem of Figure 1 (with two or more dielectrics), since it is more convenient to treat Ω_1 and Ω_2 separately, taking an

additional vector Φ_2 for the electric potentials on $\partial\Omega_2$. Therefore, the problem can be formulated as follows:

On the region Ω_1 and its boundaty:

$$\begin{aligned} &\Delta\Phi = 0 \qquad \text{on } \Omega_1 \\ &\Phi = \Phi_1 \qquad \text{on } \partial\Omega_1 \\ &\Phi = \Phi_2 \qquad \text{on } \partial\Omega_2 \\ &\mathbf{n}\cdot\mathbf{D} = g \ , \text{i.e., } -\varepsilon_1 \partial\Phi/\partial n = g \qquad \text{on } \partial\Omega_2 \end{aligned} \tag{2}$$

where Δ is the Laplacean operator
and g is an auxiliary function

On the region Ω_2 and its boundaty:

$$\begin{aligned} &\Delta\Phi = 0 \qquad \text{on } \Omega_2 \\ &\Phi = \Phi_A \qquad \text{on } \partial\Omega_A \\ &\Phi = \Phi_2 \qquad \text{on } \partial\Omega_2 \\ &\mathbf{n}\cdot\mathbf{D} = -g \ , \text{i.e., } -\varepsilon_2 \partial\Phi/\partial n = -g \qquad \text{on } \partial\Omega_2 \end{aligned} \tag{3}$$

Joining (2) and (3) together, the following problem is obtained,

$$\begin{aligned} &\text{a)}\ \Delta\Phi = 0 \qquad \text{on } \Omega_1 \cup \Omega_2 \\ &\text{b)}\ \Phi = \Phi_A \qquad \text{on } \partial\Omega_A \\ &\text{c)}\ \Phi = \Phi_1 \qquad \text{on } \partial\Omega_1 \\ &\text{d)}\ \Phi = \Phi_2 \qquad \text{on } \partial\Omega_2 \\ &\text{e)}\ \varepsilon_1 \partial\Phi/\partial n + \varepsilon_2 \partial\Phi/\partial n = 0 \quad \text{on } \partial\Omega_2 \end{aligned} \tag{4}$$

The discretization of (4a) as given by [9] for two-dimensional problems (see [10] for three-dimensional problems with axial symmetry) and applied to the general node P_0 of Figure 2 is

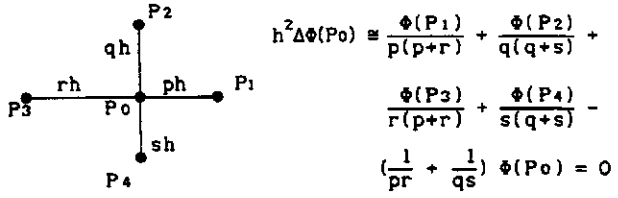

Figure 2. FDM discretization

where h = grid step; $0 < p,q,r,s \le 1.0$.

If we have $0 < p,q,r,s \le 1.0$ when P_0 is adjacent to the boundary, and $p,q,r,s = 1.0$ in the interior region (which means a grid with square elements in the interior), then the above discretization has an error of $O(h^2)$. In any other case the error is of O(h) [9].

Once the expression above is applied to every single node of the FDM region, a sparse linear system of dimension M is obtained. When P_0 is adjacent to $\partial\Omega_1$ or $\partial\Omega_2$, the corresponding equation includes some nodes from the vector Φ_1 or Φ_2 respectively. Adjacent to $\partial\Omega_A$, the equation for P_0 includes some nodes from the given electric potential vector Φ_A. Therefore, the final sparse linear system that results from (4a), (4b), (4c) and (4d) have the following matrix form (including S_3 as the additional sparse matrix associated to the new vector Φ_2)

$$\mathbf{S_2}\Phi_1 + \mathbf{D}\Phi_0 + \mathbf{S_3}\Phi_2 = \mathbf{c} \tag{5}$$

On the other hand, (4e) is applicable to the W nodes of $\partial\Omega_2$, which requires the discretization of $\partial\Phi/\partial n$ at each one of these nodes. Hence, a total of W equations can be written, including the electric potential vectors Φ_0 and Φ_2 (see section 5.2). These equations have the following matrix form:

$$\mathbf{S_4}\Phi_0 + \mathbf{E}\Phi_2 = \mathbf{0} \tag{6}$$

Finally, (1) and the suggested forms (5) and (6) are placed together in a single system,

$$\begin{bmatrix} P_1 & -I & 0 & 0 \\ P_2 & 0 & 0 & 0 \\ F & S_0 & S_1 & 0 \\ 0 & S_2 & D & S_3 \\ 0 & 0 & S_4 & E \end{bmatrix} \begin{bmatrix} q \\ \Phi_1 \\ \Phi_0 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \Phi_B \\ 0 \\ c \\ 0 \end{bmatrix} \tag{7}$$

This system is equivalent to the more explicit form of Figure 3.

The CSM equations and the FDM are treated separately for the solution of (7), as described in the following sections.

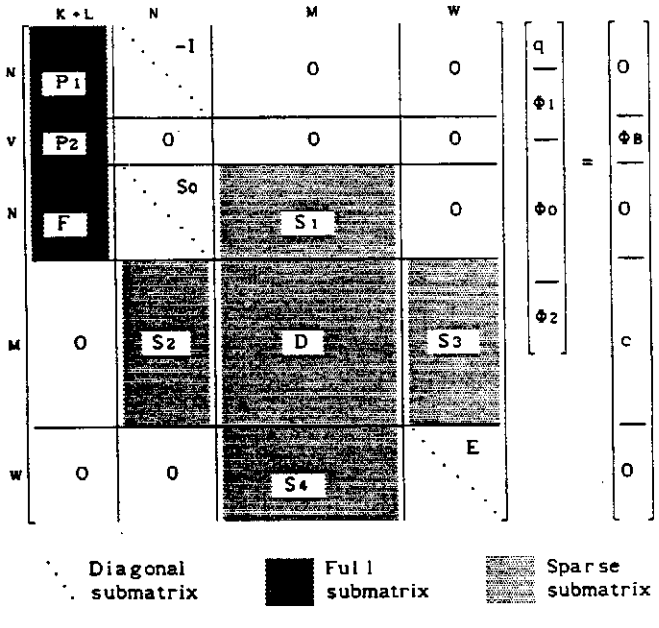

Figure 3. The complete system of linear equations

3. CSM WITH LEAST SQUARES

Suppose that the electric potentials are given on $\partial\Omega_1$ (vector Φ_1) and inside Ω_1 (vector Φ_0). Therefore, the vector of charges **q** can be calculated by means of the CSM, solving the following system obtained from (7)

$$\begin{bmatrix} P_2 \\ F \end{bmatrix} q = \begin{bmatrix} \Phi_B \\ -(S_0\Phi_1 + S_1\Phi_0) \end{bmatrix} \tag{8}$$

which may be written in the condensed form $\mathbf{C\,q = d}$.

Since (8) has V+N equations and K+L unknown charges (and provided that $V+N \ge K+L$), the least squares method can

be used to find the vector **q**. In this method, the calculated vector **q** minimizes the Euclidean norm of the error vector **C q - d**. One of the best ways of doing so is to utilize the method of the QR decomposition [4,5], calculating **QR = C**, where **Q** is a V+N x K+L matrix with orthonormal columns and **R** is a K+L x K+L upper triangular matrix. It is possible to show that the sparse linear system $\mathbf{R\,q = Q^T d}$ (where $\mathbf{Q^T}$ means the transpose of **Q**) provides the exact vector solution **q** for the least squares problem. This system is easily solved by back-substitution since **R** is upper triangular.

Q and **R** may be computed by the modified Gram-Schmidt method, whose mathematical description and algorithm are found in [4,5]. In a computer program, **Q** may be stored over the space of memory of matrix **C**, and the lines of **R** with only its non-zero elements can be sequentially stored as a vector.

The QR method does not require the computation of the matrix product $\mathbf{C^T C}$, as the traditional least squares CSM. This advantage may significantly reduce the propagation of errors during the computations.

It will be shown (section 7) that a ratio of L/N (number of simulated charges / number of contour points) from 30% to 50% is sufficient for practical purposes.

4. FDM WITH THE CONJUGATE GRADIENTS SQUARED

Supposing that vector Φ_1 is known on $\partial\Omega_1$, vectors Φ_0 and Φ_2 can be calculated with the FDM, solving the following system obtained from (7)

$$\begin{bmatrix} D & S_3 \\ S_4 & E \end{bmatrix} \begin{bmatrix} \Phi_0 \\ \Phi_2 \end{bmatrix} = \begin{bmatrix} c - S_2\Phi_1 \\ 0 \end{bmatrix} \qquad (9)$$

This system can be written in the condensed form $\mathbf{A\,\Phi = b}$. The matrix **A** has only about 1% of non-zero elements, and can be stored in a very simple scheme by means of the following vectors:

ILIN(k) - stores the line number of matrix **A**, associated to its k-th non-zero element

JCOL(k) - stores the column number of matrix **A**, associated to its k-th non-zero element

VAL(k) - stores the numeric value of A(ILIN(k),JCOL(k))

Having **A** stored this way, a matrix-vector product of the type **y = A x** can be easily computed with the following algorithm

```
For i = 1,NTOT
    y(ILIN(i)) = y(ILIN(i)) + VAL(i) * x(JCOL(i))
end
```

where NTOT is the total number of non-zero elements of **A**.

It is easy to see in this algorithm that no ordination is necessary to the elements of **A** when generating the vectors ILIN, JCOL and VAL. This property is specially suitable to problems of the type which we intend to study.

Therefore, any method that doesn't require operations more expensive than the product matrix-vector can be used for the solution of (9), in principle. To this class of methods belong the Conjugate Gradients and some derivations. Besides this simple way of storage, these methods have the advantages of reaching the exact solution after a finite number of steps (theoretically at most equal to the matrix dimension), and for any initial estimate. The following was considered:

a) Conjugate Gradients (traditional) (CG) [4,5], without preconditioning and with a diagonal preconditioning,
b) Krylov Subspace method (KS) [6], without preconditioning and with a tridiagonal preconditioning,
c) Conjugate Gradients Squared (CGS) [7], without preconditioning and with a tridiagonal preconditioning.

The CG requires **A** to be a symmetric and positive definite matrix. Since this is not the general case, it only can be applied to the modified normal system $\mathbf{A^T A\ \Phi = A^T b}$ (since $\mathbf{A^T A}$ is symmetric and positive definite).

The KS requires the calculation of a vector basis to the Krylov subspace (which dimension is arbitrary). This vector basis may demand a significant additional memory in the computer.

Th CGS doesn't require the product $\mathbf{A^T A}$ as the CG nor the additional storage as the KS and, when used with a preconditioning technique, it needs less CPU computer time (see section 7). Therefore, the CGS is suggested for the solution of (9). Appendix I includes some details as well as the algorithm used for the CGS.

5. CONTINUITY CONDITIONS ON BOUNDARIES

An adequate application of the Maxwell equations gives the necessary continuity conditions [1,2,3], i.e., Φ and $\mathbf{n\cdot D}$ are continuous across $\partial\Omega_1$ and $\partial\Omega_2$. The condition $\mathbf{n\cdot D}$ demands a specific equation to be applied at any contour node.

5.1 Discretization of $\mathbf{n\cdot D}$ on $\partial\Omega_1$

Since ε is constant on both sides of $\partial\Omega_1$, the continuity of $\mathbf{n\cdot D}$ is equivalent to the continuity of $\mathbf{n}\cdot\nabla\Phi$. As already shown by [2,3], the CSM applied to $\mathbf{n}\cdot\nabla\Phi$ leads to the expression

$$\mathbf{n}\cdot\nabla\Phi(P_i) = -\sum_{j=1}^{K+L} f_{ij}\, q_j \ , \quad i=1\ldots N \qquad (10)$$

where fij are the Maxwell electric field coefficients.

References [2,3] use only one internal node of Ω_1 for the application of the FDM (or the FEM) to the continuity of $\mathbf{n}\cdot\nabla\Phi$, which result in an expression with an error of O(h) only. A discretization with an error of $O(h^2)$ can be obtained using the Taylor series expansion as exposed in Appendix II. then

$$\mathbf{n}\cdot\nabla\Phi(P_i) = \alpha\Phi(P_i) + \beta\Phi(P_{i-1}) + \gamma\Phi(P_{i-2}) \qquad (11)$$

where Pi lies on $\partial\Omega_1$ and P_{i-1} and P_{i-2} are internal nodes of Ω_1.

The final coupling equation is a result of taking (10) = (11), or

$$\sum_{j=1}^{K+L} f_{ij}\, q_j + [\alpha\Phi(P_i) + \beta\Phi(P_{i-1}) + \gamma\Phi(P_{i-2})] = 0 \ , \quad i=1\ldots N$$

which has the equivalent matrix form $\mathbf{F\,q + S_0\Phi_1 + S_1\Phi_0 = 0}$, also included in (7). It is easy to see that **F** is a full matrix, $\mathbf{S_0}$ is a diagonal matrix and $\mathbf{S_1}$ is a sparse matrix with only two non-zero elements per line.

5.2 Discretization of $\mathbf{n\cdot D}$ on $\partial\Omega_2$

It is seen from (4e) that this continuity condition requires the discretization of $\partial\Phi/\partial\mathbf{n}$ in two stages

a) $\varepsilon_1\,\partial\Phi/\partial\mathbf{n}$, using nodes of Ω_1,
b) $\varepsilon_2\,\partial\Phi/\partial\mathbf{n}$, using nodes of Ω_2.

The method proposed by [8] uses the node on the boundary and other three internal nodes, and is considered in some details in Appendix III. Using the proposed discretization, (4e) takes the form

$$\varepsilon_1 \sum_{j=1}^{3} a_{1j}[\Phi(P_i)-\Phi(P_{1j})] + \varepsilon_2 \sum_{j=1}^{3} a_{2j}[\Phi(P_i)-\Phi(P_{2j})] = 0 \qquad (12)$$

where Pi lies on $\partial\Omega_2$,
and P_{1j} and P_{2j} are internal nodes of Ω_1,Ω_2 respectively.

The matrix form (6) is obtained applying (12) to $i=1 \ldots w$, where **E** is a diagonal matrix and $\mathbf{S_4}$ is a sparse matrix with 6 non-zero elements per line.

6. SOLUTION OF THE COMBINED SYSTEM OF LINEAR EQUATIONS

It was already shown that the system of linear equations that represents the hybrid FDM & CSM has the matrix form (7). In order to facilitate our analysis, this system is written as

$$\begin{bmatrix} P_1 & -I & 0 \\ P_2 & 0 & 0 \\ F & S_0 & Z_1 \\ 0 & Z_2 & A \end{bmatrix} \begin{bmatrix} q \\ \Phi_1 \\ \Phi \end{bmatrix} = \begin{bmatrix} 0 \\ \Phi_B \\ 0 \\ z \end{bmatrix}, \text{ where} \qquad (13)$$

$$A = \begin{bmatrix} D & S_3 \\ S_4 & E \end{bmatrix};\ \Phi = \begin{bmatrix} \Phi_0 \\ \Phi_2 \end{bmatrix};\ z = \begin{bmatrix} c \\ 0 \end{bmatrix};\ Z_1 = [S_1\ 0];\ Z_2 = \begin{bmatrix} S_2 \\ 0 \end{bmatrix}$$

Since (13) is composed by different kind of submatrices that came from distinct problems (i.e. the full rectangular submatrices from the CSM and a sparse square submatrix from the FDM), it is suggested to solve this system using an iterative method. Doing so, each part of (13) can be treated separately in an optimized way.

Notice that once the vector of charges **q** is known, the whole problem is solved. Therefore, an iterative method can be derived by using the general expression of the fixed point theory [4]

$$q^{n+1} = T\,q^n + u \qquad (14)$$

where **T** is a matrix of dimension K+L x K+L,
u is a vector of dimension K+L, and
n is the iteration number.

According to the fixed point theory applied to linear systems [4], (14) converges whenever $\sigma(\mathbf{T}) < 1$, where $\sigma(\mathbf{T})$ is the spectral radius (the largest absolute value in the set of eigenvalues) of **T**, for any initial estimate q^0 . **T** is also known as the "iteration matrix".

It is possible to show (Appendix IV) that an iterative method can be derived so that an expression for **T** results in

$$T = \theta I + (1-\theta)R^{-1}Q^T \begin{bmatrix} 0 \\ Z_1A^{-1}Z_2 - S_0 \end{bmatrix} P_1 \qquad (15)$$

where **Q** and **R** come from the decomposition $\begin{bmatrix} P_2 \\ F \end{bmatrix} = QR$

and θ is a parameter to be chosen for every problem in order to make $\sigma(\mathbf{T}) < 1$, and so allowing (14) to converge. It holds in general that $0 \le \theta < 1$.

Obviously $\sigma(\mathbf{T})$ and **T** do not need to be explicitly calculated. Figure 4 includes a flowchart for the iterative method which generates **T** precisely as expression (15).

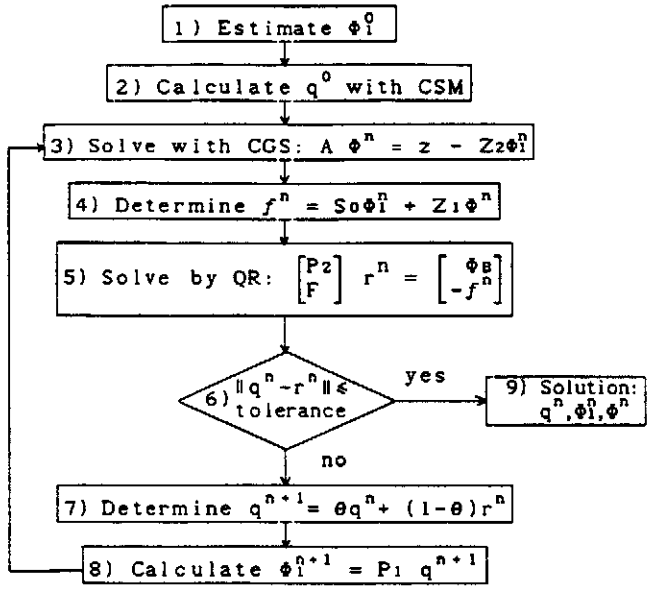

Figure 4. Iterative method for the solution of (13)

Observations:

a) The iterative method (14) is not dependent on the initial estimate q^0. However, it is possible to obtain a good initial estimate for **q** by choosing Φ_1 (step 1 in Figure 4) and solving once with the QR decomposition (step 2) the following least squares problem, taken from (7)

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} q^0 = \bar{Q}\,\bar{R}\,q^0 = \begin{bmatrix} \Phi_1^0 \\ \Phi_B \end{bmatrix} \Rightarrow \bar{R}\,q^0 = \bar{Q}^T \begin{bmatrix} \Phi_1^0 \\ \Phi_B \end{bmatrix}$$

b) The given electrode potentials are included in vectors Φ_B and **z**, which do not appear in expression (15). So the convergence to the solution (which according to the fixed point theory depends on the eigenvalues of **T**) does not depend on the given electrode potentials.

c) In view of the complexity of (15), the parameter θ can only be estimated by experience. For most of the tested problems we had convergence with $\theta \cong 0.6$ to 0.9 .

d) The iterative method proposed by [1] corresponds somewhat to the particular case of $\theta = 0$, which doesn't mean that (14) will always converge. The possibility of a choice for $\theta \neq 0$ permits the solution of a greater class of problems.

7. APPLICATIONS

We used the hybrid FDM & CSM for the calculation of the electric potential distribution close to the surface of an infinite cylinder above earth, since this case has an exact (analytical) solution. Details are reported in [11], and indicate that for this case the electric potentials calculated with the hybrid FDM & CSM have an error of less than 1% .

In a more general case, we used the hybrid FDM & CSM for the calculation of the electric potential distribution in the two-dimensional, unbounded and multi-dielectric problem as indicated in Figure 5.

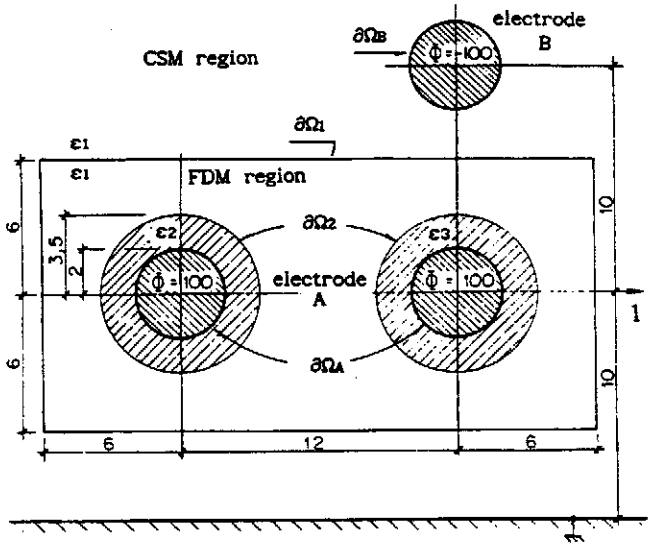

Figure 5. Two-dimensional example

Submatrices $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{F}$ of (7) include the Maxwell coefficients, calculated as indicated by [1,2,3]. In the FDM region, a rectangular grid with equally spaced nodes of h = 0.75 was placed, defining M = 15x31 = 465, N = 96 and W = 72. It was also chosen K = V = 4 .

Table 1 shows the maximum percentile deviation of the electric potentials calculated on $\partial\Omega_1$ (vector Φ_1), considered as a function of the relation between the number of simulated charges inside the FDM region (L) and the number of contour points on $\partial\Omega_1$ (N). The values were obtained by comparison with results computed with L/N ≅ 0.75 (this relation is assumed to give insignificant error for Φ_1). As a result it may be suggested that L/N ≅ 30% to 50% is a good choice.

Table 1

Error on Φ_1	
L/N	%
0.50	0.01
0.40	0.05
0.30	0.16
0.20	1.33
0.10	± 20.0

Table 2

CPU time (s)		
Method	1	2
CG	113.5	56.9
KS	453.7	17.8
CGS	36.7	9.6

1 - without precond.
2 - with precond.

Table 2 shows the CPU time for the methods CG, KS and CGS (with L/N ≅ 33%) on an IBM 3090-300S computer. The CGS with a tridiagonal preconditioning was shown to be the fastest.

As an illustration, Figure 6 shows the computed equipotential lines inside the FDM region, considering $\varepsilon_2 = 2.0$ and $\varepsilon_3 = 5.0$.

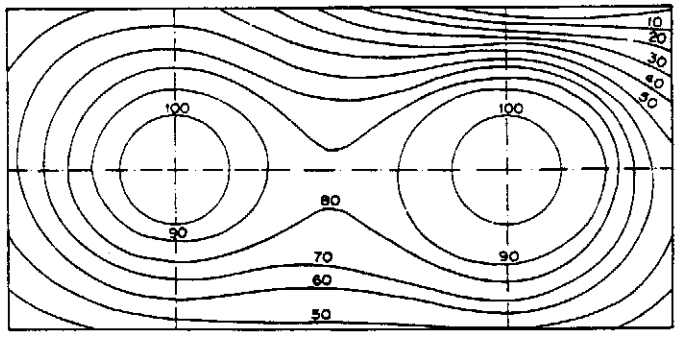

Figure 6. Equipotential lines

Figure 7 shows the electric potential distribution along the direction 1 of Figure 5, as a function of ε_2 and ε_3 . It illustrates the effect of materials with different permittivities on the electric potential distribution.

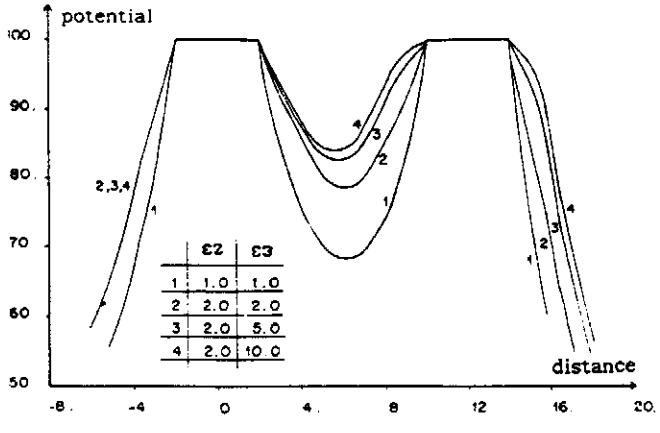

Figure 7. Electric potential as a function of ε

Additional comments:

a) The charges simulated inside the FDM region must be located at least approximately 1.0 to 2.0 times the grid step (h) distant from $\partial\Omega_1$. Charges located very close to $\partial\Omega_1$ lead to a severe loss of precision in the results (as also reported by [2]), due to the singularities of the expressions for the Maxwell coefficients.

b) When a simulated charge is located very close to another, the matrix **Q** approximates of a rank deficient matrix (see [4,5]), which may cause some difficulties in the convergence of (14). This corresponds to the case of nearly singular matrices in the traditional CSM.

c) It was considered an absolute error of 10^{-5} for the convergence of the CG, KS and CGS, and an error of 0.1% for the elements of the simulated charges vector **q** of (14) (22 iterations was necessary).

d) It is easy to see that all the mathematical approach exposed in this work applies to three-dimensional problems with axial symmetry as well (which in fact becomes reduced to a two-dimensional problem). This is feasible once the corresponding Maxwell coefficients for submatrices $\mathbf{P}_1$, $\mathbf{P}_2$ and **F** (see [1,2,3]) and the corresponding discretization for the Laplacean operator of equation (4) (see [10]) are considered. Thus, problems like the electric potential distribution across an insulator chain, or across an insulating column of a high-voltage equipment can be solved. Three-dimensional problems without any symmetry are also possible in principle.

8. CONCLUSION

New developments on the combined application of the Carge Simulation method with the Finite Difference or Finite Element method were shown.

The application of the Charge Simulation method as a least squares problem with the QR decomposition results in saving more than half of the original amount of computer memory, without loss of precision.

The use of the CGS method with a tridiagonal preconditioning for the solution of the discretised problem results in a significant reduction of the CPU computer time, with the consequence of costs reduction and reduced error propagation in the computer.

New procedures for the discretization of the boundary conditions were suggested, leading to results of higher precision.

An iterative method for the solution of the combined problem was presented, by making use of the fixed point theory of linear systems, which allows convergence for a greater class of electrostatic problems. Another feature of this new iterative method is the non-dependence on the initial estimate for electric potentials.

Finally, case studies show that the suggested innovations are effective in the mathematical formulation of the FDM & CSM or FEM & CSM, and they may represent a new motivation for the application of these hybrid methods on the computation of electric fields for unbounded problems.

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APPENDIX I

CGS algorithm with preconditioning

Consider the CGS method with tridiagonal preconditioning for the solution of the system of linear equations $\mathbf{A}\,\mathbf{x} = \mathbf{b}$. For the application of the preconditioning technique, this system is modified as

$$(\mathbf{LU})^{-1}(\mathbf{Ax}) = (\mathbf{LU})^{-1}\mathbf{b}$$

where **L,U** = matrices resulting from the LU decomposition of the tridiagonal part of **A** [4,5] .

In this case, the algorithm for the CGS is the following [7]:

```
        estimate xo
        solve (LU) ro = (b - A xo)
        q0 = p-1 = 0 ; ρ-1 = 1 ; n=0
10      if(rn < tolerance) end
        ρn  = ro^T rn ; βn = ρn/ρn-1
        un = rn + βn qn
        pn = un + βn (qn + βn pn-1)
        solve  (LU) vn =  (A pn)
        σn = ro^T vn ; αn = ρn/σn
        qn+1 = un - αn vn
        vn = αn (un + qn+1)
        xn+1 = xn + vn
        solve (LU) rn+1 = (b - A xn+1) ; n=n+1
        go to 10
```

APPENDIX II

Discretization for $\mathbf{n}\cdot\nabla\Phi$

A simple discretization for $\mathbf{n}\cdot\nabla\Phi(Pi)$ is obtained using the Taylor series expansion, assuming existence of nodes arranged on a line with the same direction of $\mathbf{n}$ (Figure 8).

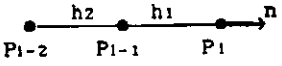

Figure 8. Discretization for $\partial\Phi/\partial n$ at P_i

Then

$$\Phi(P_{i-1}) = \Phi(P_i) - h_1\Phi'(P_i) + h_1^2\Phi''(P_i)/2 + O(h^3) \quad (16)$$

$$\Phi(P_{i-2}) = \Phi(P_i) - (h_1+h_2)\Phi'(P_i) + (h_1+h_2)^2\Phi''(P_i)/2 + O(h^3) \quad (17)$$

Taking $(h_1+h_2)^2\cdot(16) - h_1^2\cdot(17)$,

$$\mathbf{n}\cdot\nabla\Phi(P_i) = \Phi'(P_i) = \alpha\Phi(P_i)+\beta\Phi(P_{i-1})+\gamma\Phi(P_{i-2}) + O(h^2)$$

where $\alpha = 1/h_1 + 1/(h_1+h_2)$
$\beta = -(1/h_1 + 1/h_2)$
$\gamma = 1/h_2 - 1/(h_1+h_2)$
$h = \max(h_1,h_2)$

So the approximation $\mathbf{n}\cdot\nabla\Phi(P_i) = \alpha\Phi(P_i)+\beta\Phi(P_{i-1})+\gamma\Phi(P_{i-2})$ has an error of $O(h^2)$.

APPENDIX III

General discretization for $\mathbf{n}\cdot\nabla\Phi$

A general discretization for $\mathbf{n}\cdot\nabla\Phi(Pi)$ on $\partial\Omega$ is obtained, not requiring the existence of nodes arranged as those of Appendix II (Figure 9).

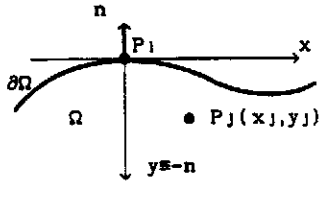

Figure 9. Discretization for $\partial\Phi/\partial n$ at P_i

As described by [8] and according to Figure 9, a discretization for $\partial\Phi/\partial n$ at P_i may be written as

$$\frac{\partial\Phi}{\partial n} = \sum_{j=1}^{3} a_j\,[\Phi(P_i)-\Phi(P_j)]$$

where $P_i \in \partial\Omega$ and $P_j \in \Omega$

The coefficients a_j are determined using the geodesic normal coordinates, what results in the solution of the simple system of equations (see [8] for details)

$$\begin{bmatrix} y_1 & y_2 & y_3 \\ x_1(1+y_1\cdot K) & x_2(1+y_2\cdot K) & x_3(1+y_3\cdot K) \\ x_1^2 - y_1^2 & x_2^2 - y_2^2 & x_3^2 - y_3^2 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

where K is the curvature of $\partial\Omega$ at Pi .

APPENDIX IV

Proposed iterative method

Suppose that $\mathbf{q}^n$ of (14) is known. The CSM gives

$$\Phi_1^n = P_1\,q^n. \quad (18)$$

Thus, vector Φ^n can be calculated from the application of the CGS method to the FDM problem. Theoretically, from (13),

$$\Phi^n = A^{-1}(z - Z_2\Phi_1^n) \quad (19)$$

Determine the auxiliary vector f^n , by using (19)

$$f^n = S_0\Phi_1^n + Z_1\Phi^n = (S_0 - Z_1A^{-1}Z_2)\Phi_1^n + Z_1A^{-1}z \quad (20)$$

A new vector of charges $\mathbf{r}^n$ can be derived from the CSM with least squares and the QR decomposition, obtained from (13)

$$\begin{bmatrix} P_2 \\ F \end{bmatrix} r^n = \begin{bmatrix} \Phi_B \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ f^n \end{bmatrix} = \begin{bmatrix} \Phi_B \\ -f^n \end{bmatrix} \quad \Rightarrow \quad r^n = R^{-1}Q^T \begin{bmatrix} \Phi_B \\ -f^n \end{bmatrix}$$

Using (18) and (20) on the last expression, we obtain

$$r^n = R^{-1}Q^T \begin{bmatrix} 0 \\ Z_1A^{-1}Z_2 - S_0 \end{bmatrix} P_1\,q^n + R^{-1}Q^T \begin{bmatrix} \Phi_B \\ -Z_1A^{-1}z \end{bmatrix} \quad (21)$$

Taking the combination $\mathbf{q}^{n+1} = \theta\,\mathbf{q}^n + (1-\theta)\,\mathbf{r}^n$, where θ is an arbitrary parameter, and by using (21), the expression (15) for $\mathbf{T}$ is immediately obtained. This combination assures that at the end of the iterative calculations (when $\mathbf{r}^n = \mathbf{q}^n$), we will have $\mathbf{q}^{n+1} = \mathbf{q}^n = \mathbf{r}^n$, for any chosen value of θ .

The flowchart of Figure 4 resumes the above procedure.

BIOGRAPHIES

João Nelson Hoffmann was born in Lapa, state of Paraná, Brazil, on April 22, 1958. He received the B.S. degree in Electrical Engineering in 1981 and the M.S. degree in Applied and Computational Mathematics in 1993. Since 1988 he is with Companhia Paranaense de Energia (COPEL), Curitiba, Paraná, Brazil, working with high voltage transmission line researches.

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