

# The FEMAX Finite-Element Package for computing three-dimensional electromagnetic fields

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*ABSTRACT. The structure and the properties of the FEMAX package are discussed. The FEMAX finite-element package is an efficient and highly accurate package especially designed for computing three-dimensional transient as well as time-harmonic electromagnetic fields in arbitrarily inhomogeneous, (an)isotropic media. The most unique features of the FEMAX package are that 1) the electric field strength is computed directly, i.e. without the intermediate use of (vector) potentials, 2) when inhomogeneities are encountered in the domain of computation, the package automatically chooses edge elements to ensure that all local continuity conditions can be met, nodal elements are used elsewhere, and 3) that the electromagnetic compatibility relations are taken into account in the formulation of the finite-element method used, thus avoiding spurious solutions. These features are included in the package in such a way that optimum results are obtained both in regard to computational efficiency (storage and time) and in regard to the desired accuracy.*

## 1 INTRODUCTION

When solving the electromagnetic field equations in a three-dimensional domain containing inhomogeneous media by using finite elements some important issues should be dealt with that are not often encountered in finite-element methods for other sets of partial differential equations.

The first important difference between the electromagnetic field equations and other sets of partial differential equations is that the field quantities in the former (the electric and the magnetic field strength) show jump discontinuities at interfaces between different media. This makes it necessary to use a computational technique that accounts for the continuity conditions of the fields across interfaces where the constitutive coefficients jump.

The second issue to be resolved lies in the fact that finite-elements methods for electromagnetic field problems often yield unwanted (spurious) solutions. Those

spurious solutions are due to the fact that some of the properties of the electromagnetic field, the electromagnetic compatibility relations [1, 2], are not represented properly in the finite-element method used.

### 1.1 Interfaces

A popular method to simplify the modeling of the continuity conditions at interfaces is to introduce electric and/or magnetic (vector) potentials [3] that are continuous along those interfaces. When using potentials in a numerical method, the finite-element method for instance, they have the disadvantage of yielding relatively inaccurate results for physical quantities like the electric or magnetic field strength. This is caused by the fact that the latter quantities can only be obtained by performing differentiations on the numerical results for the potentials which causes the loss of one order of accuracy. Obviously, much better convergence properties are obtained when numerical differentiations are avoided by formulating the problem directly in terms of the electric and/or the magnetic field strengths. When doing so, the continuity conditions along interfaces can, in principle, be dealt with by subdividing the problem space into a number of subdomains over which the constitutive coefficients are continuous functions of the spatial coordinates. The boundary conditions at the interfaces between those subdomains are imposed separately [4, 5]. This technique, however, is very difficult to implement for arbitrary configurations and yields conflicting conditions at nodes where the vector normal to the interfaces is not unique. When the electromagnetic field is computed in terms of the electric and/or the magnetic field strength, the problem of modeling fields in inhomogeneous media can be solved at the element level by using edge elements. Edge elements, however, are known to be less efficient than the commonly used nodal elements [6]. In [7] a method is described that virtually eliminates the computational disadvantage of edge elements [6] by using them only for modeling the

field along discontinuities and near reentrant corners, and by using nodal elements everywhere else. With this "mixed" method the corresponding program decides locally, for each combination of two adjacent subdomains in the discretized domain of computation, which type of element will be used. For regions in which the properties of the media are continuous functions of the spatial variables and for regions with "weak" discontinuities (i.e. discontinuities along which ignoring the jump in the normal component of the field strength would not yield unacceptably large errors) in electromagnetic properties between subdomains it will use nodal elements, for "large" differences it will use edge elements. Thus the method of modeling the field is automatically adapted to the problem at hand and all "strong" discontinuities are automatically accounted for. The degree of discontinuity above which edge elements are used is user-defined and has to be chosen in accordance with the final accuracy aimed at. In [8] a finite element code for time-harmonic electromagnetic fields was described that was designed according to the method described above.

## 1.2 Spurious solutions

Spurious solutions can be eliminated by taking into account all electromagnetic compatibility in the formulation of the finite-element method [1]. In this way the method used will accurately model the electromagnetic field equations, together with all its relevant properties, and spurious solutions are avoided.

In the present paper the version of the FEMAX finite-element code is described that was especially designed for computing transient electric fields in three spatial dimensions. It was developed on the basis of the above ideas.

## 2 NODAL ELEMENTS AND EDGE ELEMENTS

As was explained above, the FEMAX package uses a combination of nodal and edge elements. For topological reasons [9] the geometrical domain  $\mathcal{D}$ , in which the finite-element method is applied, is subdivided into tetrahedra that together span the polyhedron approximating  $\mathcal{D}$ . Consequently, the nodal and edge elements to be used will be defined on tetrahedra. The position vectors of the vertices of a particular tetrahedron  $\mathcal{T}$  are  $\{\mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ , the outwardly directed vectorial areas of the faces of  $\mathcal{T}$  are  $\{\mathbf{A}_0, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}$  and the volume of  $\mathcal{T}$  is denoted by  $V$ . Let  $\mathbf{r}_b$  be the position vector of the barycenter of  $\mathcal{T}$ , then the linear functions  $\phi_i(\mathbf{r})$  that equal unity when  $\mathbf{r} = x\mathbf{i}_1 + y\mathbf{i}_2 + z\mathbf{i}_3 = \mathbf{r}_i$  ( $i = 0, 1, 2, 3$ ) and are zero at the

remaining three vertices of  $\mathcal{T}$  can be written as

$$\phi_i(\mathbf{r}) = 1/4 - (\mathbf{r} - \mathbf{r}_b) \cdot \mathbf{A}_i/3V. \quad (1)$$

For obtaining consistency in the local degree of approximation we use consistently linear expansion functions. Therefore, the local nodal expansion functions  $\mathbf{W}_{i,j}^{(N)}(\mathbf{r})$  in  $\mathcal{T}$  are taken as

$$\mathbf{W}_{i,j}^{(N)}(\mathbf{r}) = \phi_i(\mathbf{r})\mathbf{i}_j, \quad i = 0, 1, 2, 3 \text{ and } j = 1, 2, 3, \quad (2)$$

and the local edge expansion functions  $\mathbf{W}_{i,j}^{(E)}(\mathbf{r})$  in  $\mathcal{T}$  are taken as

$$\mathbf{W}_{i,j}^{(E)}(\mathbf{r}) = -\phi_i(\mathbf{r})a_{i,j}\mathbf{A}_j/3V, \quad i, j = 0, 1, 2, 3, \quad i \neq j, \quad (3)$$

where  $a_{i,j} = |\mathbf{r}_i - \mathbf{r}_j|$  is used for dimensioning and scaling purposes. When  $\mathbf{r} \in \mathcal{T}$ , the electric field strength  $\mathbf{E}$  is locally expanded in terms of a combination of nodal and edge expansion functions through

$$\mathbf{E}(\mathbf{r}, t) = \sum_{i=0}^3 \sum_j e_{i,j}(t) \mathbf{W}_{i,j}^{(N,E)}(\mathbf{r}), \quad (4)$$

where  $e_{i,j}(t)$  are the unknown, time-dependent, field expansion coefficients and where the limits of  $j$  depend on the type of expansion used (see (2) and (3)). Note that we have, in each tetrahedron, the possibility to use arbitrary combinations of nodal and edge expansion functions. The actual choice between them is made by the program, it depends on the degree of inhomogeneity in the material properties in the immediate vicinity of  $\mathcal{T}$ . Nodal expansion coefficients are the Cartesian components of the field at the vertices of the tetrahedra (the nodes of the mesh). Edge expansion coefficients are the oriented projections of the field at the edges taken near the ends of those edges. We note that (4) always contains 12 terms. For a discussion of the properties of local expansion functions the reader is referred to [6].

We now introduce global expansion functions as the sum of all local expansion functions that are related to the same expansion coefficient. The support of the global *nodal* expansion functions is the simplicial star [9] of the node to which they are related (the simplicial star of a node is the union of all tetrahedra that have that node in common). The support of the global *edge* expansion functions is the simplicial star of the edge to which they are related (the simplicial star of an edge is the union of all tetrahedra that have that edge in common). Since the support of global nodal expansion functions is much larger than the support of global edge expansion functions, the former are computationally more efficient than the latter and the former should be used whenever

the use of the latter is not necessary because of the local degree of inhomogeneity. For an illustration the reader is referred to Fig. 2 in [8]. By using the proper combination of nodal and edge expansion functions, optimum results are obtained both in regard to accuracy and in regard to computational efficiency [6, 8].

### 3 THE FINITE-ELEMENT FORMULATION

#### 3.1 The wave equation

Eliminating the magnetic field strength  $\mathbf{H}$  from Maxwell's equations we obtain

$$\begin{aligned} \partial_t^2 \boldsymbol{\epsilon} \cdot \mathbf{E} + \partial_t \boldsymbol{\sigma} \cdot \mathbf{E} + \nabla \times (\boldsymbol{\mu}^{-1} \cdot \nabla \times \mathbf{E}) = \\ -\partial_t \mathbf{J}^{\text{imp}} - \nabla \times (\boldsymbol{\mu}^{-1} \cdot \mathbf{K}^{\text{imp}}), \end{aligned} \quad (5)$$

where  $\mathbf{J}^{\text{imp}} = \mathbf{J}^{\text{imp}}(\mathbf{r}, t)$  and  $\mathbf{K}^{\text{imp}} = \mathbf{K}^{\text{imp}}(\mathbf{r}, t)$  denote known impressed sources of electrical and magnetic current that are defined inside the domain of computation only, and where  $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}(\mathbf{r})$ ,  $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{r})$  and  $\boldsymbol{\mu} = \boldsymbol{\mu}(\mathbf{r})$  denote the permittivity, the conductivity and the permeability tensor, respectively. The wave equation is supplemented with initial conditions  $\mathbf{E}(\mathbf{r}, t_0)$  and  $\mathbf{H}(\mathbf{r}, t_0)$ . In the package the domain of computation is subdivided in tetrahedra. Now, substituting the expansion (4) for the electric field strength in (5), a system of equations in the expansion coefficients is obtained by applying the method of weighted residuals. The set of weighting functions  $\{\mathbf{W}_{p,q}^{(N,E)}(\mathbf{r})\}$  that is used is the same as the set of expansion functions. Using an integration by parts and adding the resulting equations over all tetrahedra, we obtain a system of coupled ordinary differential equations for  $\{e_{i,j}(t)\}$  that can be written as

$$\begin{aligned} \sum_{i,j} (\partial_t^2 e_{i,j}) \int_{\mathcal{D}} \mathbf{W}_{p,q} \cdot \boldsymbol{\epsilon} \cdot \mathbf{W}_{i,j} dV \\ + \sum_{i,j} (\partial_t e_{i,j}) \int_{\mathcal{D}} \mathbf{W}_{p,q} \cdot \boldsymbol{\sigma} \cdot \mathbf{W}_{i,j} dV \\ + \sum_{i,j} e_{i,j} \int_{\mathcal{D}} (\nabla \times \mathbf{W}_{p,q}) \cdot \boldsymbol{\mu}^{-1} \cdot (\nabla \times \mathbf{W}_{i,j}) dV = \\ \partial_t \int_{\partial \mathcal{D}_H} \mathbf{W}_{p,q} \cdot (\mathbf{n} \times \mathbf{H}^{\text{ext}}) dA \\ - \int_{\mathcal{D}} \mathbf{W}_{p,q} \cdot \mathbf{J}^{\text{imp}} dV \end{aligned}$$

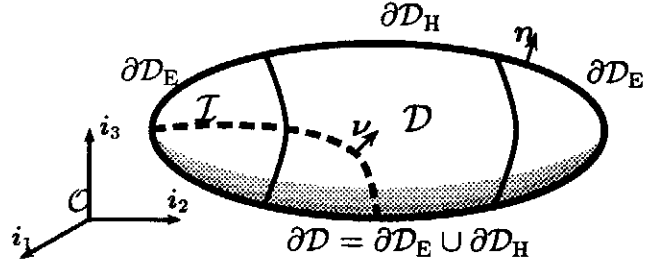


Figure 1: The domain of computation  $\mathcal{D}$ .

$$- \int_{\mathcal{D}} (\nabla \times \mathbf{W}_{p,q}) \cdot \boldsymbol{\mu}^{-1} \cdot \mathbf{K}^{\text{imp}} dV \quad \forall p, q, \quad (6)$$

where  $\partial \mathcal{D}_H$ , see Fig. 1, denotes the part of outer boundary of the domain of computation  $\mathcal{D}$  on which the tangential components of the external magnetic field strength  $\mathbf{H}^{\text{ext}} = \mathbf{H}^{\text{ext}}(\mathbf{r}, t)$  are prescribed and where  $\mathbf{n}$  denotes the unit vector along the outward normal to  $\partial \mathcal{D}$ . The tangential components of the external electric field strength  $\mathbf{E}^{\text{ext}} = \mathbf{E}^{\text{ext}}(\mathbf{r}, t)$  are prescribed on  $\partial \mathcal{D}_E$ , where  $\partial \mathcal{D} = \partial \mathcal{D}_E \cup \partial \mathcal{D}_H$  with  $\partial \mathcal{D}_E \cap \partial \mathcal{D}_H = \emptyset$ . External fields are defined outside the domain of computation only. For deriving (6) we have used the continuity of the tangential components of the magnetic field strength over all internal interfaces.

#### 3.2 The compatibility relations

For eliminating all spurious solutions (6) must be augmented with the relevant electromagnetic compatibility relations [1]. Compatibility relations are properties of a field that are direct consequences of the field equations and that should be satisfied to allow the equations to have a solution. They were first introduced by Love [10] for elastodynamics and are also known in fluid dynamics [11]. For the electric field strength, which is the fundamental unknown in FEMAX, these compatibility relations are the volume divergence condition

$$\nabla \cdot (\partial_t \boldsymbol{\epsilon} \cdot \mathbf{E} + \boldsymbol{\sigma} \cdot \mathbf{E}) = -\nabla \cdot \mathbf{J}^{\text{imp}}, \quad (7)$$

the surface divergence condition at interfaces

$$\boldsymbol{\nu} \cdot (\partial_t \boldsymbol{\epsilon} \cdot \mathbf{E} + \boldsymbol{\sigma} \cdot \mathbf{E}) + \boldsymbol{\nu} \cdot \mathbf{J}^{\text{imp}} \text{ continuous across } \mathcal{I}, \quad (8)$$

where  $\boldsymbol{\nu}$  denotes the unit vector along the normal to the interface  $\mathcal{I}$ , and the condition applying to the normal component of the electric flux density at the outer boundary  $\partial \mathcal{D}_H$

$$\mathbf{n} \cdot (\partial_t \epsilon \cdot \mathbf{E} + \sigma \cdot \mathbf{E}) = \mathbf{n} \cdot (\nabla \times \mathbf{H}^{\text{ext}} - \mathbf{J}^{\text{imp}}) \text{ on } \partial \mathcal{D}_H. \quad (9)$$

Each of these equations is added to (6) in a weighted sense. In this way the wave equation, together with its properties, is modelled in a weighted sense and spurious solutions are avoided.

Adding the weighted form of (7)-(9) to (6), a system of coupled ordinary differential equations for  $\{e_{i,j}(t)\}$  is obtained that is solved in a standard manner using either single- [12] or two-step [13] integration methods. In practice the use of two-step methods has turned out to be preferable.

#### 4 TIME-HARMONIC FIELDS IN THE TIME DOMAIN

The FEMAX code allows time-harmonic fields to be computed in the time domain. For maximum efficiency a transient is used with a transient period  $0 \leq t \leq t_{\text{tr}}$ . During this period the known right-hand side vector, representing the time harmonic excitation, is multiplied by a continuous function  $f_{\text{tr}}(t)$  that generates a smooth transient from zero to steady state. For  $f_{\text{tr}}$  we use a function of the type

$$\begin{aligned} &= 0, & -\infty < t < 0, \\ f_{\text{tr}} &= (2 - \sin(\frac{t}{t_{\text{tr}}}\frac{\pi}{2})) \sin(\frac{t}{t_{\text{tr}}}\frac{\pi}{2}), & 0 \leq t \leq t_{\text{tr}}, \quad (10) \\ &= 1, & t_{\text{tr}} < t < \infty, \end{aligned}$$

other transients being available. Using  $f_{\text{tr}}$  we obtain, at  $t = t_{\text{tr}}$ , an approximation of the time-harmonic solution with a relatively small error term and steady state is achieved very efficiently. In most cases a value of  $t_{\text{tr}}$  in the range  $5T \leq t_{\text{tr}} \leq 10T$ , where  $T$  denotes the period in time of the time-harmonic sources, turns out to yield an optimum computational efficiency. Obviously the optimum choice for  $t_{\text{tr}}$  depends both on the problem at hand (larger domains of computation requiring longer transient times) and on the accuracy requirements. In general it can be said that  $t_{\text{tr}}$  should preferably be chosen such that the solution at  $t = t_{\text{tr}}$  is already accurate enough to be used as a steady state solution and that no subsequent time stepping is required to obtain steady state. For further comments on this procedure the reader is referred to [14].

Computing time-harmonic fields in the time domain is computationally very attractive as compared with using frequency domain methods because of the fact it is usually very difficult to solve the system of equations obtained using the latter method which tends to generate

ill conditioned system matrices that usually are not positive definite. Experimentally we have found that for "large" problems (>10.000 unknowns) the time domain approach to solving time-harmonic problems is always more efficient and more accurate than the time-harmonic approach. The advantage of the time-domain, transient, approach over the time-harmonic approach increases with the number of unknowns and with the temporal frequency (for a constant size of the domain of computation).

#### 5 ABSORBING BOUNDARY CONDITIONS

The FEMAX package contains simple local absorbing boundary conditions that model the infinite homogeneous surroundings of the domain of computation. These boundary conditions assume the field locally to consist of a plane wave travelling in a specific direction, usually in the direction that is normal to the outer boundary, out of the domain of computation. Absorbing boundary conditions of this type can be used for problems in which all sources of the electromagnetic field are located inside the domain of computation.

##### 5.1 Inhomogeneous absorbing boundary conditions

In many practical problems, the field is excited by sources that are located outside the domain of computation. As an example one might think of the scattering of a plane wave by an obstacle. In cases like this we have an incident field  $\{\mathbf{E}^{\text{inc}}, \mathbf{H}^{\text{inc}}\}$ , which is the field in the absence of the scatterer, a total field  $\{\mathbf{E}, \mathbf{H}\}$ , and a scattered field  $\{\mathbf{E}^{\text{scat}}, \mathbf{H}^{\text{scat}}\}$ , which is defined as the difference between the total field and the incident field. Absorbing boundary conditions apply to scattered fields, they can be augmented to take into account the incident field in the following manner. Since absorbing boundary conditions are linear expressions in terms of the scattered field they can formally be written as a linear operator of the type

$$R(\mathbf{E}^{\text{scat}}, \mathbf{H}^{\text{scat}}) = 0. \quad (11)$$

Adding the trivial identity

$$R(\mathbf{E}^{\text{inc}}, \mathbf{H}^{\text{inc}}) = R(\mathbf{E}^{\text{inc}}, \mathbf{H}^{\text{inc}}) \quad (12)$$

to (11) we obtain

$$R(\mathbf{E}, \mathbf{H}) = R(\mathbf{E}^{\text{inc}}, \mathbf{H}^{\text{inc}}), \quad (13)$$

which relation is an inhomogeneous absorbing boundary condition that can be applied to the total field inside the domain of computation and that takes into account the

incident field as a contribution to the known excitation vector. Note that (13) applies to all absorbing boundary conditions that can be written as (11), from very simple local conditions to highly sophisticated global ones.

## 6 THE PROPERTIES OF THE FEMAX PACKAGE

The FEMAX package for computing time-domain electromagnetic fields in three spatial dimensions has the following properties and/or options:

- The electric field strength is used as the fundamental unknown quantity that will be approximated using consistently linear expansion functions.
- Because of including all relevant electromagnetic compatibility relations in the formulation, the results are free of spurious solutions.
- In the interior of the domain of computation, the user can specify arbitrary distributions of electromagnetic medium properties (permittivity, conductivity and permeability), the media may be anisotropic. It is also possible to specify specific density distributions for use in SAR computations.
- The package uses edge elements along discontinuities and nodal elements in regions with continuously varying medium properties. The contrast in properties of the media in two adjacent domains below which the medium is treated as having continuously varying properties is specified by the user.
- For a given distribution of medium properties and a given finite-element mesh, which has to be chosen fine enough to satisfy the accuracy requirements, the package automatically determines the numerically optimum distribution of the expansion functions that allow the solution to satisfy the user-defined local accuracy requirements.
- In the interior of the domain of computation, the user can prescribe arbitrary initial value distributions of the field strength as well as arbitrary time-dependent volume source distributions of imposed electric and/or magnetic current.
- The distributions of the medium properties as well as the volume source distributions can be specified either by supplying user-written subroutines for generating those quantities (in FORTRAN-77) or, much simpler, by using members of the collection of standard subdomains with standard distributions that are available in the package.

- At the outer boundary of the domain of computation, the user can prescribe arbitrary distributions of external electromagnetic fields. Local impedance boundary conditions for modeling lossy boundaries and (in)homogeneous absorbing boundary conditions for modeling radiation into an unbounded homogeneous medium surrounding the domain of computation are also available.
- Efficient sparse-matrix methods and different types of preconditioning are available for solving the system of algebraic equations.
- Field values and derived quantities can be computed at arbitrary collections of user-specified points in the configuration and as a function of time.
- FEMAX uses the SEPRAN finite-element package [18] for a number of general finite-element tasks like the generation of the mesh and the assembly of the system matrix from the FEMAX elements.

A number of programs is available for post-processing data prepared in FEMAXT. They can be used interactively for previewing and for making contour plots (FEMAXPT and FEMAXPF) of field values, and of quantities that can be expressed in terms of the field values and material properties. FEMAXH is used for making plots of the evolution in time of fields values at specified locations. Use is made of either the NAG Graphics Library [15] or MATLAB [16].

## 7 NUMERICAL RESULTS AND COMPUTATIONAL REQUIREMENTS

Early results obtained by using the FEMAX package have been presented in [14, 17]. The results presented in those papers mainly served to illustrate the accuracy and the convergence properties of the package on a simple model problem. As a more practical example we shall now compute the specific absorption rate of a field in a mathematical model of a human head (a numerical phantom). The field is generated by a dipole antenna that is located near this phantom. A quarter part of the configuration is depicted in Fig. 2. The frequency is taken as  $f=900\text{MHz}$ , which is inside the frequency range used for the European GSM (Global System for Mobile Communication) network range. The phantom has a cubic shape, it consists of a cubic region  $-0.095\text{m} \leq x, y, z \leq 0.095\text{m}$  filled with brain tissue ( $\epsilon_r=43$ ,  $\sigma=0.83\text{S/m}$ ,  $\rho=1050\text{kg/m}^3$ ). The region outside the above region, but inside the cube  $-0.1\text{m} \leq x, y, z \leq 0.1\text{m}$  has properties for modelling the skull ( $\epsilon_r=17$ ,  $\sigma=0.25\text{S/m}$ ,  $\rho=1200\text{kg/m}^3$ ). Outside this region the properties of vacuum are assumed.

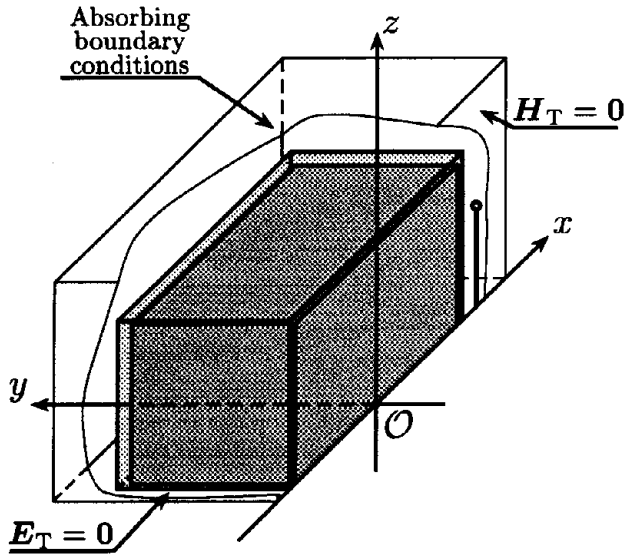


Figure 2: The numerical phantom

The antenna is modelled by a  $\lambda/2$  dipole of length 0.166m, having its center at  $(x, y, z) = (0.115, 0, 0)$  and its orientation parallel to the  $z$ -axis.

The problem was solved under the condition that edge elements are used when the relative contrast  $C_r$  in the numerical values of  $\sigma(\mathbf{r}) + j\omega\epsilon_r(\mathbf{r})\epsilon_0$  in two adjacent tetrahedra exceeds 0.1 (with this choice, edge elements are used along all interfaces between the different domains), nodal elements are used otherwise. By using the symmetry of the configuration with respect to the planes  $y = 0$  and  $z = 0$  we can solve the problem for positive  $y$  and  $z$  values only, using the condition  $\mathbf{E}_T(\mathbf{r}, t) = \mathbf{i}_3 \times \mathbf{E}(\mathbf{r}, t) = \mathbf{0}$  (the subscript T refers to the tangential components) at the plane  $z = 0$  and the condition  $\mathbf{H}_T(\mathbf{r}, t) = \mathbf{i}_2 \times \mathbf{H}(\mathbf{r}, t) = \mathbf{0}$  at the plane  $y = 0$ . The domain of computation  $-0.11 \leq x \leq 0.2, 0 \leq y \leq 0.16, 0 \leq z \leq 0.16$  is surrounded by a mathematical boundary on which absorbing boundary conditions are applied. For the problem at hand the solution is expected to have a "singularity" in the immediate vicinity of the dipole antenna. Therefore we have used a non-uniform mesh that has its highest density near this antenna. A side view of this mesh, consisting of 144000 tetrahedra, is given in Fig. 3. Note that the mesh is such that the interfaces between different media coincide with interfaces between tetrahedra, in this way we have avoided staircasing in the modeling of the material properties.

For this configuration the Specific Absorption Rate

(SAR in W/kg) is required. In Fig. 4 the normalized SAR distribution is depicted for a cross-section in the plane  $y=0$ . The normalization was carried out such that the maximum SAR value anywhere in the configuration was set to 0dB. In Fig. 5 the SAR distribution is depicted along the line  $(-0.1 \leq x \leq 0.1, y = 0, z = 0)$ , the same normalization was used. The almost linear decay of the SAR (dB) values with decreasing  $x$ -values reflects the exponential decay of the field in the highly lossy tissues. The "roughness" of the result in Fig. 5 is due to the fact that a relatively coarse mesh had to be used away from the antenna. For  $x \leq 0.05$  about 5 elements per wavelength were used, near the antenna a much denser mesh was used and, consequently, much smoother results were obtained for  $x > 0.05$ .

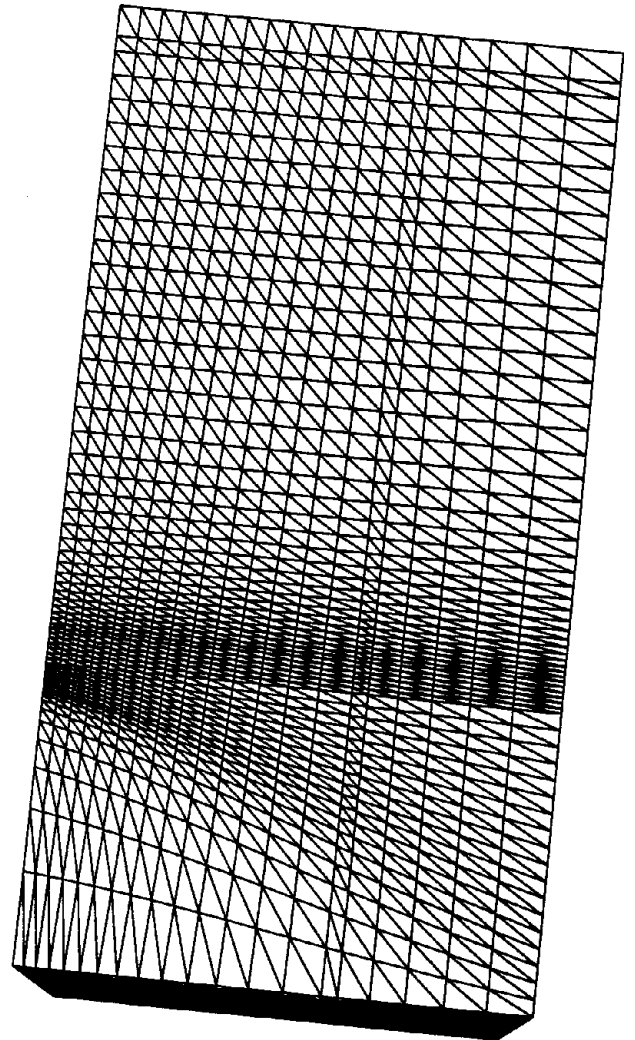


Fig. 3. The mesh used

The computational effort for solving a problem follows from the effort required for generating the system matrices, the effort for solving the equations each time step and the number of time steps. Since the matrices are stored in compact mode (ignoring all zero's) the solution of the system of equations can be obtained only by using iterative methods because of which the solution time depends on the number of iterations required. The actual number of iterations for solving a problem iteratively each time step is relatively small and depends on the type of preconditioning available for the problem at hand. Typically 2 - 20 iterations are required for solving a system of equations with 100,000 unknowns. The computational effort increases only slightly faster than linear with the number of unknowns and linear with the number of time steps required. For large problems with many time steps the computational effort for generating the system matrices is negligible in comparison to the computational effort required for the integration of the system of differential equations along the time axis.

## 8 THE PACKAGE

The FEMAX finite-element package consists of approximately 250 subroutines. It uses the SEPRAN [18] finite element package as a general finite-element background package that carries out tasks like the generation of the mesh, the assembly of the matrices and vectors, the solution of the systems of algebraic equations and the data management. Either the NAG Graphics Library [15] or MATLAB [16] is used for postprocessing. The storage requirements for the executable are 6.5Mbytes. FEMAX runs on both VAX-VMS and UNIX platforms. The FEMAX source code, together with an extensive User's Guide, are commercially available. The SEPRAN source code should be acquired separately.

The time-harmonic version of the FEMAX packages [8] is also commercially available. A FEMAX package for static and stationary electric or magnetic fields based on the theory presented in [2] is presently under development [19].

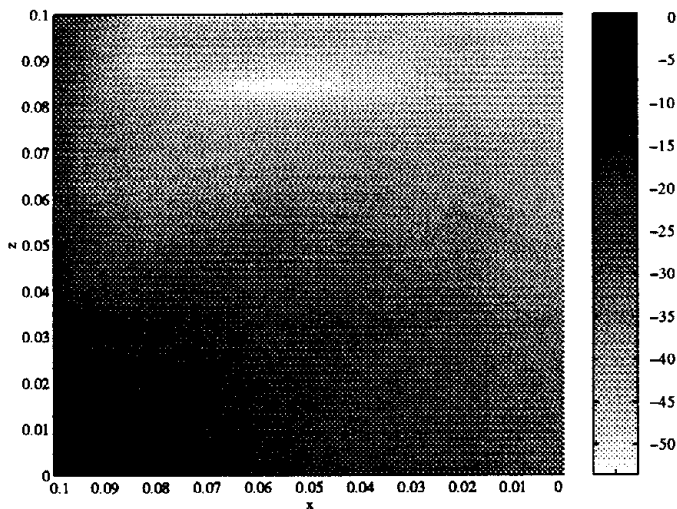


Fig. 4. The normalized SAR distribution in the plane  $y = 0$  in dB.

For the problem at hand the total number of unknowns amounted to 114085, 3082 of them being prescribed through essential boundary conditions. Each of the three matrices contained 4116764 non-zero elements. The time-harmonic problem was solved with a transient of 10 periods in time, with 20 time steps on each period a total of 1055 iterations were required to solve the problem. All computations were carried out on a HP 9000-735/125 workstation, with 400Mbyte main memory space. The total computation time for obtaining the solution amounted to about 100 min.

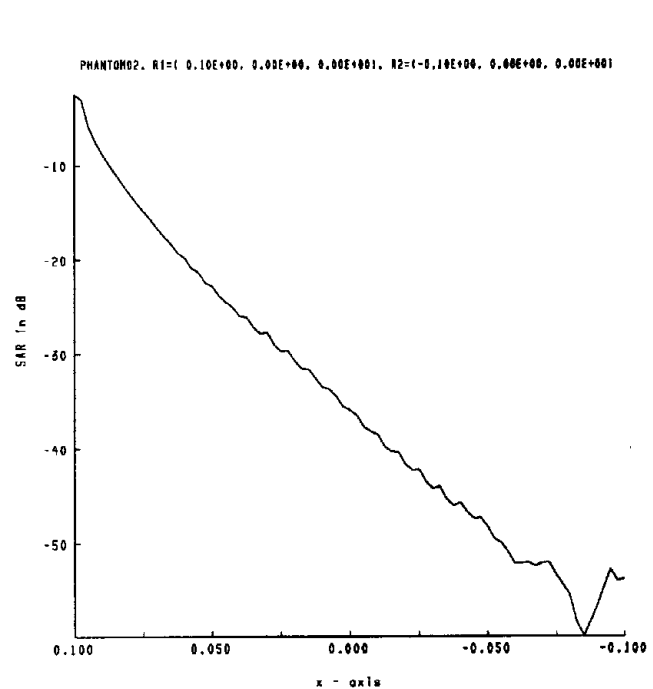


Fig. 5. The normalized SAR distribution along the line  $y = 0, z = 0$  in dB.

## 9 CONCLUSION

The FEMAX finite-element package was described. This package was especially designed for computing transient as well as time-harmonic electromagnetic fields in three-

dimensional configurations containing inhomogeneous and (an)isotropic media. Because of formulating the problem in terms of the electric field strength directly, all problems that are inherent to potential formulations are avoided. The modeling of the conditions along internal interfaces in the configuration is performed automatically by the package, and the user can influence this process by prescribing the minimum degree of discontinuity  $C_r$  for which the discontinuity in the medium properties should be modelled to obtain the required accuracy.

## 10 ACKNOWLEDGEMENT

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