Compatibility Relations for Time-Domain and Static Electromagnetic Field Problems

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Abstract - **When computing an electromagnetic field using a numerical method, e. g. the finite element method, it is possible that, although Maxwell's equations are discretized accurately, highly inaccurate computational results are obtained. In those cases it can easily be shown that (some of) the electromagnetic compatibility relations (field properties that follow from Maxwell's equations) are not satisfied. The divergence condition on the fluxes, for instance, follows directly from the field equations but not necessarily from their discretized counterparts. This necessitates inclusion of the compatibility relations in the finite-element formulation of the field problem. First a survey is given of all electromagnetic compatibility relations for the time-domain electromagnetic field equations. Subsequently the compatibility relations for the static field equations are discussed.**

I. Introduction

Because of its flexibility, the finite-element method seems to be the most suitable method for computing electromagnetic fields in inhomogeneous media and/or complicated geometries. In the finite-element formulation of an electromagnetic field problem the field equations can only be satisfied approximately. As a consequence of this, field properties that follow from Maxwell's electromagnetic field equations, the electromagnetic compatibility relations [1], may not be reflected accurately in a numerical solution. In earlier papers [2, 3] Mur presented methods for computing the electric and/or the magnetic field directly, using a combination of linear edge and linear nodal expansion functions for obtaining optimum computational efficiency. In these papers the importance of including the divergence condition, which is one of the compatibility relations, in the formulation of the problem was discussed. The equations applying to the divergence of the electric and magnetic flux densities follow directly from the electromagnetic field equations. They are satisfied whenever the field equations are satisfied exactly.

In the present paper the use of the divergence condition will be generalized to the use of the compatibility relations for electromagnetic fields. It will also be shown that the use of divergence-free edge elements, which is advocated by some authors (see [4] and the references contained in it) for satisfying some of these compatibility relations, as well as the use of face elements, introduces the possibility of violating additional relations of the compatibility type. The importance of including the electromagnetic compatibility relations explicitly in the finite-element formulation of the problem is stressed.

The analysis of the compatibility relations is carried out first for methods for computing time-domain (transient) electromagnetic fields. The analysis of methods for time-harmonic electromagnetic field problems runs along similar lines and leads to similar conclusions. It can be shown (and verified experimentally) that the importance of the compatibility relations increases with decreasing frequencies (slower variations of the solution in time) and, consequently, the compatibility relations are of the utmost importance for the numerical computation of static electric and magnetic fields. These cases will be discussed separately showing the connections between the compatibility relations for transient fields and those for static fields.

II. The Basic Equations

As the point of departure for our analysis we use the time-domain electromagnetic field equations

$$\partial_t \boldsymbol{D} \quad + \boldsymbol{J} \quad - \nabla \times \boldsymbol{H} = -\boldsymbol{J}^{\text{ext}}, \tag{1}$$

$$\partial_t \boldsymbol{B} \qquad\quad + \nabla \times \boldsymbol{E} = -\boldsymbol{K}^{\text{ext}}, \tag{2}$$

where $\boldsymbol{J}^{\text{ext}}$ and $\boldsymbol{K}^{\text{ext}}$ are sources of electric and magnetic current that are known, throughout the domain of computation $\mathcal{D}$ (see Fig. 1), as a function of the time coordinate t. In (2) we have included the magnetic current $\boldsymbol{K}^{\text{ext}}$ for symmetry reasons. $\boldsymbol{J}^{\text{ext}}$ and $\boldsymbol{K}^{\text{ext}}$ may also represent contrast sources used in a contrast-source formulation that replaces a transparent obstacle in a known external field by equivalent sources. The field equations are supplemented by the interface conditions

$$\boldsymbol{\nu} \times \boldsymbol{E} \text{ continuous across sourcefree interfaces,} \quad (3)$$

$$\boldsymbol{\nu} \times \boldsymbol{H} \text{ continuous across sourcefree interfaces,} \quad (4)$$

and the boundary conditions

$$\boldsymbol{\nu} \times \boldsymbol{E} = \boldsymbol{\nu} \times \boldsymbol{E}^{\mathrm{ext}} \text{ on } \partial\mathcal{D}_{\mathrm{E}}, \quad (5)$$

$$\boldsymbol{\nu} \times \boldsymbol{H} = \boldsymbol{\nu} \times \boldsymbol{H}^{\mathrm{ext}} \text{ on } \partial\mathcal{D}_{\mathrm{H}}, \quad (6)$$

where $\boldsymbol{\nu}$ is the unit vector along the normal to either the interface $\mathcal{I}$ or the outer boundary $\partial\mathcal{D} = \partial\mathcal{D}_{\mathrm{E}} \cup \partial\mathcal{D}_{\mathrm{H}}$ (with $\partial\mathcal{D}_{\mathrm{E}} \cap \partial\mathcal{D}_{\mathrm{H}} = \emptyset$) of the domain of computation $\mathcal{D}$, and where $\boldsymbol{\nu} \times \boldsymbol{E}^{\mathrm{ext}}$ and $\boldsymbol{\nu} \times \boldsymbol{H}^{\mathrm{ext}}$ are known, along the relevant parts of this outer boundary, as a function of t.

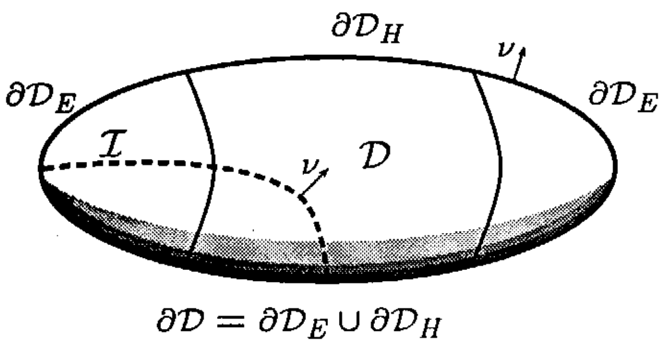

Fig. 1. The domain of computation $\mathcal{D}$.

Together with the constitutive equations and the initial conditions at $t = t_0$, (1)-(6) define an electromagnetic-field problem with a unique solution [5]. Note that the source terms in (1) and (2) are not related to the boundary conditions in (5) and (6).

III. The Compatibility Relations

Compatibility relations [1] are properties of a field that are direct consequences of the field equations and that must be satisfied to allow them to have a solution. For the electromagnetic field equations they are discussed below.

A. Interior

Applying the divergence operator $\nabla\cdot$ to (1) and (2) it follows that

$$\nabla \cdot (\partial_t \boldsymbol{D} + \boldsymbol{J}) = -\nabla \cdot \boldsymbol{J}^{\mathrm{ext}}, \quad (7)$$

$$\partial_t \nabla \cdot \boldsymbol{B} = -\nabla \cdot \boldsymbol{K}^{\mathrm{ext}}. \quad (8)$$

The electromagnetic compatibility (divergence) relations (7) and (8) apply to subdomains of the domain of computation in which the electromagnetic field vectors are continuously differentiable functions of the spatial coordinates.

B. Interfaces

The field vectors are not differentiable with respect to the spatial coordinates at the interfaces between regions with different medium properties. In that case (7) and (8) are replaced by

$$\boldsymbol{\nu} \cdot (\partial_t \boldsymbol{D} + \boldsymbol{J}) + \boldsymbol{\nu} \cdot \boldsymbol{J}^{\mathrm{ext}} \text{ continuous across interface,} \quad (9)$$

$$\boldsymbol{\nu} \cdot \partial_t \boldsymbol{B} + \boldsymbol{\nu} \cdot \boldsymbol{K}^{\mathrm{ext}} \text{ continuous across interface,} \quad (10)$$

where $\boldsymbol{\nu}$ is the unit vector normal to the interface.

Note that (9) and (10) express the continuity condition applying to the normal components of the electric and the magnetic flux densities across an interface between different media.

C. Outer boundary

A third type of compatibility relation is found when studying the behavior of the field near the outer boundary of the domain of computation. Applying the operator $\boldsymbol{\nu}\cdot$, where $\boldsymbol{\nu}$ denotes the unit vector along the normal to the outer boundary, to (1) and (2) we obtain, using (5) and (6), the relations

$$\boldsymbol{\nu} \cdot (\partial_t \boldsymbol{D} + \boldsymbol{J}) = \boldsymbol{\nu} \cdot (\nabla \times \boldsymbol{H}^{\mathrm{ext}} - \boldsymbol{J}^{\mathrm{ext}}) \text{ on } \partial\mathcal{D}_{\mathrm{H}}, \quad (11)$$

$$\boldsymbol{\nu} \cdot \partial_t \boldsymbol{B} = -\boldsymbol{\nu} \cdot (\nabla \times \boldsymbol{E}^{\mathrm{ext}} + \boldsymbol{K}^{\mathrm{ext}}) \text{ on } \partial\mathcal{D}_{\mathrm{E}}. \quad (12)$$

These equations express the fact that prescribing the tangential components of the electric (magnetic) field strength at a given part $\partial\mathcal{D}_{\mathrm{E}}$ ($\partial\mathcal{D}_{\mathrm{H}}$) of the outer boundary $\partial\mathcal{D}$ implies a related behavior of the normal components of the magnetic (electric) flux densities at that part of the boundary.

Note that these equations have the form of additional boundary conditions applying at the outer boundary of the domain of computation. They follow, however, directly from the fact that the field inside the domain of computation should satisfy Maxwell's equations.

D. Compatibility relations and edge elements

Some authors use divergence-free edge elements (e.g. Whitney 1) for imposing the divergence conditions exactly. Edge elements cause the tangential components of the fields to be continuous, they leave the normal components free to jump. Apart from the fact that divergence-free edge elements can only be used in the simple case where the compatibility relations (7) and (8) reduce to $\nabla \cdot \boldsymbol{D} = 0$ (or $\nabla \cdot \boldsymbol{J} = 0$) and $\nabla \cdot \boldsymbol{B} = 0$, respectively, the resulting freedom of the normal component of the field at the interface between two adjoining tetrahedra to jump, even when it should be continuous, is unwanted. Consequently, the continuity of the normal flux has to be added to our list of compatibility relations to be imposed upon the solution. Failing to do so may be the cause of undesired surface charge distributions in between edge elements. When adjoining finite elements contain identical materials, and assuming that the external sources of current are continuous between those finite elements, the following relations hold

$$\boldsymbol{\nu} \cdot \boldsymbol{E} \text{ continuous between edge elements,} \tag{13}$$

$$\boldsymbol{\nu} \cdot \boldsymbol{H} \text{ continuous between edge elements.} \tag{14}$$

In the alternative cases, (9) and (10) still apply. Imposing these relations results in an increase of the connectivity of the system matrices. Note that the need for imposing the continuity relations (13) and (14) is caused solely by the use of edge expansion functions and not by the electromagnetic field problem or the finite-element formulation used.

E. Compatibility relations and face elements

Some authors propose the use of face (also called facet) elements for modeling flux distributions. Face elements cause the normal fluxes between tetrahedra to be continuous, they have the disadvantage of leaving tangential components free to jump, even when they should be continuous. Assuming that no surface sources of current are present at the interface between those finite elements, the following continuity relations should hold

$$\boldsymbol{\nu} \times \boldsymbol{E} \text{ continuous between face elements,} \tag{15}$$

$$\boldsymbol{\nu} \times \boldsymbol{H} \text{ continuous between face elements,} \tag{16}$$

otherwise the proper jump condition should be implemented. Imposing these relations results in an increase of the connectivity of the system matrix (matrices). Note that the need for imposing the continuity relations (15) and (16) is caused solely by the use of face expansion functions that do not automatically satisfy the continuity conditions (3) and (4).

F. In summary

Equations (7)-(12) are a set of six electromagnetic compatibility relations that are direct consequences of Maxwell's equations. In exact methods for solving the electromagnetic field equations they are automatically accounted for. In numerical methods, for instance in the finite-element method, for solving the electromagnetic field equations they should be taken into account explicitly whenever the method used does not automatically account for them.

Equations (13)-(16) are additional compatibility relations the need for which is caused by the use of either edge or face elements in homogeneous domains. In those domains edge and face elements allow unphysical discontinuities in the solution and compatibility relations have to be added to the formulation of the field problem for restricting those discontinuities to acceptable values.

Note that (7)-(16) do not contain any extra information that is not contained in the field equations. However, failing to include them in the finite-element formulation of an electromagnetic-field problem, either exactly or numerically, may be the cause of highly inaccurate results. Errors of this type are often referred to as "spurious solutions" or "vector parasites".

IV. Application to Static Electric Fields

For static electric fields the basic equations (1) - (6) reduce to

$$\nabla \times \boldsymbol{E} = -\boldsymbol{K}^{\mathrm{ext}}, \tag{17}$$

together with the interface condition

$$\boldsymbol{\nu} \times \boldsymbol{E} \text{ continuous across sourcefree interfaces,} \tag{18}$$

and the boundary condition

$$\boldsymbol{\nu} \times \boldsymbol{E} = \boldsymbol{\nu} \times \boldsymbol{E}^{\mathrm{ext}} \text{ on } \partial\mathcal{D}_{\mathrm{E}}. \tag{19}$$

Note that we have lost the boundary condition on $\partial\mathcal{D}_{\mathrm{H}}$ which has to be replaced by the compatibility relation applying to this part of the outer boundary.

A. Interior compatibility

In case of a conducting medium (7) reduces to

$$\nabla \cdot \boldsymbol{J} = -\nabla \cdot \boldsymbol{J}^{\mathrm{ext}}, \tag{20}$$

In case of a non-conducting (dielectric) medium (7) reduces to

$$\nabla \cdot \boldsymbol{D} = \rho^{\mathrm{e,ext}}, \tag{21}$$

where $\rho^{\mathrm{e,ext}}$ denotes the known external electric volume charge density.

B. Interface compatibility

In case of a conducting medium (9) reduces to

$$\boldsymbol{\nu} \cdot \boldsymbol{J} + \boldsymbol{\nu} \cdot \boldsymbol{J}^{\mathrm{ext}} \text{ continuous across interface,} \tag{22}$$

in case of a non-conducting (dielectric) medium (9) reduces to

$$\boldsymbol{\nu} \cdot \boldsymbol{D}|_1^2 = \sigma^{\mathrm{e,ext}}, \tag{23}$$

where $\boldsymbol{D}|_1^2$ denotes the jump in $\boldsymbol{D}$ across the interface and where $\sigma^{\mathrm{e,ext}}$ denotes the known external electric surface charge density. In case of an interface between a conducting and a non-conducting (dielectric) medium (9) reduces to

$$\boldsymbol{\nu} \cdot \boldsymbol{J} = \boldsymbol{\nu} \cdot \boldsymbol{J}^{\mathrm{e,ext}}|_1^2 \tag{24}$$

at the conducting side of the interface.

C. Outer boundary compatibility

In case of a conducting medium (11) reduces to

$$\boldsymbol{\nu} \cdot \boldsymbol{J} = \boldsymbol{\nu} \cdot (\nabla \times \boldsymbol{H}^{\mathrm{ext}} - \boldsymbol{J}^{\mathrm{ext}}) \text{ on } \partial\mathcal{D}_{\mathrm{H}}. \tag{25}$$

Recall that $\boldsymbol{H}^{\mathrm{ext}}$ and $\boldsymbol{J}^{\mathrm{ext}}$ are not related. In case of a non-conducting (dielectric) medium (11) reduces to

$$\boldsymbol{\nu} \cdot \boldsymbol{D} = \sigma^{\mathrm{e,ext}} \text{ on } \partial\mathcal{D}_{\mathrm{H}}, \tag{26}$$

where $\sigma^{\mathrm{e,ext}}$ denotes the known external electric surface charge density at the outer boundary.

D. Compatibility relations and edge elements

In case adjoining tetrahedra contain identical materials as regards their electric properties (13) applies, otherwise (22)-(24) apply.

E. Compatibility relations and face elements

In case no surface souces of magnetic current are present at the interface between adjoining face elements (15) applies, otherwise the proper jump condition should be implemented.

V. Application to Static Magnetic Fields

For static magnetic fields the basic equations (1) - (6) reduce to

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}^{\mathrm{ext}}, \tag{27}$$

together with the interface condition

$$\boldsymbol{\nu} \times \boldsymbol{H} \text{ continuous across sourcefree interfaces,} \tag{28}$$

and the boundary condition

$$\boldsymbol{\nu} \times \boldsymbol{H} = \boldsymbol{\nu} \times \boldsymbol{H}^{\mathrm{ext}} \text{ on } \partial\mathcal{D}_{\mathrm{H}}. \tag{29}$$

Note that we have lost the boundary condition on $\partial\mathcal{D}_{\mathrm{E}}$ which has to be replaced by the compatibility relation applying to this part of the outer boundary.

A. Interior compatibility

For static magnetic fields equation (8) reduces to

$$\nabla \cdot \boldsymbol{B} = \rho^{\mathrm{m,ext}}. \tag{30}$$

where $\rho^{\mathrm{m,ext}}$ denotes the known external magnetic volume charge density.

B. Interface compatibility

For static magnetic fields equation (10) reduces to

$$\boldsymbol{\nu} \cdot \boldsymbol{B}|_1^2 = \sigma^{\mathrm{m,ext}}, \tag{31}$$

where $\sigma^{\mathrm{m,ext}}$ denotes the known external magnetic surface charge density at the interface.

C. Outer boundary compatibility

For static magnetic fields the continuity of the normal flux (12) reduces to

$$\boldsymbol{\nu} \cdot \boldsymbol{B} = -\sigma^{\mathrm{m,ext}} \text{ on } \partial\mathcal{D}_{\mathrm{E}}. \tag{32}$$

where $\sigma^{\mathrm{m,ext}}$ denotes the known external magnetic surface charge density at the outer boundary.

D. Compatibility relations and edge elements

In case adjoining tetrahedra contain identical materials as regards their magnetic properties (14) applies, otherwise (31) applies.

E. Compatibility relations and face elements

In case no surface sources of electric current are present at the interface between adjoining face elements (16) applies, otherwise the proper jump condition should be implemented.

VI. Conclusions

When the electromagnetic field equations are solved numerically using expansions that do not themselves exactly satisfy these equations, which is the case in the finite-element method, it is necessary to include the compatibility relations in the formulation in order to obtain correct results. Attempts to solve this difficulty by using edge elements merely complicate the situation by introducing the need to impose additional compatibility relations. In our analysis we have first presented the compatibility relations applying to time-domain (transient) fields and, subsequently, those for static fields. In doing so, the relation between those two cases is clarified and a better understanding is obtained of the function of the compatibility relations and their application in the entire range from static to high frequency applications.

In summary, we conclude that we have presented the electromagnetic field compatibility relations. To obtain reliable computational results from finite-element methods for solving the electromagnetic field equations, these relations should be made a part of the formulation of the problem.

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