A RECURSIVE TECHNIQUE TO AVOID ARITHMETIC OVERFLOW AND UNDERFLOW WHEN COMPUTING SLOWLY CONVERGENT EIGENFUNCTION TYPE EXPANSIONS

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Abstract

Eigenfunction expansions for fields scattered by large structures are generally very slowly convergent. The summation often consists of two factors where one factor approaches zero and the other factor grows in magnitude without bound as the summation index increases. Each term of the expansion is bounded; however, due to the extreme magnitude of the individual factors, computational overflow and underflow errors can limit the number of terms that can be computed in the summation thereby forcing the summation to be terminated before it has converged. In this paper an exact technique that circumvents these problems is presented. An auxiliary function is introduced which is proportional to the original factor with its asymptotic behavior factored out. When these auxiliary functions are introduced into the summation, we are left with the task of numerically summing products of well behaved factors. A recursion relationship is developed for computing this auxiliary function.

1 Introduction

When solving for the fields scattered by canonical geometries, the exact solution is often available in eigenfunction form. The eigenfunction form is a viable representation for the fields providing that some characteristic dimension of the structure is "small" with respect to the wavelength, otherwise, the eigenfunction expansion is very slowly convergent and additionally can exhibit the following computational difficulty. These pathological eigenfunction expansions are in the form of infinite summations of products and quotients. Each term of the summation is well-behaved, however, the magnitude of the individual factors and/or divisors become either too small or too large to handle on the computer resulting in overflow/overflow errors. Ideally, one should find an alternate representation of the series that is more quickly convergent using a technique such as Watson's transformation [*Tyras*, 1969]. For many geometries the topology of the characteristic plane may be too complicated to perform the necessary function theoretic manipulations. Therefore, one may be forced to sum the slowly convergent series.

The overflow/underflow problem can be alleviated by implementing an auxiliary function that is proportional to the original pathological function with the asymptotic behavior factored out. These auxiliary functions can be calculated "exactly" from new recursion relationships which are derived from the recursion relationship of the original function. In Section 2, the development of these auxiliary functions and the corresponding recursive techniques will be presented. Section 3 contains an example of the plane wave scattering by a circular cylinder using the techniques developed in this paper and Section 4 contains some concluding remarks. Throughout this paper an $e^{j\omega t}$ time dependence is assumed and suppressed.

2 Analytic Formulation

Symbolically, an eigenfunction expansion often has the form

$$\phi = \sum_n C_n \, S_n(x) \, L_n(x), \tag{1}$$

where C_n is a well-behaved constant, $S_n(x)$ is a factor that becomes increasingly small as n grows,

$$S_n(x) \overset{n\to\infty}{\longrightarrow} 0, \tag{2}$$

and $L_n(x)$ is a factor that grows without bound with increasing n,

$$L_n(x) \overset{n\to\infty}{\longrightarrow} \infty, \tag{3}$$

such that the product, $S_n(x)L_n(x)$, is bounded and the sum $\sum_n C_n S_n(x) L_n(x)$ is convergent.

2.1 Computation by Recursion

Recursion relationships are very convenient, and are a common way to calculate the $S_n(x)$ and $L_n(x)$ functions. Typically, the functions that arise in eigenfunction solutions to electromagnetic problems satisfy a three term recursion relationship which can be expressed as follows:

$$A(n,x)\, y_n(x) + B(n,x)\, y_{n+1}(x) + C(n,x)\, y_{n-1}(x) = 0. \tag{4}$$

For a fixed value of x, the recursion relation can be treated as a difference equation in n. A three term difference equation has 2 independent solutions [*Press, et al.*, 1988], i.e:

$$y_n(x) = \{P_n(x), Q_n(x)\}. \tag{5}$$

So the general solution to the recursion relation is:

$$y_n(x) = \alpha \, P_n(x) + \beta \, Q_n(x) \tag{6}$$

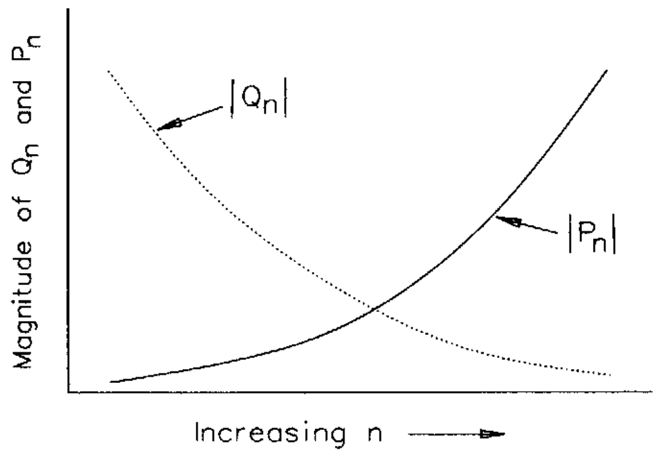

Figure 1: Relative magnitude trends of P_n and Q_n.

where α and β are constants that need to be determined by physical considerations.

Miller's algorithm [*Press, et al.*, 1988] is a commonly used technique to generate the sequence $\{P_n(x)\}$ or $\{Q_n(x)\}$. The algorithm begins by arbitrarily choosing two successive values to substitute into the recursion relationship which is used to generate the entire sequence. For example, let

$$y_0(x) = C_0(x) \quad \text{and} \quad y_1(x) = C_1(x), \tag{7}$$

which when substituted into Equation (6) yields:

$$C_0(x) = \alpha P_0(x) + \beta Q_0(x), \tag{8}$$
$$C_1(x) = \alpha P_1(x) + \beta Q_1(x). \tag{9}$$

Since $P_n(x)$ and $Q_n(x)$ are known to be independent, then by Equations (8)–(9) the values of α and β are determined uniquely, hence $y_n(x)$ is well-defined. Note that $y_n(x)$ contains components of both solutions, $\{P_n(x), Q_n(x)\}$, providing that the choice of $C_0(x)$ and $C_1(x)$ are not proportional to either $P_0(x)$ and $P_1(x)$, respectively or $Q_0(x)$ and $Q_1(x)$, respectively.

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} P_0 & Q_0 \\ P_1 & Q_1 \end{bmatrix}^{-1} \begin{bmatrix} C_0 \\ C_1 \end{bmatrix}; \quad \begin{vmatrix} P_0 & Q_0 \\ P_1 & Q_1 \end{vmatrix} \neq 0. \tag{10}$$

At this point it is necessary to examine the stability of the desired solution. Stability refers to the relative rate of growth of the magnitude of the desired solution relative to the non-desired solution. Let's examine the common circumstance where $|P_n(x)|$ increases as n increases and $|Q_n(x)|$ decreases as n increases as shown in Figure 1. It is clear that since our solution, $y_n(x)$, contains components of both $P_n(x)$ and $Q_n(x)$, then if we consider larger and larger values of n, $Q_n(x)$ will be less and less significant compared to $P_n(x)$, therefore:

$$y_n(x) \sim \alpha P_n(x); \qquad n \to \infty. \tag{11}$$

Under these circumstances, $P_n(x)$ is said to be stable recursing up (in n). Similarly we could have arbitrarily chosen values for $y_n(x)$ for two large successive values of n, and then recursed down thereby recovering the solution for $Q_n(x)$. In this case $Q_n(x)$ is said to be stable recursing down. Mathematically,

$$y_n(x) \sim \beta Q_n(x); \qquad n \to -\infty. \tag{12}$$

This process generates a sequence proportional to the desired $P_n(x)$ and $Q_n(x)$ sequences. $P_n(x)$ or $Q_n(x)$ can be recovered by multiplying the entire sequence $\{\alpha P_n(x)\}$ or $\{\beta Q_n(x)\}$ by $\frac{1}{\alpha}$ or $\frac{1}{\beta}$, respectively. α or β can be determined by calculating one value of $\{P_n(x)\}$ or $\{Q_n(x)\}$ and comparing it to the corresponding value of $\{\alpha P_n(x)\}$ or $\{\beta Q_n(x)\}$ which was calculated by recursion. An alternative is to use a normalization relationship of the form:

$$\sum_n \gamma_n P_n(x) = 1 \qquad \text{or} \qquad \sum_n \delta_n Q_n(x) = 1. \tag{13}$$

In the introduction of *Abramowitz and Stegun* [1972, p. XIII], there is a listing of many functions and their direction of recursive stability, and in the various chapters corresponding to the functions of interest, normalization relations of the form in Equation (13) can be found.

An alternative to Miller's algorithm can be applied if two successive values of the solution are known. If the desired solution contains only one component of the two independent solutions $\{P_n(x), Q_n(x)\}$, then it is necessary to recurse in the direction of recursive stability. Ideally, the direction of recursion should not matter, however, due to round off error in the computer, the undesired solution will be present and eventually grow to a significant value relative to the desired component of the solution.

As mentioned previously, the magnitude of the individual functions which we need to calculate can be either too large or too small for the computer to handle. This is why we introduce the auxiliary functions in Section 2.2.

2.2 The Auxiliary Functions

Since $S_n(x)$ and $L_n(x)$ are computed separately, before the sum converges individually they can become either too small or too large to calculate (due to computer limitations). To remove this upper bound limitation on the index of the summation, n, we first note the asymptotic behavior of $S_n(x)$ and $L_n(x)$.

$$S_n(x) \propto \sigma_n(x), \quad n \to \infty, \tag{14}$$

and,

$$L_n(x) \propto \Lambda_n(x), \quad n \to \infty. \tag{15}$$

Introduce the auxiliary functions $A_n^S(x)$ and $A_n^L(x)$ which are defined for an appropriate interval over n by the following expressions:

$$S_n(x) = A_n^S(x)\,\sigma_n(x), \tag{16}$$

and,

$$L_n(x) = A_n^L(x)\,\Lambda_n(x). \tag{17}$$

Since the auxiliary functions are equal to the original pathological functions with the asymptotic behavior factored out, the auxiliary functions remain well-behaved for all n and are therefore computationally preferable over the original $S_n(x)$ and $L_n(x)$ functions. These expressions (Equations (16) and (17)) for $S_n(x)$ and $L_n(x)$ can be substituted into the eigenfunction expansion, Equation (1), yielding:

$$\phi = \sum_n C_n\,A_n^S(x)\,\sigma_n(x)\,A_n^L(x)\,\Lambda_n(x), \tag{18}$$

or,

$$\phi = \sum_n C_n'(x)\,A_n^S(x)\,A_n^L(x), \tag{19}$$

where,

$$C_n'(x) = C_n\,\sigma_n(x)\,\Lambda_n(x). \tag{20}$$

The expression for $C_n'(x)$ can usually be significantly simplified which eliminates the necessity to compute the extremely small and extremely large values for $\sigma_n(x)$ and $\Lambda_n(x)$, respectively as $n \to \infty$.

The three factors in each term of Equation (19) are well-behaved for large values of n which makes this procedure convenient for computing a slowly convergent eigenfunction expansion.

It is common practice to extract a factor from functions that grow without bound. The main contribution of this paper is the computation procedure for the auxiliary functions which is presented in the next section.

2.2.1 Computation of the Auxiliary Functions by Recursion

The direction of recursive stability of an auxiliary function is the same as the function from which it was derived, therefore Miller's algorithm can also be applied to auxiliary functions. We begin by assuming that the recursion relationship for the original function (Equation (4)) is known. If we are trying to calculate $A_n^L(x)$, the auxiliary function for $L_n(x)$, then we simply substitute Equation (17) into the recursion relationship, Equation (4).

$$A_n(x)\,\Lambda_n(x)\,A_n^L(x) + B_n(x)\,\Lambda_{n+1}(x)\,A_{n+1}^L(x) + C_n(x)\,\Lambda_{n-1}(x)\,A_{n-1}^L(x) = 0 \tag{21}$$

In this form, the recursion relationship would experience the same overflow and underflow difficulties as the original function (due to the asymptotic factors $\Lambda_n(x)$, $\Lambda_{n+1}(x)$ and

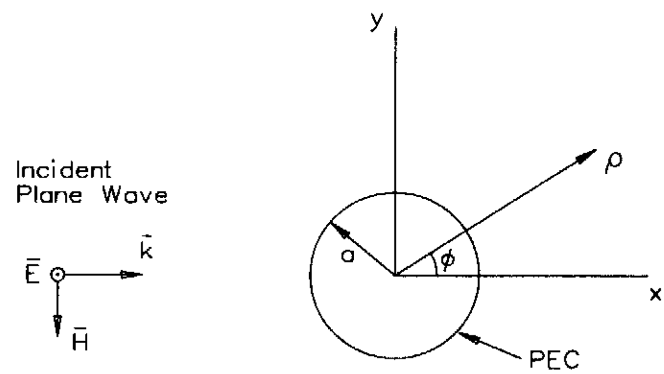

Figure 2: Plane wave incident upon PEC circular cylinder.

$\Lambda_{n-1}(x))$. It can, and must, be analytically simplified at this stage to avoid taking differences of very large numbers which is a computational faux pas. Section 3 illustrates this procedure by presenting an example which calculates the plane wave scattering by a circular cylinder.

3 Example: Scattering by a PEC Circular Cylinder

In this section we are presenting an example that applies the general technique outlined in this paper to the specific problem of determining the total (incident + scattered) z-directed electric field when a TM_z plane wave is incident upon a perfect electric conducting (PEC) circular cylinder which has its axis along the z-axis. The incident field is given by:

$$E_z^i = E_0 \, e^{-jkx} = E_0 \, e^{-jk\rho\cos\phi} = E_0 \sum_{n=-\infty}^{\infty} j^{-n} J_n(k\rho) \, e^{jn\phi}, \tag{22}$$

where E_0 is the complex amplitude and the coordinates x, ρ and ϕ are shown in Figure 2. The total z-directed field is given by [*Harrington*, 1961]:

$$E_z = E_0 \sum_{n=-\infty}^{\infty} j^{-n} \left[J_n(k\rho) - \frac{J_n(ka)}{H_n^{(2)}(ka)} H_n^{(2)}(k\rho) \right] e^{jn\phi} \tag{23}$$

It is possible to convert this series into a more quickly converging representation, however, this will not be done since the goal of this section is to illustrate the recursive technique outlined in this paper by a simple example. An additional numerical difficulty which will not be addressed in detail here, with the form of Equation (23) is that when computing the total fields near the cylinder, $\rho \approx a$, there can be a loss of significant digits. This may be overcome by computing the cross product directly by means of recursion relations

[*Abramowitz and Stegun*, 1972, p. 361]. Note that this problem does not occur when computing the scattered fields alone.

Using the relationship that Z_n, any integer order Bessel function $(J_n, Y_n, H_n^{(1)}, H_n^{(2)})$, satisfies:

$$Z_{-n} = (-1)^n Z_n, \tag{24}$$

we can express the total fields as:

$$E_z = E_0 \sum_{n=0}^{\infty} \epsilon_n \, j^{-n} \left[J_n(k\rho) - \frac{J_n(ka)}{H_n^{(2)}(ka)} H_n^{(2)}(k\rho) \right] \cos(n\phi), \tag{25}$$

where,

$$\epsilon_n = \begin{cases} 1 & ; \quad n = 0, \\ 2 & ; \quad n \neq 0. \end{cases} \tag{26}$$

Notice that the second term in the brackets exhibits the behavior that is discussed in this paper, namely that each individual factor grows without bound $(H_n^{(2)})$, or approaches zero (J_n), as n increases. The asymptotic behavior of the Bessel functions of the first and second kind are given by [*Abramowitz and Stegun*, 1972, p. 365]:

$$J_n(x) \sim \frac{1}{\sqrt{2\pi n}} \left(\frac{ex}{2n} \right)^n = \sigma_n(x), \qquad n \to \infty, \tag{27}$$

and,

$$Y_n(x) \sim -\sqrt{\frac{2}{\pi n}} \left(\frac{ex}{2n} \right)^{-n}, \qquad n \to \infty, \tag{28}$$

and since,

$$H_n^{(2)}(x) = J_n(x) - j\, Y_n(x). \tag{29}$$

Then,

$$H_n^{(2)}(x) \sim j\sqrt{\frac{2}{\pi n}} \left(\frac{ex}{2n} \right)^{-n} = \Lambda_n(x), \qquad n \to \infty. \tag{30}$$

So then the auxiliary functions are defined by:

$$J_n(x) = \frac{1}{\sqrt{2\pi n}} \left(\frac{ex}{2n} \right)^n A_n^J(x), \qquad n = 1, 2, 3, \ldots, \infty, \tag{31}$$

and,

$$H_n^{(2)}(x) = j\sqrt{\frac{2}{\pi n}} \left(\frac{ex}{2n} \right)^{-n} A_n^{H^{(2)}}(x), \qquad n = 1, 2, 3, \ldots, \infty. \tag{32}$$

All Bessel functions satisfy the same recursion relationship [*Abramowitz and Stegun*, 1972, p. 365]:

$$Z_{n-1}(x) + Z_{n+1}(x) = \frac{2n}{x} Z_n(x). \tag{33}$$

By comparing the asymptotic forms of $J_n(x)$ and $Y_n(x)$, $J_n(x)$ is downward stable and $Y_n(x)$ is upward stable [*Abramowitz and Stegun*, 1972, p. XIII]. Since the Hankel function consists of both $J_n(x)$ and $Y_n(x)$ and they differ in their direction of recursive stability, it is necessary to decompose the Hankel function to determine $J_n(x)$ and $Y_n(x)$ separately.

The definitions for the auxiliary functions (Equations (31) and (32)) are indeterminate for $n = 0$. For this reason, we will extract the $n = 0$ term of Equation (25) and begin the sum from $n = 1$.

$$\begin{aligned} E_z &= E_0 \left[J_0(k\rho) - \frac{J_0(ka)}{H_0^{(2)}(ka)} H_0^{(2)}(k\rho) \right] \\ &\quad + 2E_0 \sum_{n=1}^{\infty} \frac{j^{-n}}{\sqrt{2\pi n}} \left[\left(\frac{ek\rho}{2n} \right)^n A_n^J(k\rho) - \left(\frac{eka^2}{2n\rho} \right)^n \frac{A_n^J(ka)}{A_n^H(ka)} A_n^H(k\rho) \right] \cos(n\phi). \end{aligned} \tag{34}$$

Calculation of $A_n^J(x)$

The recursion relation which defines $A_n^J(x)$ is found by substituting Equation (31) into Equation (33). After algebraic simplification, the following form of the recursion relation is obtained:

$$A_n^J(x) = e \left(\frac{n}{n+1} \right)^{n+1/2} A_{n+1}^J(x) - \frac{(\frac{1}{2}ex)^2}{(n+2)^2} \left(\frac{n}{n+2} \right)^{n+1/2} A_{n+2}^J(x) \tag{35}$$

where e is the base of the natural logarithm. This is in a form suitable for downward recursion.

We will apply Miller's algorithm to determine a sequence denoted by $\tilde{A}_n^J(x)$ which is proportional to the desired $A_n^J(x)$ sequence. We choose N to be larger than the maximum number of terms expected to be summed by M. Also, let:

$$\tilde{A}_N^J(x) = 1, \tag{36}$$

and,

$$\tilde{A}_{N+1}^J(x) = 0. \tag{37}$$

Use the recursion relationship (Equation (35)) to determine $\{\tilde{A}_n^J(x)\}$ for $n = 1, 2, 3, \ldots N-1$. $\{\tilde{A}_n^J(x)\}$ is a sequence proportional to the desired sequence $\{A_n^J(x)\}$. There are many ways to normalize the sequence $\{\tilde{A}_n^J(x)\}$. Here we will use Equation (31) to determine $A_1^J(x)$ which will then be compared to $\tilde{A}_1^J(x)$ to determine the constant of proportionality, β.

$$\beta = \frac{ex}{2\sqrt{2\pi}} \frac{\tilde{A}_1^J(x)}{J_1(x)}, \tag{38}$$

So then,

$$\{A_n^J(x)\} = \frac{1}{\beta}\{\tilde{A}_n^J(x)\}; \qquad n = 1, 2, 3, \ldots N - M. \tag{39}$$

The following table was constructed, using these procedures, to emphasize the favorable behavior of the magnitude of the auxiliary functions compared to the magnitude of the Bessel functions. We are showing two arguments of the auxiliary function for a wide range of orders. This table illustrates the difficulty in computing the Bessel functions directly. The triple asterisk indicates that this term is larger than 10^{308}, the largest number our computer can handle.

n	$J_n(1)$	$A_n^J(1)$	$J_n(5)$	$A_n^J(5)$
1	4.4005×10^{-1}	0.81157	-3.2758×10^{-1}	-0.12083
2	1.1490×10^{-1}	0.88200	4.6565×10^{-2}	0.014297
5	2.4976×10^{-4}	0.94323	2.6114×10^{-1}	0.31559
20	3.8735×10^{-25}	0.98405	2.7703×10^{-11}	0.73798
50	2.9060×10^{-78}	0.99345	2.2942×10^{-45}	0.88306
100	8.4318×10^{-189}	0.99670	6.2678×10^{-119}	0.93919
200	***	0.99834	4.7600×10^{-296}	0.96898
500	***	0.99927	***	0.98737
1000	***	0.99965	***	0.99367

Calculation of $A_n^{H^{(2)}}(x)$

As mentioned previously, the Hankel function, $H_n^{(2)}(x)$, consists of Bessel functions of the first and second kind, $J_n(x)$ and $Y_n(x)$, respectively. $J_n(x)$ and $Y_n(x)$ differ in their direction of recursive stability, therefore, they must be computed separately. We note that as a consequence of Equations (29) and (32):

$$J_n(x) = -\sqrt{\frac{2}{\pi n}}\left(\frac{ex}{2n}\right)^{-n} \mathcal{I}m\{A_n^{H^{(2)}}(x)\} \tag{40}$$

and,

$$Y_n(x) = -\sqrt{\frac{2}{\pi n}}\left(\frac{ex}{2n}\right)^{-n} \mathcal{R}e\{A_n^{H^{(2)}}(x)\}. \tag{41}$$

From these relationships, we determine that $\mathcal{I}m\{A_n^{H^{(2)}}(x)\}$ is stable recursing down and that $\mathcal{R}e\{A_n^{H^{(2)}}(x)\}$ is stable recursing up. Substituting Equation (40) into Equation (33) yields the following recursion relation in a suitable form for downward recursion.

$$\begin{aligned}\mathcal{I}m\{A_n^{H^{(2)}}(x)\} = {} & \frac{1}{e}\left(\frac{2n}{x}\right)^2\left(\frac{n+1}{n}\right)^{n+\frac{3}{2}} \mathcal{I}m\{A_{n+1}^{H^{(2)}}(x)\} \\ & - \left(\frac{2n}{ex}\right)^2\left(\frac{n+2}{n}\right)^{n+\frac{3}{2}} \mathcal{I}m\{A_{n+2}^{H^{(2)}}(x)\}\end{aligned} \tag{42}$$

We can use this recursion relation along with Equation (40) with $n = 1$ (for normalization), to apply Miller's algorithm.

$$\mathcal{I}m\{A_1^{H^{(2)}}(x)\} = -\sqrt{\frac{\pi}{2}\frac{ex}{2}}\, J_1(x) \tag{43}$$

The calculation of $\mathcal{R}e\{A_n^{H^{(2)}}(x)\}$ requires two starting values and the recursion relationship since it is upward stable. The two starting values are obtained by substituting $n = 1$ and $n = 2$ into Equation (41).

$$\mathcal{R}e\{A_1^{H^{(2)}}(x)\} = -\sqrt{\frac{\pi}{2}\frac{ex}{2}}\, Y_1(x), \tag{44}$$

and,

$$\mathcal{R}e\{A_2^{H^{(2)}}(x)\} = -\sqrt{\pi}\left(\frac{ex}{4}\right)^2 Y_2(x). \tag{45}$$

The recursion relation is obtained by substituting Equation (41) into Equation (33) yielding the following recursion relationship for $\mathcal{R}e\{A_n^{H^{(2)}}(x)\}$ in a suitable form for upward recursion.

$$\begin{aligned}\mathcal{R}e\{A_n^{H^{(2)}}(x)\} &= e\left(\frac{n-1}{n}\right)^{n-1/2} \mathcal{R}e\{A_{n-1}^{H^{(2)}}(x)\} \\ &\quad - \frac{\left(\frac{1}{2}ex\right)^2}{(n-2)^2}\left(\frac{n-2}{n}\right)^{n-1/2} \mathcal{R}e\{A_{n-2}^{H^{(2)}}(x)\}\end{aligned} \tag{46}$$

The following table is presented for a comparison of the Hankel function of the second kind with its auxiliary function. We are showing two arguments of the auxiliary function for a wide range of orders.

n	$H_n^{(2)}(1)$	$A_n^{H^{(2)}}(1)$	$H_n^{(2)}(5)$	$A_n^{H^{(2)}}(5)$
1	$(4.4005 + j7.8121) \times 10^{-1}$	$1.3307 - j0.74960$	$(-3.2758 - j1.4786) \times 10^{-1}$	$-1.2594 + j2.7900$
2	$(0.11490 + j1.6507) \times 10^{0}$	$1.3511 - j0.0940$	$(0.46565 - j3.6766) \times 10^{-1}$	$-7.5237 - j0.9529$
5	$(0.0000 + j2.6041) \times 10^{2}$	$1.0831 + j0.0000$	$(2.6114 + j4.5369) \times 10^{-1}$	$5.8970 - j3.3942$
20	$(0.0000 + j4.1140) \times 10^{22}$	$1.0175 + j0.0000$	$(0.0000 + j5.9340) \times 10^{8}$	$1.3996 + j0.0000$
50	$(0.0000 + j2.1911) \times 10^{77}$	$1.0068 + j0.0000$	$(0.0000 + j2.7888) \times 10^{42}$	$1.1381 + j0.0000$
100	$(0.0000 + j3.7753) \times 10^{185}$	$1.0034 + j0.0000$	$(0.0000 + j5.0849) \times 10^{115}$	$1.0661 + j0.0000$
200	* * *	$1.0017 + j0.0000$	$(0.0000 + j3.3446) \times 10^{292}$	$1.0323 + j0.0000$
500	* * *	$1.0007 + j0.0000$	* * *	$1.0128 + j0.0000$
1000	* * *	$1.0003 + j0.0000$	* * *	$1.0064 + j0.0000$

4 Conclusion

In this paper we introduce a technique which circumvents the need to compute very large or very small special functions that commonly appear in eigenfunction expansions. This was accomplished by introducing a set of well-behaved auxiliary functions which are equal to the original special function with its asymptotic behavior factored out. When the eigenfunction expansion is expressed in terms of these auxiliary functions, the summation can be simplified resulting in well-behaved factors in the sum. The auxiliary functions can be computed via modified recursion formulas.

This procedure is formally exact since we are not making any approximations – only substitutions. Typically, when calculating summations of the type addressed here without the use of the auxiliary functions, at some point in the summation (which needs to be determined), an asymptotic form is substituted into the expression. The procedure described herein avoids the need to switch functional representations thereby eliminating the need to determine the value of the index to implement the asymptotic representation.

In this paper we have restricted our development of the auxiliary functions to the extraction of the asymptotic form of the function. The procedure is not limited to this asymptotic extraction. Any functional form that is convenient for the formulation of the problem at hand can be extracted in a similar manner.

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