NUMERICAL INTEGRATION OF MARCUSE'S POWER LOSS FORMULA

JAMES P COUGHLIN, PhD
MATHEMATICS DEPARTMENT
TOWSON STATE UNIVERSITY
TOWSON, MARYLAND

ALBERT D. KRALL
ADVANCED TECHNOLOGY AND RESEARCH CORP.
LAUREL, MARYLAND

ROBERT H. BARAN
U.S. NAVAL SURFACE WEAPONS CENTER
SILVER SPRING, MARYLAND

MEMBERS, I.E.E.E.

Introduction: D. Marcuse [1] has derived a power loss formula to calculate the power losses from an electromagnetic wave traveling down a tapered dielectric rod. He first considers the losses at a step and then approximates the tapered rod by a series of steps. He assumes that, when the radius of the rod is a, the constants describing the bound mode will be the same as for an infinite rod. When the radius changes, so do the coefficients. Further, there is an additional change due to power taken from the bound mode and converted into the radiation modes. For a full discussion of the model, the interested reader is referred to [1]. Here, we are concerned only with the numerical integration of his formula.

In order to understand his formula, it is necessary to include lengthy abstracts from Marcuse's original paper. To begin with, the bound mode fields inside the rod are:

$$E_z = A\, J_\nu(\kappa r) \cos(\nu\phi)$$

$$H_z = B\, J_\nu(\kappa r) \sin(\nu\phi)$$

$$E_r = -i/\kappa^2 \{ \kappa\beta_0 A\, J_\nu'(\kappa r) + \omega\, \mu\, B\, \nu/r\; J_\nu(\kappa r) \} \cos(\nu\phi) \qquad (1)$$

$$E_\phi = i/\kappa^2 \{ \beta_0 A \nu / r \; J_\nu(\kappa r) + \kappa \omega \mu B J'_\nu(\kappa r) \} \sin(\nu\phi)$$

$$H_r = -i/\kappa^2 \{ n^2 \omega \epsilon_0 A \nu/r \, J_\nu(\kappa r) + \kappa \beta_0 B J'_\nu(\kappa r) \} \sin(\nu\phi)$$

$$H_\phi = -i/\kappa^2 \{ n^2 \kappa\omega\epsilon_0 A J'_\nu(\kappa r) + \beta_0 B \nu/r \; J_\nu(\kappa r) \} \cos(\nu\phi)$$

While the fields outside the rod may be written:

$$E_z = C H_\nu^1(i\gamma r) \cos(\nu\phi)$$

$$H_z = D H_\nu^1(i\gamma r) \sin(\nu\phi)$$

$$E_r = i/\gamma^2 \{ i\gamma\beta_0 C H_\nu^{1'}(i\gamma r) + \omega\mu\nu/r \, D H_\nu^1(i\gamma r) \} \cos(\nu\phi) \qquad (2)$$

$$E_\phi = -i/\gamma^2 \, [\beta_0 \nu/r \; C H_\nu^1(i\gamma r) + i\omega\gamma\mu D H_\nu^{1'}(i\gamma r)] \sin(\nu\phi)$$

$$H_r = i/\gamma^2 [\omega\epsilon_0 \nu/r \; C H_\nu^1(i\gamma r) + i\gamma\beta_0 D H_\nu^{1'}(i\gamma r) \,] \sin(\nu\phi)$$

$$H_\phi = i/\gamma^2 \, [i\gamma\omega\epsilon_0 C H_\nu^{1'}(i\gamma r) + \beta_0 \nu/r \, D H_\nu^1(i\gamma r)] \cos(\nu\phi)$$

It is necessary that these fields satisfy the usual continuity conditions where they meet, namely that the tangential components of H and E are continuous and the normal component of D. This gives rise to the "eigenvalue" equation (We specialize to the lowest order mode, $\nu=1$)

$$\left\{ n^2 a\gamma^2/\kappa \, [J_0(\kappa a)/J_1(\kappa a) - 1/\kappa a] + \gamma a \, i \, H_0^1(i\gamma a)/H_1^1(i\gamma a) - 1 \right\} *$$

$$\left\{ a\gamma^2/\kappa [J_0(\kappa a)/J_1(\kappa a) - 1/\kappa a] + \gamma a i \, H_0^1(i\gamma a) \; / H_1^1(i\gamma a) - 1 \right\}$$

$$= [\; (n^2-1)\beta_0 k/\kappa^2 \;]^2 \qquad (3)$$

where $k^2 = 4\pi^2 f^2 \mu_0 \epsilon_0$

$\beta_0 =$ *the solution of the eigenvalue problem*

$\kappa^2 = n^2 k^2 - \beta_0^2 \qquad \mu = \mu_r * \mu_0$

n^2 = the dielectric constant

$\gamma^2 = \beta_0^{\,2} - k^2$ $\qquad$ μ_r = relative magnetic permeability

$H = iH_0^{\,1}(i\gamma a)\ /H_1^{\,1}(i\gamma a)$

When this equation is satisfied we can compute the values of A and B and therefore the fields inside the rod from:

$$B/A = -\sqrt{\epsilon_0/\mu_0}\ \ ka\ (\kappa a)^2\ \Big[\ n^2/\kappa a\ \{J_0(\kappa a)/J_1(\kappa a)\ -1/\kappa a\ \}\ + \quad 1/\gamma a\ \{\ H-1/\gamma a\ \}\ \Big]\ /\ [\beta_0 a(1+\kappa^2/\gamma^2)\ ] \qquad (4)$$

and

$$p=\pi/4\ \star \Big[\ k\beta_0/\kappa^4 \star \Big\{ (\kappa a)^2 [\ J_0^{\,2}(\kappa a)\ +\ J_1^{\,2}(\kappa a)]\ -2J_1^{\,2}(\kappa a)\ \Big\} \star \{n^2+\mu_0/\epsilon_0\ \ B^2/A^2\}$$

$$+\ k\beta_0/\gamma^4 \star \{\ \ (\gamma a)^2 \star \{1-H^2)\}\ +2\ \}\ \star\ J_1^{\,2}(\kappa a)\ \star \{\ 1\ +\mu_0/\epsilon_0\ B^2/A^2\}$$

$$+2\ \sqrt{\mu_0/\epsilon_0}\ B/A\ \{\ (\beta_0^{\,2}+n^2k^2)/\kappa^4\ -\ (\beta_0^{\,2}+k^2)/\gamma^4\}\ \ J_1^{\,2}(\kappa a)\ \ \Big] \qquad (5)$$

$$\star A^2\sqrt{\epsilon_0/\mu_0}$$

The equations thus far describe the "so called" bound mode. In addition to this, radiation modes will be created by the taper and, if we write:

$\sigma^2 = n^2k^2 - \beta^2$

$\rho^2 = k^2 - \beta^2$

Then:

$E_z = F\ J_\nu(\sigma r)\ \cos\nu\phi$

$H_z = G\ J_\nu(\sigma r)\ \sin\nu\phi$

$E_r =\ -i/\sigma^2\ \{\ \sigma\beta\ F\ J_\nu'(\sigma r)\ \ +\ \omega\mu\nu/r\ \ G\ J_\nu(\sigma r)\ \}\ \cos\nu\phi$

$$E_\phi = 1/\sigma^2 \{\beta\nu/r\ F\ J_\nu(\sigma r) + \sigma\mu\omega\ G\ J_\nu'(\sigma r)\} \sin\nu\phi \qquad (6)$$
$$H_r = -1/\sigma^2 \{ n^2\omega\epsilon_0\nu/r\ F\ J_\nu(\sigma r) + \sigma\beta\ G\ J_\nu'(\sigma r)\} \sin\nu\phi$$
$$H_\phi = -1/\sigma^2 \{ n^2\sigma\omega\epsilon_0\ F\ J_\nu'(\sigma r) + \beta\nu/r\ G\ J_\nu(\sigma r)\} \cos\nu\phi$$

give the radiation modes inside the rod while:

$$E_z = [H\ J_\nu(\rho r) + I\ Y_\nu(\rho r)] \cos\nu\phi$$
$$H_z = [K\ J_\nu(\rho r) + M\ Y_\nu(\rho r)] \sin\nu\phi \qquad (7)$$
$$E_r = -1/\rho^2 \Big\{ \rho\beta[H\ J_\nu'(\rho r) + I\ Y_\nu'(\rho r)] + \omega\mu\nu/r\ [K\ J_\nu(\rho r) + M\ Y_\nu(\rho r)] \Big\} \cos\nu\phi$$

$$E_\phi = 1/\rho^2 \Big\{ \beta\nu/r\ [H\ J_\nu(\rho r) + I\ Y_\nu(\rho r)] + \rho\omega\mu\ [K\ J_\nu'(\rho r) + M\ Y_\nu'(\rho r)] \Big\} \sin\nu\phi$$

$$H_r = -1/\rho^2 \Big\{ \omega\epsilon_0\nu/r\ [H\ J_\nu(\rho r) + I\ Y_\nu(\rho r)] + \rho\beta\ [K\ J_\nu'(\rho r) + M\ Y_\nu'(\rho r)] \Big\} \sin\nu\phi$$
$$H_\phi = -1/\rho^2 \Big\{ \rho\omega\epsilon_0\ [H\ J_\nu'(\rho r) + I\ Y_\nu'(\rho r)] + \beta\nu/r\ [K\ J_\nu(\rho r) + M\ Y_\nu(\rho r)] \Big\} \cos\nu\phi$$

give the fields outside the rod.

Let:

$$b = \rho/\sigma\ J_1'(\sigma a)\ Y_1(\rho a)$$
$$c = (n^2-1)k\beta/\ a\rho\sigma^2\ \ J_1(\sigma a)\ Y_1(\rho a)$$
$$d = \rho/\sigma\ J_1'(\sigma a)\ J_1(\rho a) \qquad (8)$$
$$e = J_1(\sigma a)\ J_1'(\rho a)$$

$$f = (n^2-1)k\beta/\ ap\sigma^2\ \ J_1(\sigma a)\ \ J_1(pa)$$

$$g = J_1(\sigma a)\ \ Y_1'\ (pa)$$

Then:

$$F/G = +/-\ \sqrt{\mu_0\epsilon_0}\ \sqrt{\frac{(g-b)^2+(e-d)^2+c^2+f^2}{(g-n^2b)^2+(e-n^2d)^2+c^2+f^2}} \qquad (9)$$

and

$$\begin{aligned} P=(\pi/2)^3a^2\beta/p\ \omega\epsilon_0\{\ & [g-n^2b+c\sqrt{\mu_0/\epsilon_0}\ \ G/F]^2 + \\ & [e-n^2d+f\sqrt{\mu_0/\epsilon_0}\ \ G/F]^2 \\ & [c+(g-b)\sqrt{\mu_0/\epsilon_0}\ \ G/F]^2 \qquad (10) \\ & [f+(e-d)\sqrt{\mu_0/\epsilon_0}\ G/F\ ]^2\ \ \}\ F^2 \end{aligned}$$

where F and G are the analogs of A and B in the bound modes. The square root in the first formula introduces an ambiguity of sign which we shall comment on later.

For the present, we compute the four partial derivatives:

$$\begin{aligned} \frac{\partial H}{\partial a} = \pi p/2\ \Big[\ & \Big\{a(\sigma^2-n^2p^2)/\sigma\ J_0(\sigma a)\ [Y_0(pa)-\ Y_1(pa)/pa\ ] \\ & +[\ 2/pa\ -pa\ +n^2(pa-2pa/a^2\sigma^2)]\ J_1(\sigma a)\ Y_1(pa) \\ & +[n^2p^2/\sigma^2-1]\ J_1(\sigma a)\ Y_0(pa) \Big\}\ F \\ & +(n^2-1)k^2\beta/\omega\epsilon_0\sigma^2p\ \ * \qquad (11) \\ & \Big\{\sigma J_0(\sigma a)\ Y_1(pa)\ +pJ_1(\sigma a)\ Y_0(pa) \\ & \qquad -2/a\ J_1(\sigma a)\ Y_1(pa)\ \Big\}\ G\ \ \Big] \end{aligned}$$

$$\frac{\partial I}{\partial a} = -\pi\rho/2 \Big[\Big\{ a(\sigma^2-n^2\rho^2)/\sigma\ J_0(\sigma a)\ \{ J_0(\rho a)-J_1(\rho a)/\rho a \}$$
$$+[2/\rho a-\rho a+n^2(\rho a-2\rho a/a^2\sigma^2)\ ]\ J_1(\sigma a)J_1(\rho a)$$
$$+(n^2\rho^2/\sigma^2 -1)\ J_1(\sigma a)\ J_0(\rho a) \Big\} F$$

$$+(n^2-1)k^2\beta/\omega\epsilon_0\sigma^2\rho \star \tag{12}$$
$$\Big\{ \sigma J_0(\sigma a)\ J_1(\rho a) +\rho\ J_1(\sigma a)\ J_0(\rho a)-2/a\ J_1(\sigma a)\ J_1(\rho a) \Big\} \star\ G\Big]$$

$$\frac{\partial K}{\partial a} = \pi\rho/2\sigma\ \ (n^2-1)k^2 \Big[$$
$$\beta\ /\ \omega\mu\rho\sigma \Big\{ \sigma J_0(\sigma a)\ Y_1(\rho a) + \rho J_1(\sigma a) Y_0(\rho a)\ -2/aJ_1(\sigma a)\ Y_1(\rho a) \Big\} F$$
$$+ \Big\{ aJ_0(\sigma a)\ (\ Y_0(\rho a)-Y_1(\rho a)/\rho a\ ) +2\ /\ \rho\sigma a\ J_1(\sigma a) Y_1(\rho a) \tag{13}$$
$$-1/\sigma\ J_1(\sigma a) Y_0(\rho a) \Big\} \star G\ \Big]$$

$$\frac{\partial M}{\partial a} = -.5\pi\rho/\sigma\ (n^2-1)k^2 \Big[\beta/\omega\mu\rho\sigma$$
$$\Big\{ \sigma J_0(\sigma a)\ J_1(\rho a)\ +\rho J_1(\sigma a) J_0(\rho a)\ -2/a\ J_1(\sigma a) J_1(\rho a) \Big\} F$$
$$+\Big\{\ aJ_0(\sigma a)\ [\ J_0(\rho a)-J_1(\rho a)/\rho a\ ] \tag{14}$$

$$+2/\rho\sigma a\ \ J_1(\sigma a)J_1(\rho a)\ -\ 1/\sigma\ \ J_1(\sigma a)J_0(\rho a)$$
$$\Big\}\ \star\ \ G\Big]$$

And now we are in a position to compute the first integrand:

$$I(\rho,a) = \frac{\pi}{4\rho^2\gamma^2\ P}\ J_1(\kappa a)\ \star$$

$$\Big\{ (\beta_0+\beta)\gamma\rho\omega\ (\epsilon_0 A \frac{\partial H}{\partial a} + \mu_0 B \frac{\partial K}{\partial a}) \star$$

$$\Big[a \frac{\gamma J_0(\rho a) + \rho J_1(\rho a)\ iH_0{}^1(i\gamma a) / H_1{}^1(i\gamma a)}{\gamma^2+\rho^2} - \frac{J_1(\rho a)}{\gamma\rho} \Big]$$

$$+\{ (\beta_0+\beta)\gamma\rho\omega\ (\epsilon_0 A \frac{\partial I}{\partial a} + \mu_0 B \frac{\partial M}{\partial a}) \} \star \qquad (15)$$

$$\Big[a \frac{\gamma Y_0(\rho a) + \rho Y_1(\rho a)\ iH_0{}^1(i\gamma a) / H_1{}^1(i\gamma a)}{\gamma^2+\rho^2} - \frac{Y_1(\rho a)}{\gamma\rho} \Big]$$

$$+(k^2+\beta_0\beta) \Big[(A\frac{\partial K}{\partial a} + B\frac{\partial H}{\partial a})\ J_1(\rho a) + (A\frac{\partial M}{\partial a} + B\frac{\partial I}{\partial a})\ Y_1(\rho a) \Big] \Big\}$$

Corresponding to each choice of sign for the square root in equation 9, there will be a function, $I(\rho,a)$. By $I_+(\rho,a)$ we mean the function corresponding to the positive choice of the sign and by $I_-(\rho,a)$ we mean the function corresponding to the negative choice. This corresponds completely to the even and odd symmetry present in the slab dielectrics. Now calculate:

$$p(\rho) = \int_0^L I_+(\rho,a)\ a'(z)\ \exp[-i \int_0^z (\beta_0-\beta)\ ds]\ dz \qquad (16)$$

$$q(\rho) = \int_0^L I_-(\rho,a)\ a'(z)\ \exp[-i \int_0^z (\beta_0-\beta)\ ds]\ dz \qquad (17)$$

And then the fractional power loss is given by:

$$\Delta P/P = \oint \{ |p|^2 + |q|^2 \}\ |\beta|/\rho\ d\beta \qquad (18)$$

<u>The Eigenvalue Equation:</u> The first step in integrating this expression is the evaluation of the integrand and this begins with a solution of the eigenvalue equation. In practice it was found easier to solve for γ ($= \sqrt{\beta_0{}^2 - k^2}$ than for β_0 because there is one root at $\gamma = 0$ and we are looking for the smallest positive root. The function on the right was

expanded in a power series in the two variables γa and $\ln(\gamma a)$, as shown in appendix I. (See also [2] and [3].) Only the lowest order terms were retained and the resulting formula led to the approximation

$$\gamma a = 1.123 \exp \left[-\frac{1}{2} \frac{n^2+1}{\kappa_0 a} \frac{J_0(\kappa_0 a)}{J_1(\kappa_0 a)} \right]$$

This formula was extremely accurate for $ka < 1$ and provided a useful starting point for an iteration scheme whenever $\kappa_0 a$ was less than 2.4 (ie: the first zero of the Bessel function), that is: as long as all higher order modes are choked off.

When the eigenvalue is determined, the bound mode is completely described by equations (4) and (5). In the calculation, fifty values of γ, A and B were calculated at points spaced at equal intervals down the rod. These values were then stored in a table to be used as needed.

The Singularities: With the bound mode available, the inner integrand may be calculated from equations (7)-(14). A value of $I(\rho,a)$ is needed for each value of β from -k to +k inclusive. At each of these (end) points, $\rho = 0$, the denominator of (18) vanishes and a singularity (which must be dealt with) is created. It is to be hoped that $p(\rho)$ and $q(\rho)$ will be proportional to $\sqrt{\rho}$ or better. Unfortunately equation (14) does not have $\sqrt{\rho}$ but $1/\rho^2$. Now a factor of $\rho^{5/2}$ must be sought from the rest of the terms and the search leads us a merry chase. The details are included in appendix I. When $|p|$ and $|q|$ were plotted for values of β close to k, the graph rose steeply. (Indeed, it was this that led to the calculations described in appendix II.) After the results of appendix II were available, the integrand was calculated for values of β each of which was only half as far from k as its predecessor. The graph rose sharply revealing a burst of power very close to $\beta = k$. The preliminary calculations had missed this entirely and underestimated the power lost. With the aid of appendix II, the revised calculations used much smaller steps near $\beta = k$ and were much more accurate as we shall show.

Another, albeit simpler, singularity was found at $\beta = 0$. It was easily mended. Now a straightforward Simpson's law integration scheme could be (and was) written. It worked moderately well for short rods but failed utterly when applied to long rods. This led to an investigation which showed that the problem lay in the periodic nature of the integrand.

The Inverse Square Law and the Validation of the Program: Starting with the power loss formula:

$$\Delta P/P = \oint \{ |p|^2 + |q|^2 \} \, |\beta|/\rho \, d\beta \qquad (18)$$

where:

$$p(\rho) = \int_0^L I_+(\rho,a)\, a'(z) \exp[-i \int_0^z (\beta_0-\beta)\, ds]\, dz \qquad (16)$$

$$q(\rho) = \int_0^L I_-(\rho,a)\, a'(z) \exp[-i \int_0^z (\beta_0-\beta)\, ds]\, dz \qquad (17)$$

If z is large, so is the argument of the imaginary exponential function and that creates the problem. The period of the integrand is small compared to the spacing, Δx. So we wind up taking a quasi-random sample of the integrand.

Let us suppose that the antenna is self-similar in that the radius depends only on the ratio $x=z/L$. Substituting this in the integral:

$$p(\rho) = \int_0^1 I_+(\rho,a[x]) \frac{da}{dx} L \exp\{ -i\int_0^x [\beta_0(a[x]) - \beta] L\, ds\}\, dx$$

Now set $u = \int_0^x [\beta_0(a[x]) -\beta]\, ds$.

$$p(\rho) = \int_0^{u_1} \phi(u) \exp\{-iLu\}\, du \qquad (19)$$

where

$$\phi(u) = I_+(\rho, a[x]) \frac{da}{dx} / [\beta_0(a[x]) - \beta] \qquad (20)$$

and the quantities to be evaluated are understood to be functions of u. Integrating by parts and multiplying by L:

$$Lp(\rho) = i \exp(-iLu)\, \phi(u)\, \Big|_0^{u_1} -i \int_0^{u_1} \phi(u) \exp\{-iLu\}\, du$$

$$Lp(\rho) = i \exp\{-iLu_1\}\, \phi(u_1) - i\phi(0) -iT_+$$

and

$$Lq(\rho) = i \exp\{-iLu_1\}\, \psi(u_1) -i\psi(0) - iT_-$$

$$L^2\Delta P/P= \int_{-k}^{k} [\phi^2(u_1) + \phi^2(0) + \psi^2(u_1) + \psi^2(0)]\, |\beta|/\rho\, d\beta$$

$$-2\int_{-k}^{k}[\phi(u_1)\, \phi(0) + \psi(u_1)\, \psi(0)] \cos(Lu_1)\, |\beta| /\rho\, d\beta$$

$$\int_{-k}^{k} \Big[\phi(u_1) \{ T_+ \exp(iLu_1) + T_+^* \exp(-iLu_1) \} + \psi(u_1) \{ T_- \exp(iLu_1) + T_-^* \exp(-iLu_1) \} - \phi(0) \{ T_+ + T_+^* \} - \psi(0) \{ T_- + T_-^* \} \; T_+^* T_+ + T_- T_-^* \Big] |\beta| / \rho \; d\beta$$

Now the limits of T_+ and T_- as L goes to infinity are both zero as noted above and the third integral grows smaller as L grows larger. We claim that the same is true for the second integral as well. Observe that the integrand :

$$\{ \phi(u_1) \, \phi(0) + \psi(u_1) \, \psi(0) \} \, |\beta| / \rho$$

is continuous except possibly at β=0 and ρ=0. By appendix II, even in these three exceptional cases the integrand is continuous . Accordingly the integral goes to zero (Riemann- Lebesgue Lemma)

Hence, as L goes to infinity, $L^2 \Delta P/P$ approaches:

$$\int_{-k}^{k} \{ \phi^2(u_1) + \phi^2(0) + \psi^2(u_1) + \psi^2(0) \} \, |\beta| / \rho \; d\beta$$

and by using this formula we can compute the value of the constant appearing in the inverse square law. The agreement initially was not very good, and a more sophisticated integration -- the one of van der Vooren and van Linde [4] (which owes some filial respect to the scheme of Filon) was introduced.. This scheme is designed for integrating:

$$\int_0^{2N\pi} f(x) \cos(\omega x) \, dx$$

and

$$\int_0^{2N\pi} f(x) \sin(\omega x) \, dx$$

by approximating f(x) by an appropriate polynomial.

The first hurdle that we must clear in order to use this scheme stems from the fact that our integral is not exactly in this form since

u_1 is not, in general, a multiple of 2π. So we choose N to be the greatest integer that does not exceed $u_1 / 2\pi$. The integral from $2n\pi$ to u_1 is then found by a Simpson Law scheme. Before evaluating the remainder by the van der Vooren - van Linde scheme, we have one more hurdle to cross. This scheme requires that the integrand be evaluated at integral submultiples of $2N\pi$. But u must be constructed from a numerical integration scheme and the values of u that are available are not always at the required points. To overcome this difficulty, the most straightforward imaginable solution was used: linear interpolation. We did not have time to carry out a thorough analysis of the errors involved in such a scheme, but we did test it by generating a value of u from the quadratic function:

$$u = 2\pi x - \pi x^2/10$$

By setting $x = n/5$ for $n=1$ to 50, we generated a table of values of u The following four integrals were then evaluated:

$$I_1 = \oint_0^{10\pi} u^5 \cos(u)\, du \qquad I_2 = \oint_0^{10\pi} u^5 \sin(u)\, du$$

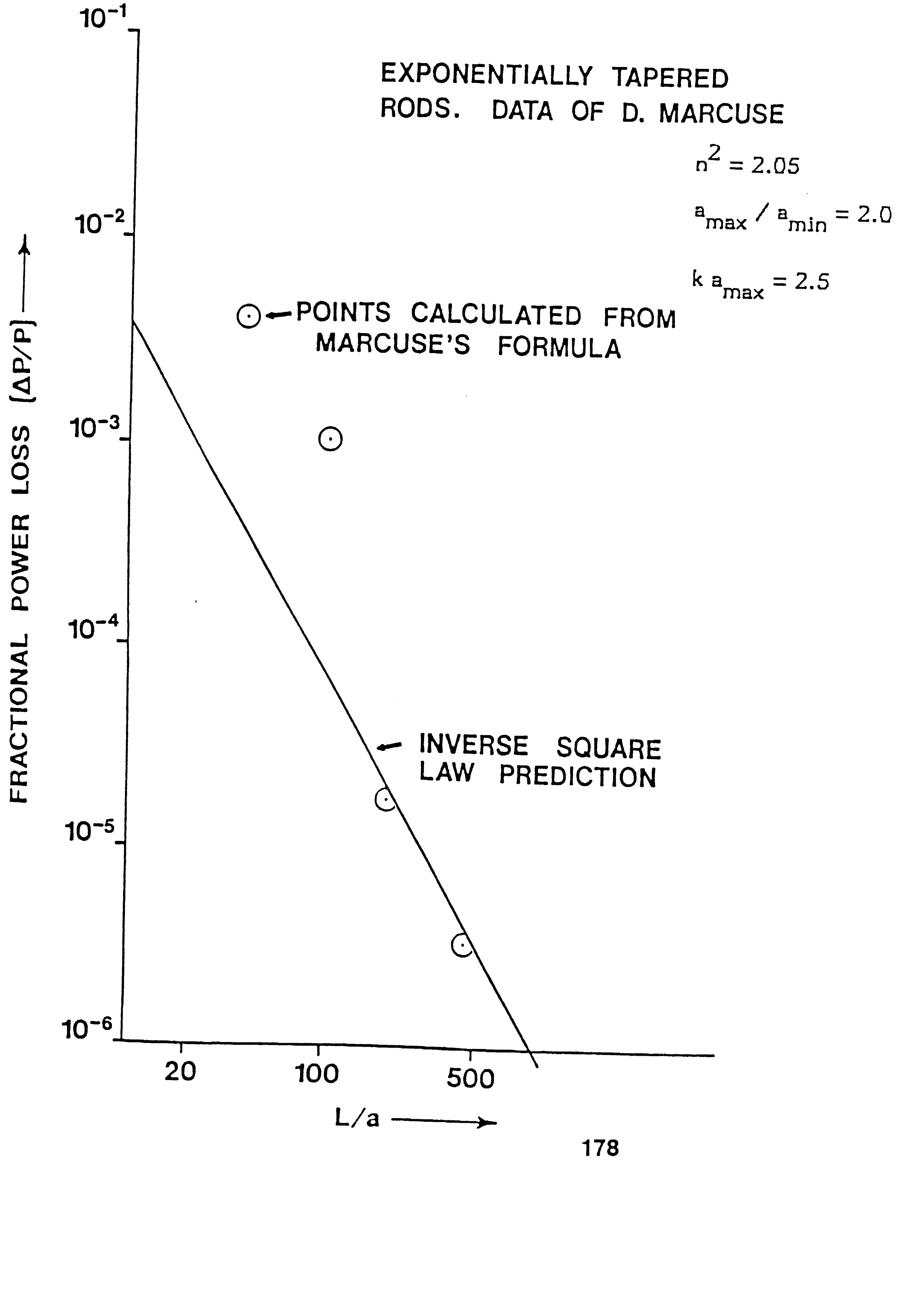
$$I_3 = \oint_0^{10\pi} u^4 \cos(u)\, du \qquad I_4 = \oint_0^{10\pi} u^4 \sin(u)\, du$$

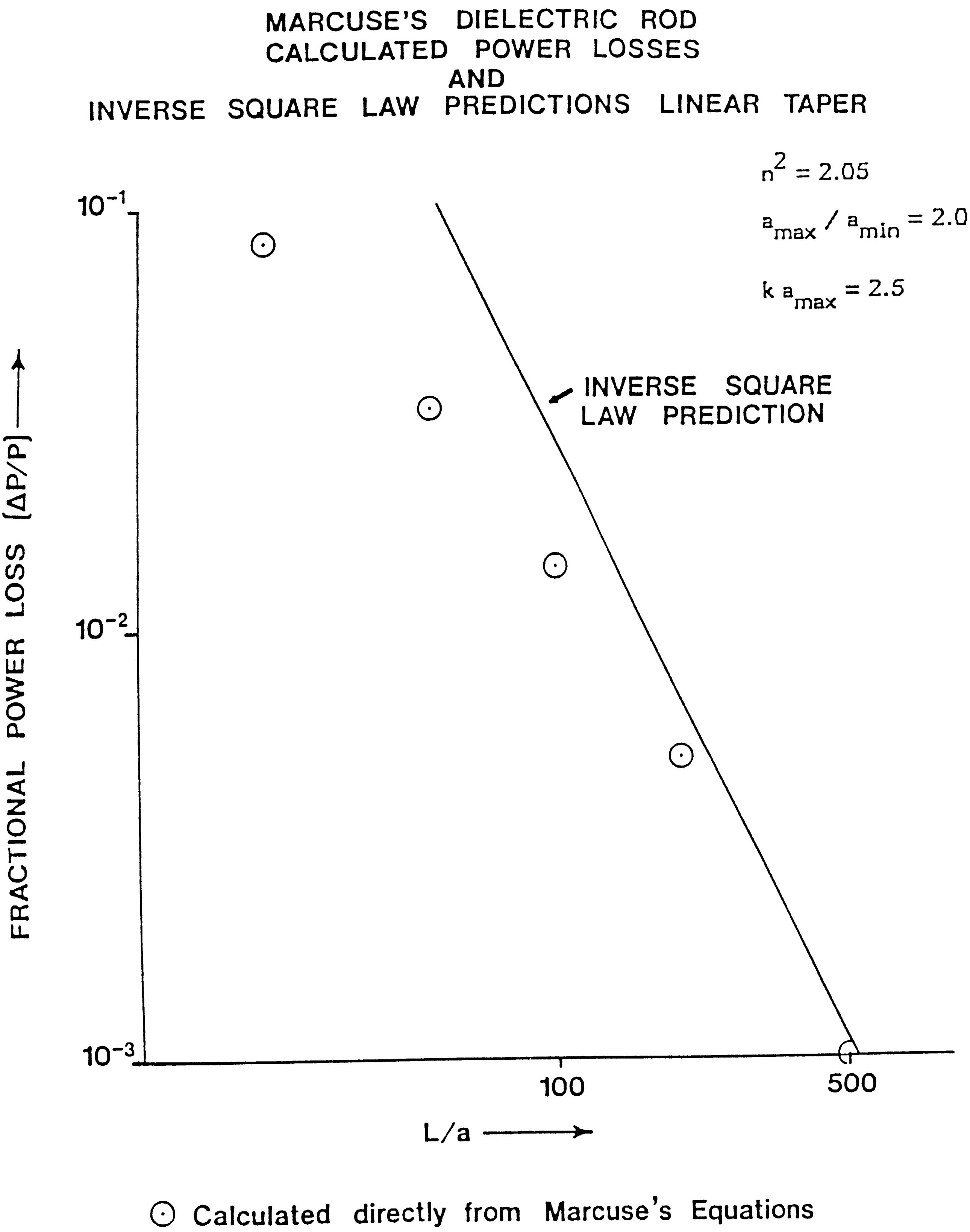
in three different ways. The first was a straightforward application of integral calculus and repeated integrations by parts (exact). The second used the vdVvL scheme just as it stands and the third chose a value of u as the scheme demands, but then interpolated linearly in the table to find the value of z and then got back to u from the interpolated value of z by using the formula. The results are in the table below:

	exact	vdVvL	mod. vdVvL
I_1	$.48112369\times10^7$	$.48218\times10^7$	$.48183\times10^7$
I_2	$-.29985613\times10^8$	$-.29986\times10^8$	$-.29983\times10^8$
I_3	$.1232712\times10^6$	$.12340\times10^6$	$.12332\times10^6$
I_4	$-.96224739\times10^6$	$-.96225\times10^6$	$-.96214\times10^6$

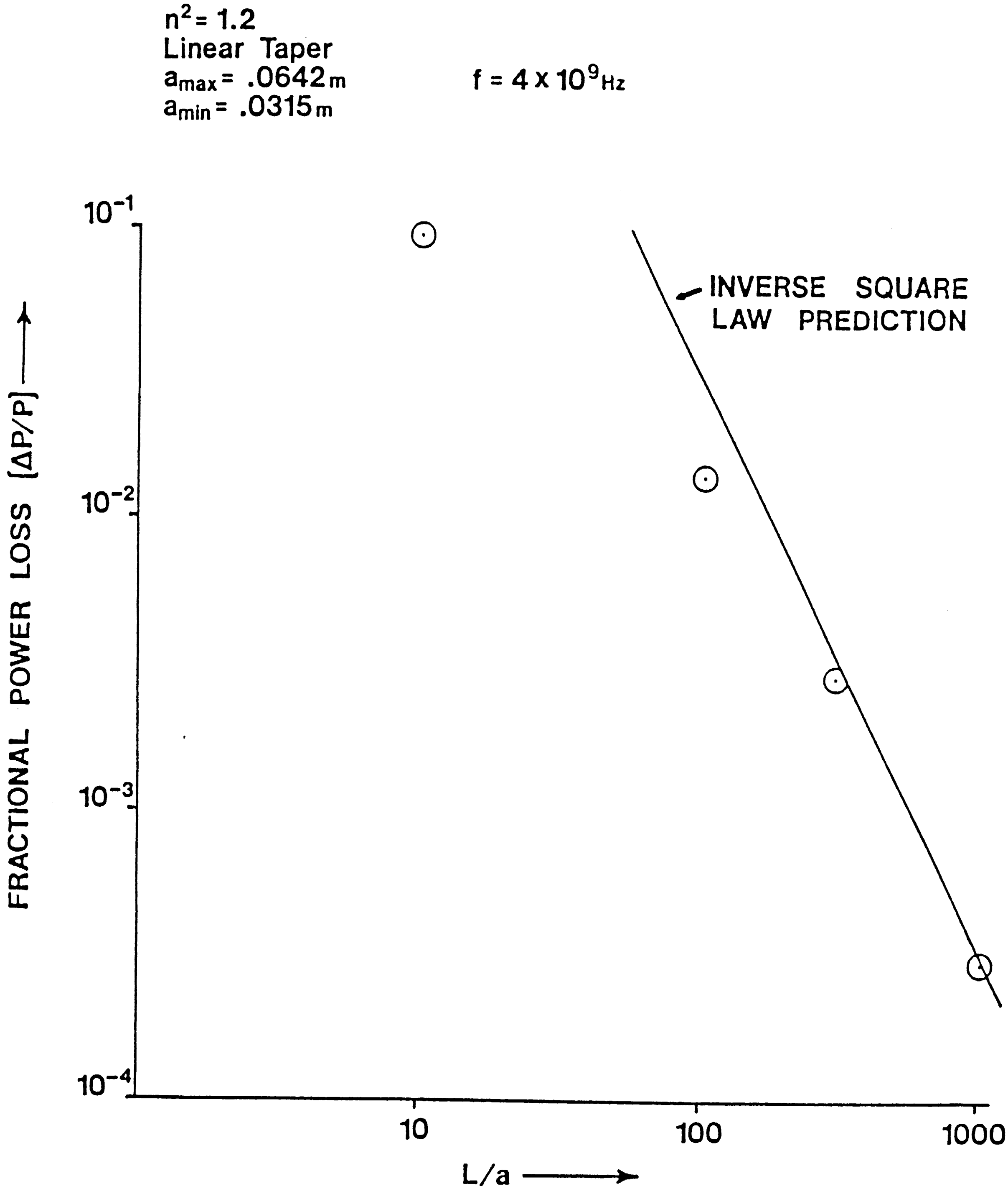
The modified scheme does not seem markedly inaccurate - the errors are all less than a quarter of a percent. To obtain exact error bounds a complete analysis would have to be undertaken and the results might well be too complicated to apply to our problem in any simple way This table seems sufficient to us to warrant our confidence in the results obtained by the use of this scheme.

It was hoped that the introduction of this scheme would remedy all the problems and bring the long antenna results and the inverse square law results together. This did not occur and a further search revealed the power spike just below $\beta = k$ alluded to earlier. When both programs were corrected to use much smaller steps near $\beta = k$, the desired agreement was forthcoming, as the following figures show.

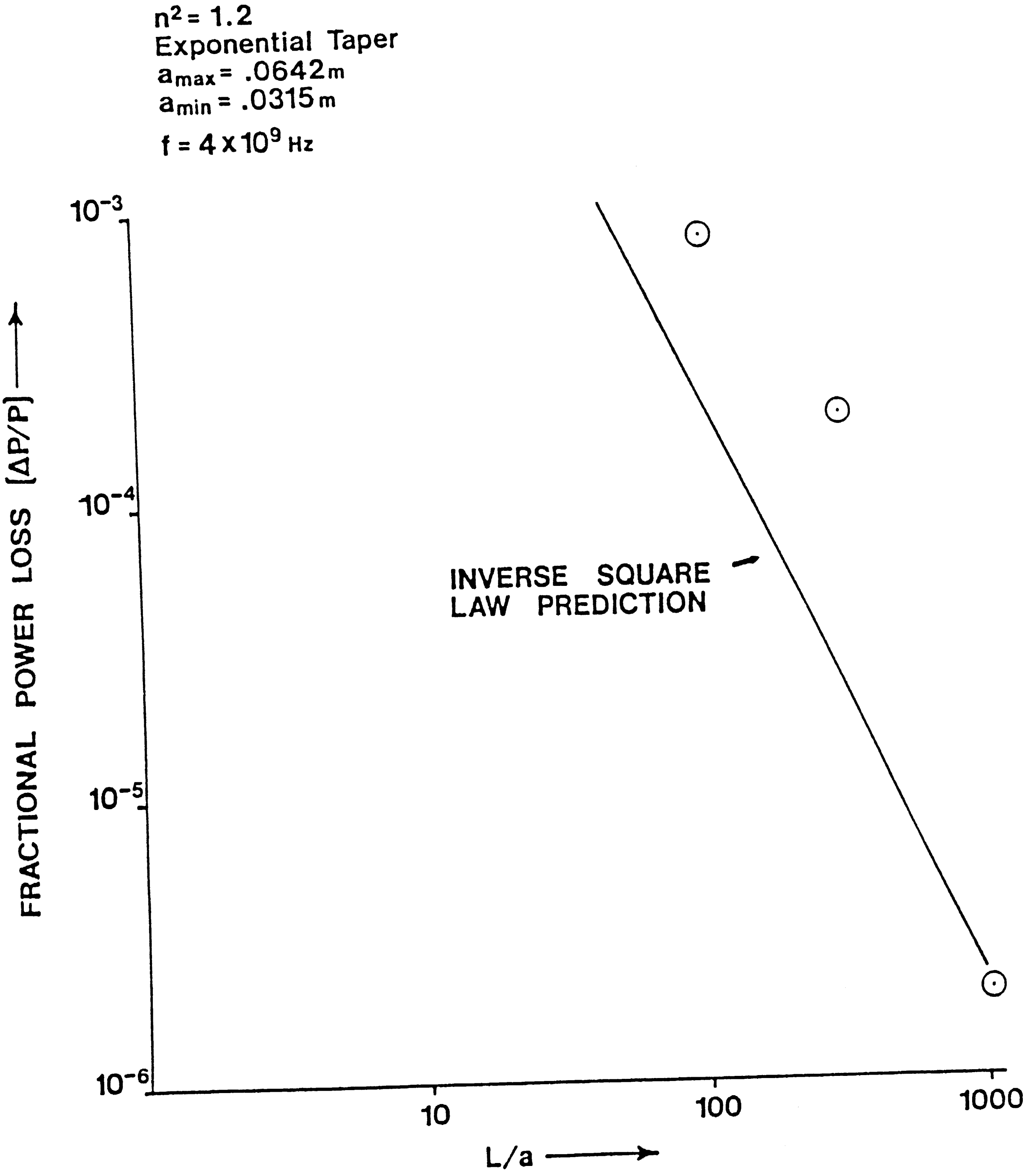

EXPONENTIALLY TAPERED RODS. DATA OF D. MARCUSE
$n^2 = 2.05$
$a_{max} / a_{min} = 2.0$
$k\,a_{max} = 2.5$
POINTS CALCULATED FROM MARCUSE'S FORMULA
INVERSE SQUARE LAW PREDICTION
FRACTIONAL POWER LOSS [ΔP/P]
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
20
100
500
L/a



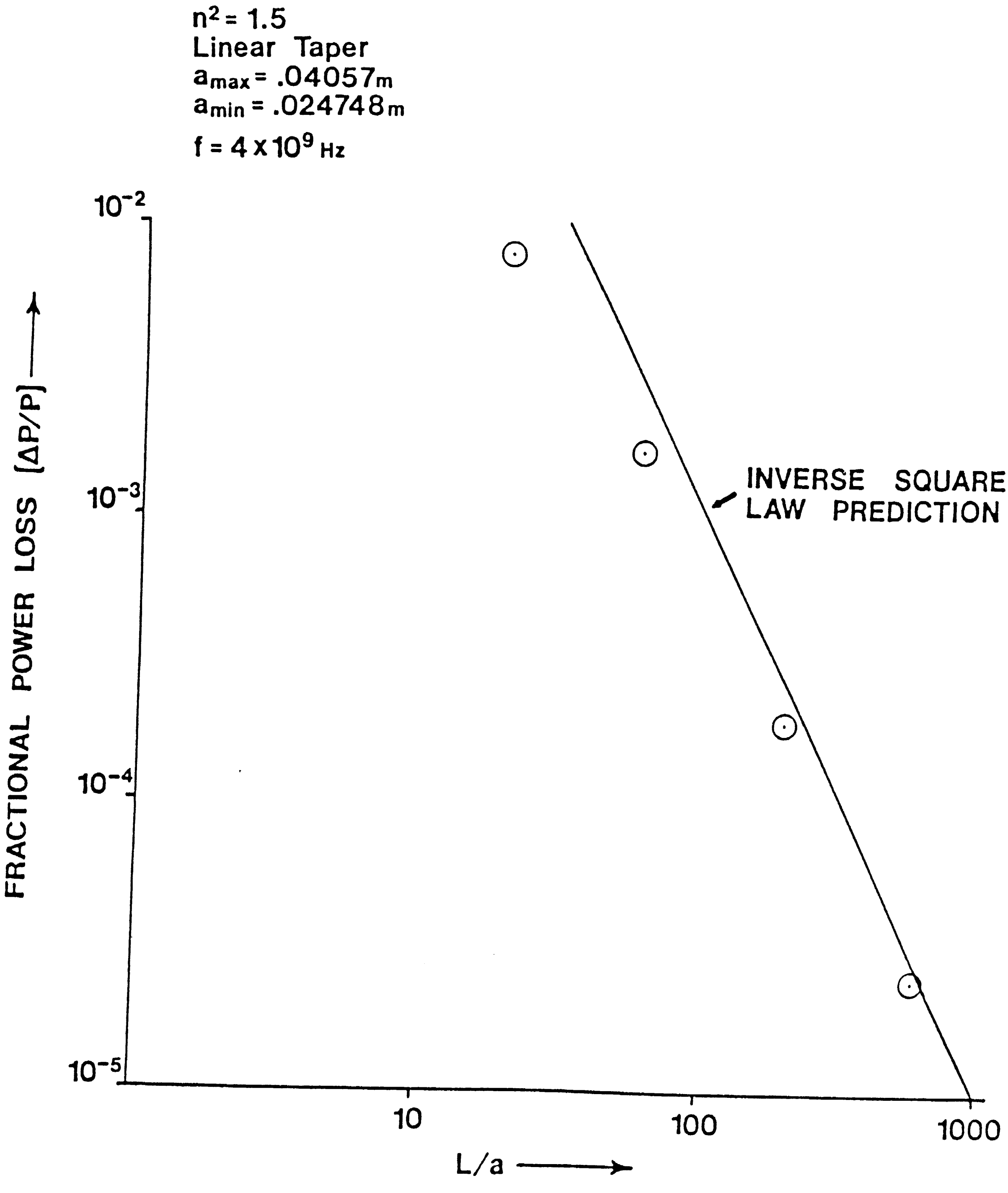
MARCUSE'S DIELECTRIC ROD
CALCULATED POWER LOSSES
AND
INVERSE SQUARE LAW PREDICTIONS LINEAR TAPER
$n^2 = 2.05$
$a_{max} / a_{min} = 2.0$
$k\, a_{max} = 2.5$
INVERSE SQUARE LAW PREDICTION
FRACTIONAL POWER LOSS [ΔP/P]
10^{-1}
10^{-2}
10^{-3}
100
500
L/a
⊙ Calculated directly from Marcuse's Equations



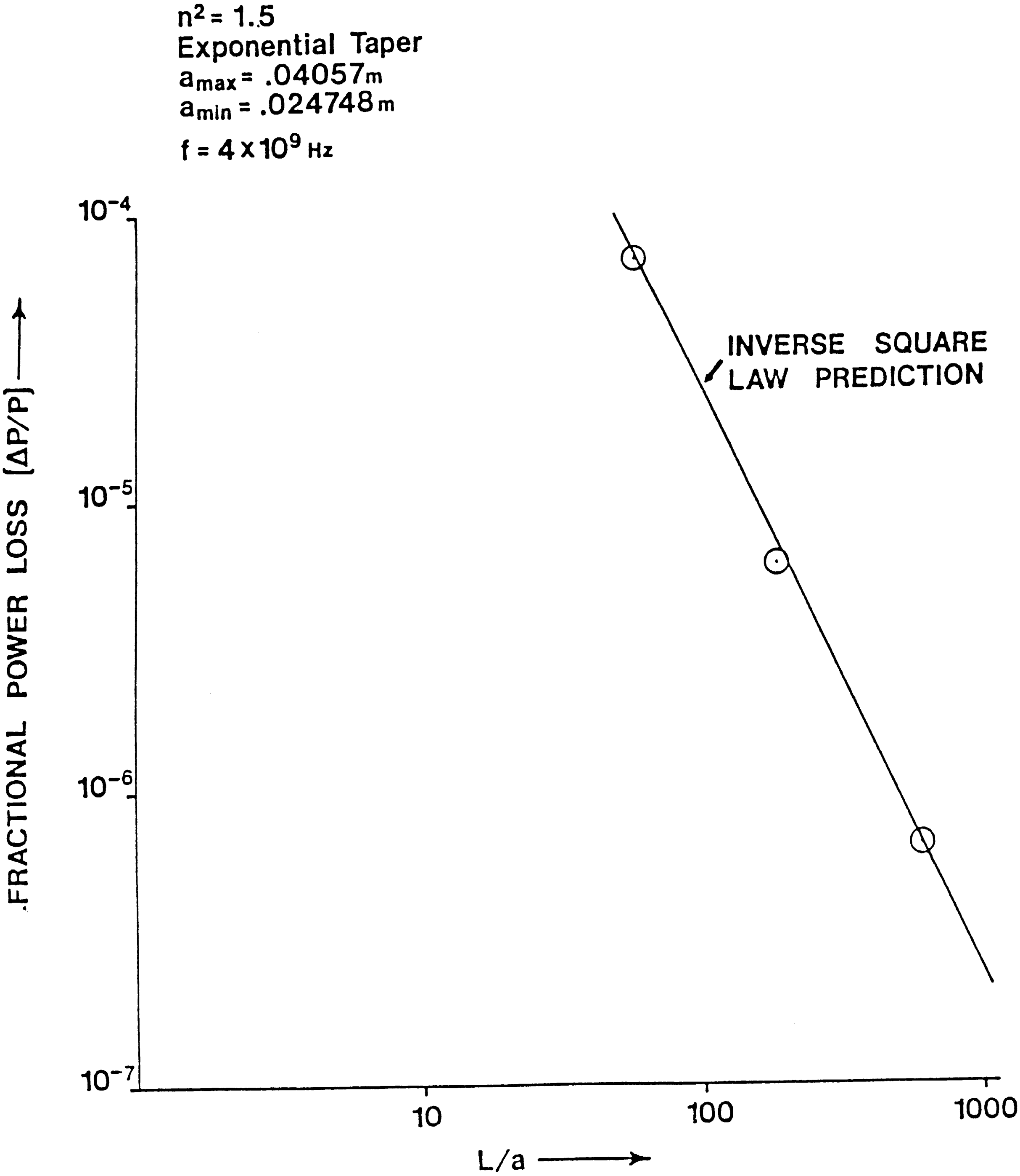
⊙ Calculated directly from Marcuse's Equations



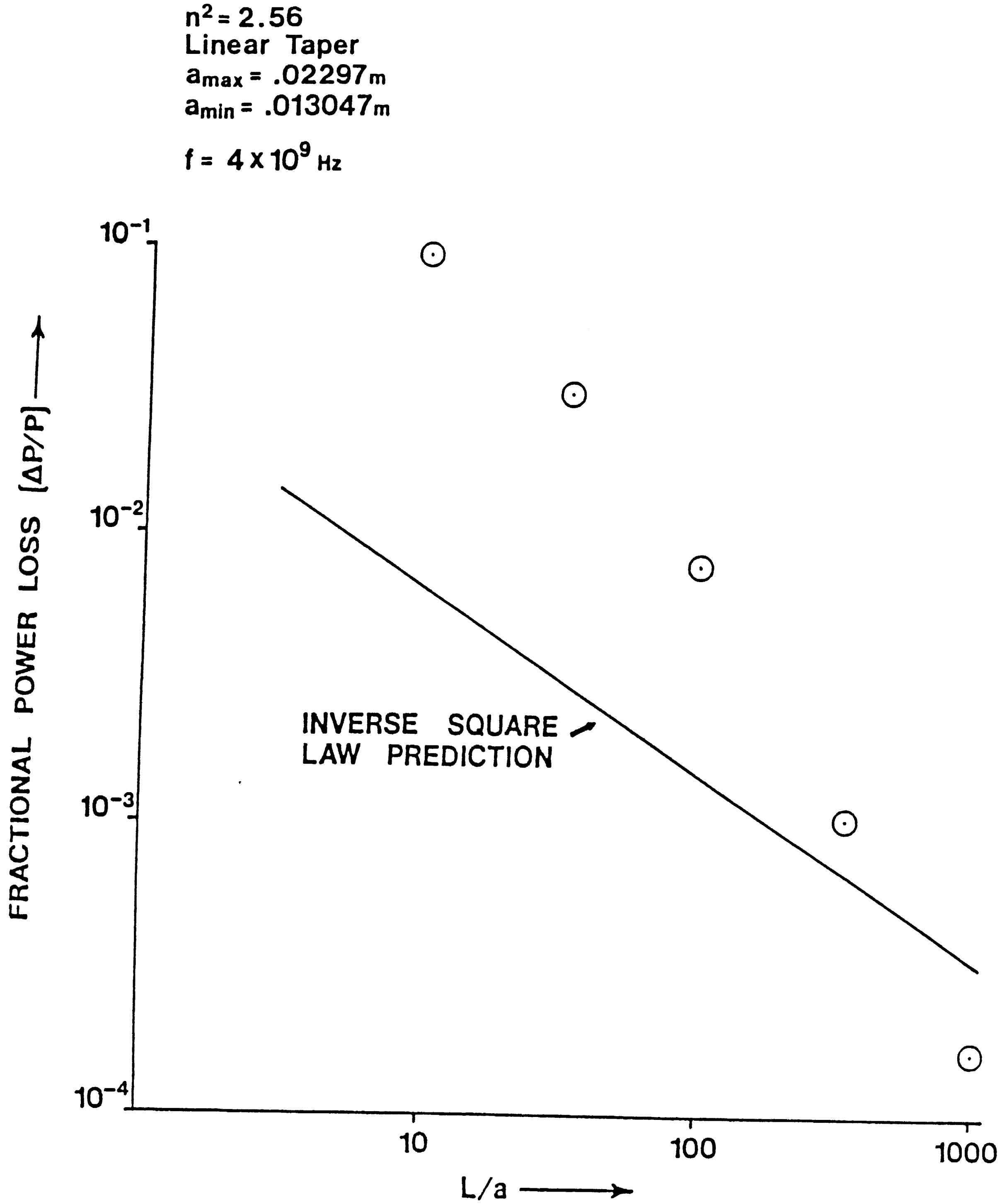
⊙ Calculated directly from Marcuse's Equations



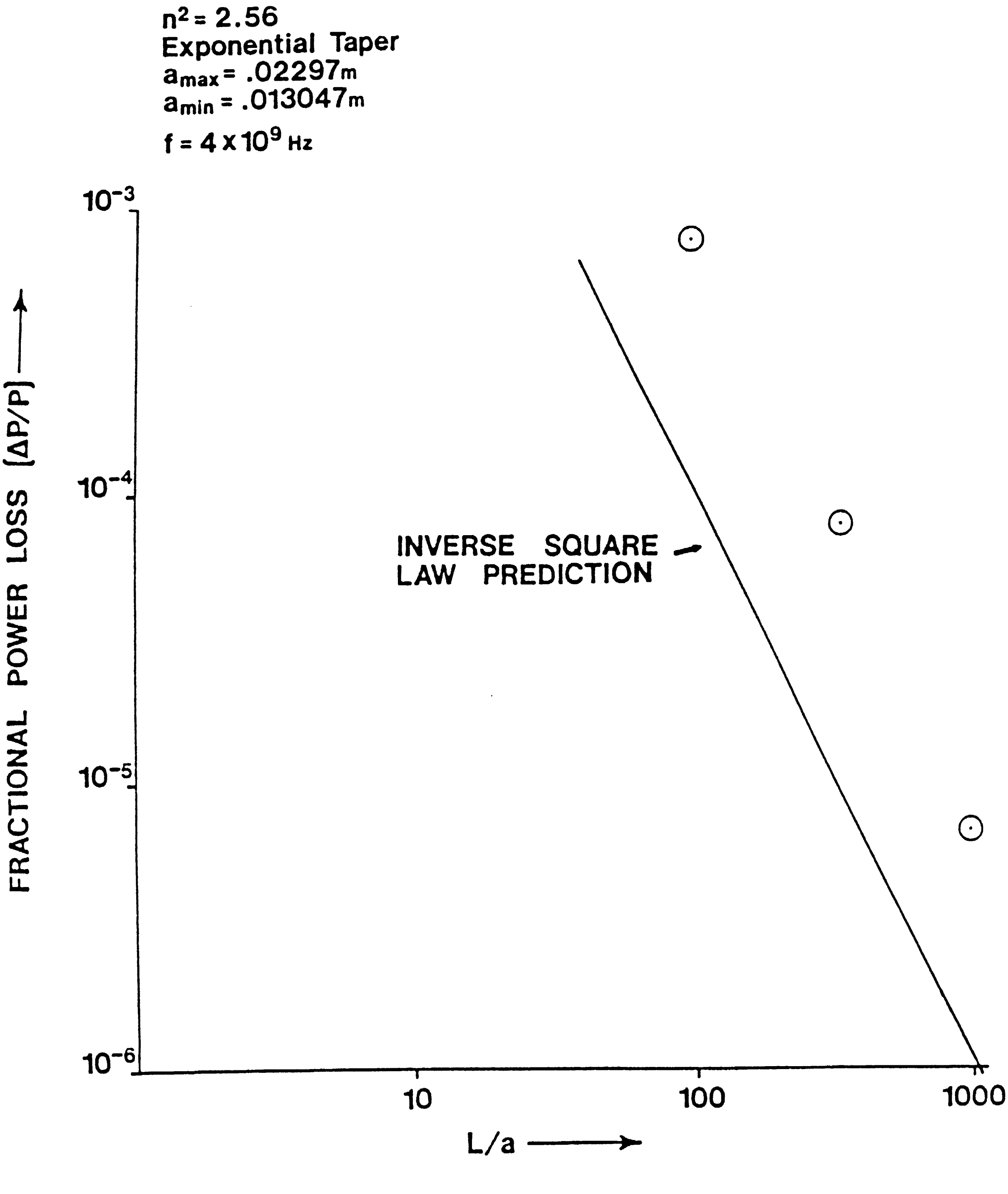
⊙ Calculated directly from Marcuse's Equations



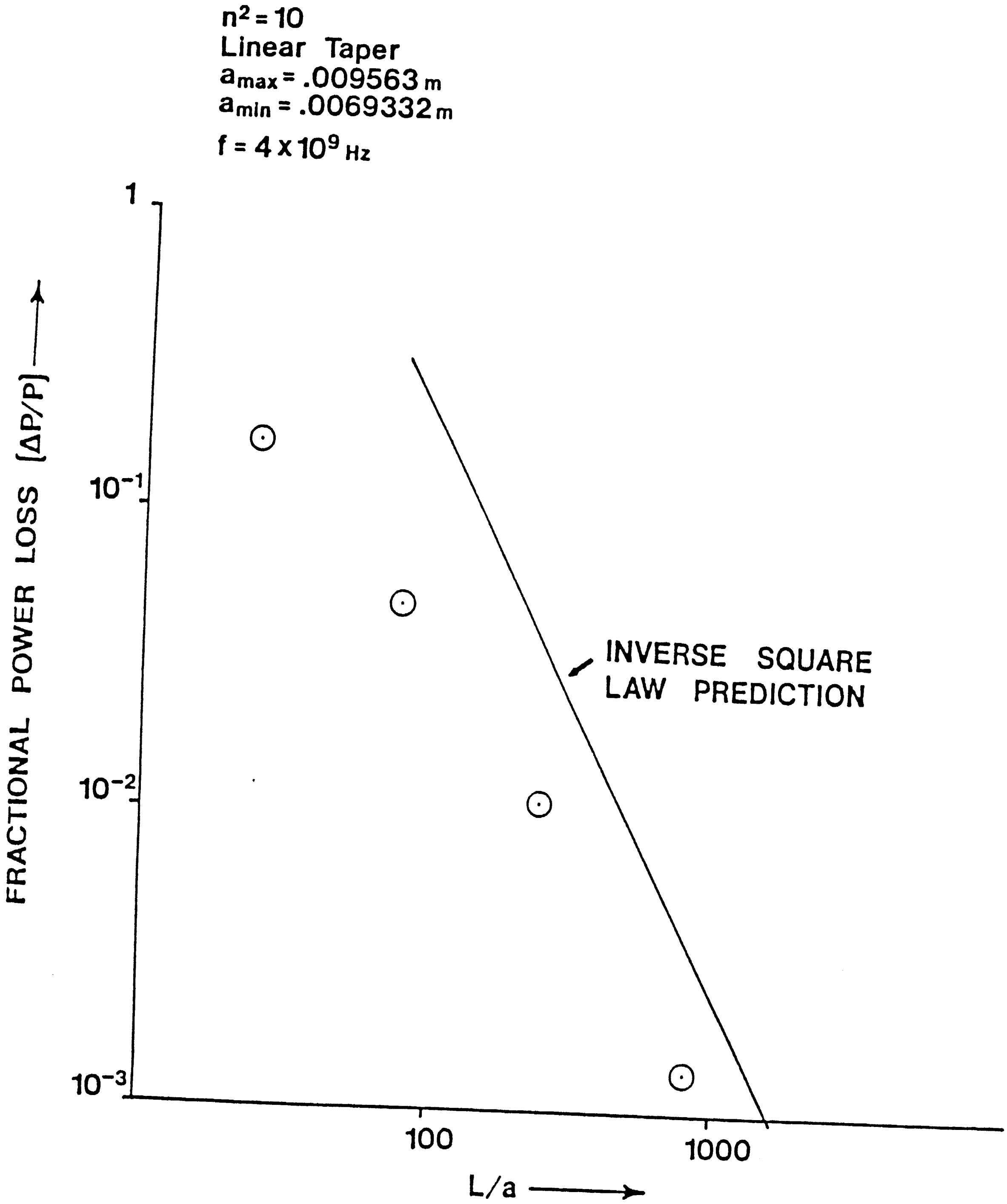
⊙ Calculated directly from Marcuse's Equations



⊙ Calculated directly from Marcuse's Equations

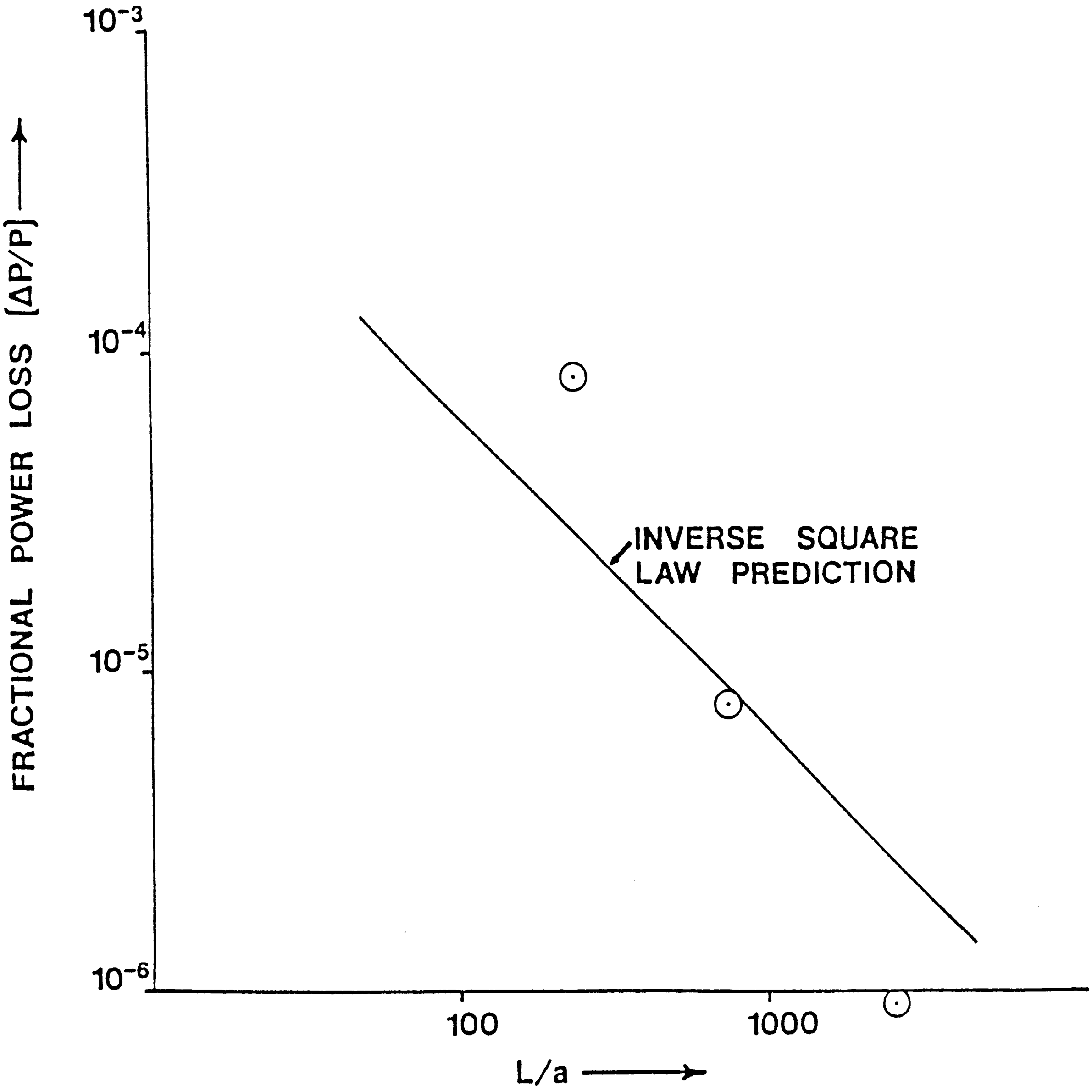


⊙ Calculated directly from Marcuse's Equations



⊙ Calculated directly from Marcuse's Equations

$n^2 = 10$
Exponential Taper
$a_{max} = .009563\,m$
$a_{min} = .006932\,m$
$f = 4 \times 10^9$ Hz



⊙ Calculated directly from Marcuse's Equations

Bibliography

[1] Marcuse, Dietrich: Radiation Losses of the Dominant Mode in a Round Dielectric Waveguide. Bell System Technical Journal 49 (8) October 1970

[2] Coughlin, James P.: Baran, Robert H. and Krall, Albert D. : The Mathematics of Marcuse's Power Loss Equation. IEEE Antennas and Propagation Society / URSI Radio Science Meeting. Syracuse, New York 6-10 June 1988

[3] Coughlin, James P. Calculation of Losses from a Dielectric Rod NSWC - TR 81-299 July 1981

[4] van de Vooren, A.I. and van Linde, H.J.: Numerical Calculation of Integrals with strongly Oscillating Integrand. Mathematics of Computation 20 (1966)

[5] Abramowitz, Milton and Stegun, Irene : Handbook of Mathematical Functions. U.S. Department of Commerce. 1964

APPENDIX I

$$\left\{\frac{\gamma^2 a^2\, J_\nu'(\kappa a)}{\kappa a\;\; J_\nu(\kappa a)} + i\gamma a\,\frac{H_\nu^{(1)'}(i\gamma a)}{H_\nu^{(1)}(i\gamma a)}\right\} *$$

$$\left\{\frac{n^2\gamma^2 a^2\, J_\nu'(\kappa a)}{\kappa a\, J_\nu(\kappa a)} + i\gamma a\frac{H_\nu^{(1)'}(i\gamma a)}{H_\nu^{(1)}(i\gamma a)}\right\} =$$

$$\frac{\nu^2(n^2-1)^2k^2\beta_0^{\,2}}{\kappa^4}$$

This is the equation that determines whether the ν^{th} mode can propagate down the rod. A careful study of this equation will reveal that it is not possible to satisfy it for arbitrary ν and a. Indeed unless a has a certain minimum value, there will, in general, be no solution. The only exception to this rule is the case $\nu=1$, where the above equation has a solution for all values of a. Accordingly, $\nu=1$ is the lowest order mode that can propogate down the rod (lower than ν =0 !). If we specialize to this case, the resulting equation becomes:

$$\left[\frac{n^2a^2\gamma^2}{\kappa a}\left\{\frac{J_0(\kappa a)}{J_1(\kappa a)} - \frac{1}{\kappa a}\right\} + \gamma a\,\frac{iH_0^{(1)}(i\gamma a)}{H_1^{(1)}(i\gamma a)} - 1\right] *$$

$$\left[\frac{a^2\gamma^2}{\kappa a}\left\{\frac{J_0(\kappa a)}{J_1(\kappa a)} - \frac{1}{\kappa a}\right\} + \gamma a\,\frac{iH_0^{(1)}(i\gamma a)}{H_1^{(1)}(i\gamma a)} - 1\right]$$

$$= \quad (n^2-1)^2\,\beta_0^{\,2}k^2\,/\,\kappa^4$$

where the derivatives of the Bessel functions have been eliminated by using well known identities (cf. Abramowitz and Stegun[5])

We can expand the Hankel functions as follows:

$$i\gamma a H_0^{(1)}(i\gamma a)\,/\,H_1^{(1)}(i\gamma a) \;\simeq\; -\,\gamma^2a^2\,(\gamma_1 + \ln(\frac{\gamma a}{2}) + O(\gamma^2a^2)$$

where γ_1 is Euler's constant. This expansion is not, strictly speaking, a valid MacLaurin series for the Hankel functions, since the natural logarithm term is extremely large near $\gamma a = 0$. But we will make the substitution anyway, leading to:

$$\left\{1-\gamma^2a^2\left(\ln\left(\frac{\gamma a}{2}\right) + \gamma_1\right) + \frac{n^2\gamma^2a^2}{\kappa_0 a} t_1 \qquad +O(\gamma^2) \right\}$$

$$* \left\{1-\gamma^2a^2\left(\ln\left(\frac{\gamma a}{2}\right) +\gamma_1\right) + \frac{\gamma^2a^2}{\kappa_0 a} t_1 + O(\gamma^2) \right\} = \frac{(n^2-1)^2k^2(k^2+\gamma^2)}{[(n^2-1)k^2-\gamma^2]^2}$$

where

$$t_1 = \left\{ \frac{J_0(\kappa_0 a)}{J_1(\kappa_0 a)} - \frac{1}{\kappa_0 a} \right\}$$

Expand the left hand side in a power series in γa and subtract the right hand side to obtain:

$$\frac{1+2/(n^2-1)}{k^{*2}} + 2\gamma_1 +2 \ln\frac{\gamma a}{2} \qquad \frac{(n^2+1)}{\kappa_0 a} t_1 = O(\gamma^2)$$

(where $k^* = ka$) and in view of the behavior of the logarithm near $\gamma=0$, this equation cannot be satisfied unless

$$\ln\frac{\gamma a}{2} = - \left[\gamma_1 + \frac{n^2+1}{2\kappa_0 a} t_1 + 2\frac{n^2+1}{n^2-1} k^{*2} \right]$$

And,if this equation is solved for γa, we get:

$$\gamma a \simeq 2\exp(-\gamma_1) \exp\left(- \frac{n^2+1}{2\kappa_0 a} \quad \frac{J_0(\kappa_0 a)}{J_1(\kappa_0 a)} \right)$$

This is a very useful approximate formula for computing the solution to the eigenvalue problem. As the table below shows, the results are quite good for small values of ka and even for large values they form a good starting point for an iteration scheme.

TABLE II

ka	γa (formula)	γa (computer calculation)
.5	1.5×10^{-5}	1.48×10^{-5}
.625	9.74×10^{-4}	9.76×10^{-5}
.75	9.45×10^{-3}	9.50×10^{-3}
.875	3.77×10^{-2}	3.77×10^{-2}
1.0	1.08×10^{-1}	9.32×10^{-2}
1.5	4.72×10^{-1}	5.48×10^{-1}

APPENDIX II

We now return to the expression for $I(\rho,a)$ given by formula 15. Corresponding to each choice of sign for the square root in equation 9, there will be a function, $I(\rho,a)$. By $I_+(\rho,a)$ we mean the function obtained from the positive choice for the sign and by $I_-(\rho,a)$, we mean the function corresponding to the negative choice. Now calculate:

$$p(\rho) = \int_0^L I_+(\rho,a)\, a'(z)\, \exp[-i \int_0^z (\beta_0-\beta)\, ds]\, dz$$

$$q(\rho) = \int_0^L I_-(\rho,a)\, a'(z)\, \exp[-i \int_0^z (\beta_0-\beta)\, ds]\, dz$$

And then the fractional power loss will be given by:

$$\Delta P/P = \int_{-k}^{k} \{ |q|^2 + |p|^2 \}\, |\beta| / \rho \; d\beta$$

Now since $\rho = \sqrt{(k^2-\beta^2)}$, it is clear that this last named integrand has a singularity at each end point of the interval of integration. This singularity is rendered doubly difficult because each limit as β approaches k or -k must be handled separately for each choice of sign for F/G.

$$\lim_{\beta \Rightarrow k} I_+(\rho,a) / \sqrt{\rho}$$

$$\lim_{\beta \Rightarrow -k} I_+(\rho,a) / \sqrt{\rho}$$

$$\lim_{\beta \Rightarrow k} I_-(\rho,a) / \sqrt{\rho}$$

$$\lim_{\beta \Rightarrow -k} I_-(\rho,a) / \sqrt{\rho}$$

vanish. To see this we shall outline the derivation for the first of these four limits, since that is the most difficult of the four. First we show:

$$\lim_{\beta \Rightarrow k} F/G = \sqrt{\epsilon_0/\mu}$$

and, then:

$$\lim_{\beta \Rightarrow k} [\, F/G - \sqrt{\epsilon_0/\mu}\,] \,/\, \rho^2 \;=\; L_1 = 0$$

With the aid of these, we can show

$$\lim_{\beta \Rightarrow k} [\, \ln(\rho) \star F/\rho^{-1/2} \,] = F_1 = 0$$

And now we are prepared to look at the integrand. To save writing, we will use $H^\star$ in place of $iH_0{}^1(i\gamma a)/H_1{}^1(i\gamma a)$. Then

$$\lim_{\beta \Rightarrow k} I(\rho,a)/\sqrt{\rho} \;=\; \lim_{\beta \Rightarrow k} \pi J_1(\kappa a) \,/\, 4\, P\, \gamma^2\, \rho^{5/2} \star$$

$$\Big\{ (\beta_0+\beta)\gamma\rho\, (\omega\epsilon_0 A \frac{\partial H}{\partial a} + \omega\mu B \frac{\partial K}{\partial a}) \star \Big[a\, \frac{\gamma J_0(\rho a) + H^\star \rho J_1(\rho a)}{\gamma^2 + \rho^2} - \frac{J_1(\rho a)}{\gamma\rho} \Big]$$

$$+\, (\beta_0+\beta)\gamma\rho (\omega\epsilon_0 A \frac{\partial I}{\partial a} + \omega\mu B \frac{\partial M}{\partial a}) \star \Big[a\, \frac{\gamma Y_0(\rho a) + H^\star \rho Y_1(\rho a)}{\gamma^2 + \rho^2} - \frac{Y_1(\rho a)}{\gamma\rho} \Big]$$

$$+ (k^2+\beta_0\beta) \star \Big[(A \frac{\partial K}{\partial a} + B \frac{\partial H}{\partial a}) \star J_1(\rho a) \;+\; (A \frac{\partial M}{\partial a} + B \frac{\partial I}{\partial a}) Y_1(\rho a) \Big]$$

This limit gives rise to the following four limits:

$$L_1 = \lim \rho^{-5/2} \{ (\beta_0+\beta)\gamma\rho\omega\epsilon_0 \frac{\partial H}{\partial a} \Big[a \frac{\gamma J_0(\rho a) + \rho J_1(\rho a) H^\star}{\gamma^2 + \rho^2} - \frac{J_1(\rho a)}{\gamma\rho} \Big]$$
$$+ (k^2+\beta_0\beta) \frac{\partial K}{\partial a}\; J_1(\rho a) \quad \}$$

$$L_2 = \lim_{\beta_0 \Rightarrow k} \rho^{-5/2} \{ (\beta_0+\beta)\gamma\rho\omega\epsilon_0 \frac{\partial I}{\partial a} \Big[a \frac{\gamma Y_0(\rho a) + \rho H^\star Y_1(\rho a)}{\gamma^2 + \rho^2} - \frac{Y_1(\rho a)}{\gamma\rho} \Big]$$

$$+ (k^2+\beta_0\beta) \frac{\partial M}{\partial a}\; Y_1(\rho a) \;\}$$

$$L_3 = \lim_{\beta \Rightarrow k} \rho^{-5/2} \left\{ (\beta_0+\beta)\gamma\rho\omega\mu \frac{\partial K}{\partial a} \left[a\frac{\gamma J_0(\rho a) + \rho H^* J_1(\rho a)}{\gamma^2 + \rho^2} - \frac{J_1(\rho a)}{\gamma\rho} \right] \right.$$

$$\left. +(k^2+\beta\beta_0) J_1(\rho a) \frac{\partial H}{\partial a} \right\}$$

$$L_4 = \lim_{\beta \Rightarrow k} \rho^{-5/2} \left\{ (\beta_0+\beta)\gamma\rho\omega\mu \frac{\partial M}{\partial a} \left[a\frac{\gamma Y_0(\rho a) + \rho H^* Y_1(\rho a)}{\gamma^2+\rho^2} - \frac{Y_1(\rho a)}{\rho\gamma} \right] \right.$$

$$\left. +(k^2+\beta\beta_0)\frac{\partial I}{\partial a} Y_1(\rho a) \right\}$$

Then the limit of $I(\rho,a)/\sqrt{\rho}$ will be

$$I_0 = \pi J_1(\kappa a)/\ (4\gamma^2 P) \left\{ A(L_1+L_2) + B(L_3+L_4) \right\}$$

$$L_1 = L_1' + L_1''$$

$$L_1' = \lim_{\beta_0 \Rightarrow k} \rho^{-5/2} \left[(\beta_0+\beta)\gamma\rho\omega\epsilon_0 \frac{\partial H}{\partial a} \right]^*$$

$$\left\{ a \frac{\gamma J_0(\rho a) + \rho H^* J_1(\rho a)}{\gamma^2+\rho^2} - \frac{J_1(\rho a)}{\rho\gamma} - \frac{a}{2\gamma} \right\}$$

$$=0$$

$$L_1 = L_1'' = \lim_{\beta \Rightarrow k} \rho^{-5/2} \left\{ (\beta_0+\beta)\omega\epsilon_0 a\rho/2\gamma \frac{\partial H}{\partial a} + (k^2+\beta_0\beta)\ J_1(\rho a) \frac{\partial K}{\partial a} \right\}$$

$$= \ .25\ \pi a F_1\ (L^{iii} + L^{iv} + L^{v})$$

$$L^{v}=0$$

$$L_1^{iii} = \lim[\ \ln(\rho)]^{-1} * \frac{Y_1(\rho a)}{\rho a} \{\ \sigma a J_0(\sigma a) - 2J_1(\sigma a)\ \}*$$

$$\Big[-(\beta_0+\beta)\omega\epsilon_0 + (\beta_0+\beta)(n^2-1)k^2\beta G/F/\sigma^2 +$$

$$(k^2+\beta_0\beta)\left\{2J_1(\rho a)/\rho a\right\}\left\{\beta/\omega\mu\right\}\left\{(n^2-1)k^2/\sigma^2\right\}$$

$$-(k^2+\beta_0\beta)\left\{2J_1(\rho a)/\rho a\right\}G/F$$

$$L_1^{iii} = -2/\pi a^2 * [\sigma_0 a J_0(\sigma_0 a) - 2J_1(\sigma_0 a)] * \{L_1^{vi} + L_1^{vii}\}$$

$$L_1^{vi} = \lim_{\beta \Rightarrow k} \{ \frac{(k^2+\beta_0\beta)\ [2J_1(\rho a)/\rho a]\ [(n^2-1)k^2/\sigma^2]\ [\beta/\omega\mu] - (\beta_0+\beta)\omega\epsilon_0 \}}{\rho^2 \ln(\rho)}$$

$$L_1^{vi} = 0$$

$$L_1^{vii} = \lim_{\beta \Rightarrow k} \{ \frac{(\beta_0+\beta)(n^2-1)k^2/\sigma^2\ \beta\ G/F - (k^2+\beta_0\beta)\ G/F\ [2J_1(\rho a)/\rho a]\}}{\rho^2 \ln(\rho)}$$

$$L_1^{vii} = 0$$

$$L_1 = .25\ \pi a\ F_1 * \lim_{\beta \Rightarrow k} \frac{Y_0(\rho a)}{\ln(\rho)} *$$

$$\{ \quad (\beta_0+\beta)\omega\epsilon_0 [\sigma a J_0(\sigma a) - J_1(\sigma a)]$$

$$+(\beta_0+\beta)\omega\epsilon_0 * (n^2-1) * k^2/\sigma^2\ \ \beta/\ \omega\epsilon_0\ G/F\ J_1(\sigma a)$$

$$+(k^2+\beta_0\beta)\ [2J_1(\rho a)/\rho a]\ [(n^2-1)k^2/\sigma^2]\ [\beta/\omega\mu]\ J_1(\sigma a)$$

$$+(k^2+\beta_0\beta)\ [2J_1(\rho a)/\rho a]\ [(n^2-1)k^2/\sigma^2]\ G/F\ *$$

$$[\sigma a J_0(\sigma a) - J_1(\sigma a)] \quad \}$$

$$L_1 = F_1 a\ \sigma_0 a J_0(\sigma a)\ (\beta_0 + k) k\ \underline{V}\ \epsilon_0/\mu$$

$$L_2 = -L_1$$

$$L_3 = F_1\ k\ (\beta_0 + k) \sigma_0 a J_0(\sigma_0 a)\ a$$

$$L_4 = -L_3$$

1988 INSTITUTIONAL MEMBERS

ANTENNA TECHNOLOGIES
6 Shields Dr Box 618
Bennington, VT 05201

NOKIA-MOBIRA
P O Box 86 24101 Salo
Finland

LOCKHEED MISSILES & SPACE CO
Adv Systems Tech Laboratory
3251 Hanover St.
Palo Alto, CA 94304

W RICHARD GREEN & ASSOC.
3200 Wilkinson Rd
Cameron Park, CA 95682

HUNTING ENGINEERING LTD
Redding Wood, Ampthill
Bedford England MK45 2HD

ROSEMONT AEROSPACE
14300 Judicial Rd
Burnsville, MN 55337

TEXAS INSTRUMENTS
2501 W University
P O Box 801/MS 8019
McKinney, TX 75069

KERSHNER, WRIGHT & HAGAMAN
5730 General Washington Dr.
Alexandria, VA 22312

DELFIN SYSTEMS
1349 Moffett Park Dr
Sunnyvale, CA 94089

SCALA ELECTRONIC CORP
P O Box 4580
Medford, OR 97501

HELSINKI UNIV OF TECHNOLOGY
Otakaari 5A Espoo 15
Finland

AWA DEFENCE & AEROSPACE
P O Box 96
North Ryde NSW,
Australia 2113

NEW MEXICO STATE UNIVERSITY
Box 3548/Physics/Science Lab
Las Cruces, NM 88003

TRW
One Space Park
Redondo Beach, CA 90278

CULHAM LABORATORY
UK Atomic Energy Authority
Abingdon Oxfordshire
England OX14 3DB

U.S. AIR FORCE
Base Library Bldg 437
485 EIG/EIEUS
Griffiss, AFB, NY 13441

KATHREIN-WERKE KG
Postfach 260
D-8200 Rosenheim 2
West Germany

OAR CORPORATION
10447 Roselle St.
San Diego, CA 92121

SCIENCE APPLICATIONS INTERNATIONAL CORP
5151 East Broadway, Suite 900
Tucson, AZ 85711

U.S. COAST GUARD
2100 2nd St. SW
Washington, DC 20593-0001

STG NATIONAL LIGHT-EN
Ruimtevvart Laboratorium
1059 CM Amsterdam
Netherlands

TELECOM RESEARCH LABS
770 Blackburn Rd
CLayton, Victoria, Australia 3168

HONEYWELL INC
401 DEFENSE HIGHWAY
Annapolis, MD 21401

IIT RESEARCH INSTITUTE
185 Admiral Cochrane Dr
Annapolis, MD 21401

FORD AEROSPACE
3939 Fabian Way MS T96
Palo Alto, CA 94303

ITT AEROSPACE/OPT DIVISION
P O Box 3700
Ft Wayne, IN 46801

NYNEX CORPORATION
500 Westchester Ave
White Plains, NY 10604

BUSINESS MANAGEMENT SYSTEMS
Sproughton House
Sproughton, Ipswich
Suffolk, England Ip8 3AW

FFV ELEKTRONIK AB
S-351 80 VAXJO
Ljungadalsgatan 2
Sweden 0470-420-0

VISTA RESEARCH
100 View St Box 998
Mountain View, CA 94042

TASC
55 Walkers Brook Dr.
Reading, MA 01867

PILKINGTON
Pilkington Tech Center, L
Lathom, Ormskirk
Lancashire, England L40-SUF

TCI
1625 Stierlin Rd
Mountain View, CA 94043

TELEX COMMUNICATIONS INC.
8601 Northeast Hwy 6
Lincoln, NE 68505

ROCKWELL INTERNATIONAL
1745 Jeff Davis Hwy, Suite 1
Arlington, VA 22202

KATHREIN INCORPORATED
26100 Brush Ave, Suite 319
Euclid, OH 44132

I.D.S. INGEGNERIA SISTEMI
Via Roma 50
Pisa, Italy 56100

CELWAVE RF
Frejasvey 30
Hillerod, Denmark 3400

MIT, LINCOLN LAB
P.O. Box 73
Lexington, MA 02173-0073

FGAN/FHP
Neuenahrer Str. 20
Wachtberg-Werthoven
W Germany, D5307

GEORGIA INSTITUTE OF TECHNOLOGY
GTRI/ECSL/ERB Bldg, Rm 242
Atlanta, GA 30332

RADIO FREQUENCY SYSTEM
AUSTRALIA PTY LTD.
Box 191 Croydon
Victoria, Croydon, Australia 3136