Complete Radiation Boundary Conditions for Maxwell's Equations

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***Abstract*—We describe the construction, analysis, and implementation of arbitrary-order local radiation boundary condition sequences for Maxwell's equations. In particular we use the complete radiation boundary conditions which implicitly apply uniformly accurate exponentially convergent rational approximants to the exact radiation boundary conditions. Numerical experiments for waveguide and free space problems using high-order discontinuous Galerkin spatial discretizations are presented.**

***Index Terms*—radiation boundary conditions, time-domain methods.**

I. Introduction

The radiation of energy to the far field is a central feature of electromagnetism. As such, efficient, convergent domain truncation algorithms are a necessary component of any software for simulating electromagnetic waves in the time domain. Complete radiation boundary conditions (CRBC), introduced for acoustics in [1], are, in our view, an ideal solution to this problem. In particular they have a number of advantages relative to the popular perfectly matched layers (PML) [2]. Most importantly:

i. They are provably spectrally convergent and the required parameters can be chosen automatically to guarantee any required accuracy;
ii. The computational boundary can be placed arbitrarily close to scatterers or other inhomogeneities.

In this paper we will outline the theory behind the method, discuss the auxiliary equations which must be solved, and show some results from simple numerical experiments using high-order discontinuous Galerkin (DG) discretizations [3].

II. Exact Radiation Conditions and Local Approximations

Consider Maxwell's equations in a uniform dielectric half-space, $x_1 > 0$:

$$\epsilon \frac{\partial E}{\partial t} = \nabla \times H, \quad \mu \frac{\partial H}{\partial t} = -\nabla \times E.$$

We imagine the computational domain to be located in $x_1 < 0$ with $x_1 = 0$ being the radiation boundary. The model used in $x_1 < 0$ may contain complex scatterers, dispersive media, and other complexities. It is possible to generalize our construction to the case of stratified media (see [4] for the acoustic case) and dispersive models [5] extending into the far field, but here we will restrict ourselves to the simplest case.

An exact radiation condition with $c = 1/\sqrt{\epsilon\mu}$ and $\alpha = \sqrt{\frac{\mu}{\epsilon}}$ is given by [6]:

$$\frac{2}{c}\frac{\partial}{\partial t}\left(E_2 - \alpha H_3\right) + \mathcal{R}\left(E_2 - \alpha H_3\right) = \alpha \frac{\partial H_1}{\partial x_3} - \frac{\partial E_1}{\partial x_2}, \quad (1)$$

$$\frac{2}{c}\frac{\partial}{\partial t}\left(E_3 + \alpha H_2\right) + \mathcal{R}\left(E_3 + \alpha H_2\right) = -\alpha \frac{\partial H_1}{\partial x_2} - \frac{\partial E_1}{\partial x_3}. \quad (2)$$

Here $\mathcal{R}$ is a nonlocal operator defined in terms of the spatial Fourier transform $\mathcal{F}$ on the hyperplane $x_1 = 0$ and a convolution in time with a Bessel kernel:

$$\mathcal{R}w = \mathcal{F}^{-1} \;\; c|k|^2 K(c|k|t) * (\mathcal{F}w)\big), \quad K(z) = \frac{J_1(z)}{z}.$$

(A similar formula holds on a spherical boundary [6].)

It is possible to construct efficient, low-memory algorithms to evaluate these nonlocal operators [7], [8], which could be a useful alternative for waveguide geometries or scatterers which can be snugly fit by a spherical radiation boundary. With CRBC we approximate the nonlocal operator $\mathcal{R}$ using a sequence of auxiliary fields which satisfy hyperbolic equations on the radiation boundary. Advantages of the local approach are relative ease of implementation (no spatial transforms are required) and the possibility to use a rectangular cuboid or a more general polyhedron as the radiation boundary.

Fundamentally the local methods implement rational approximations in frequency space to the Laplace transform of the temporal convolution kernel K,

$$\hat{K}(s) = \frac{1}{s + \left(s^2 + c^2|k|^2\right)^{1/2}}.$$

We demand an accuracy τ uniformly on an inversion contour $\Re s = T^{-1}$ where T is the simulation time. Assuming a separation $\delta > 0$ from sources and scatterers we *guarantee* this accuracy with a CRBC using P auxiliary fields and [1]:

$$P \propto \ln\left(\frac{cT}{\delta}\right) \cdot \ln\left(\frac{1}{\tau}\right).$$

Optimal approximants are easily computed. They are defined via certain parameters a_j and a code for their computation given the error tolerance τ and the dimensionless parameter

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$\eta = cT/\delta$ can be found at **www.rbcpack.org**. The approximations are extraordinarily efficient. For example, if we take $\tau = 10^{-4}$ and $\eta = 10^3$ then $P = 9$ suffices.

III. CRBC System

The CRBC system on a face with normal e_1 is defined via a collection of parameters a_j mentioned above. It is most easily understood using the normal characteristic variables, which we also note are what appear in (1)-(2). They are:

$$w^{\pm} = \begin{pmatrix} E_2 \pm \alpha H_3 \\ E_3 \mp \alpha H_2 \end{pmatrix}, \quad w^{\tan} = \begin{pmatrix} E_1 \\ \alpha H_1 \end{pmatrix}.$$

Written in terms of these variables Maxwell's equations take the form:

$$\begin{aligned} \frac{\partial w^+}{\partial t} + c\frac{\partial w^+}{\partial x_1} + cS^+\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) w^{\tan} &= 0 \\ \frac{\partial w^-}{\partial t} - c\frac{\partial w^+}{\partial x_1} + cS^-\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) w^{\tan} &= 0 \\ \frac{\partial w^{\tan}}{\partial t} + cS^{\tan}\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) \left(w^+, w^-\right) &= 0, \end{aligned}$$

where $S^{+,-,0}$ are linear partial differential operators. We now introduce auxiliary fields $(w_j^+, w_j^-, w_j^{\tan})$, $j = 0, \ldots, P$ and solve for $j = 1, \ldots, P$:

$$\begin{aligned} (1 + a_{2j})\frac{\partial w_j^+}{\partial t} + \frac{1 - a_{2j}^2}{Ta_{2j}} w_j^+ + S^+ w_j^{\tan} &= \\ (1 - a_{2j-1})\frac{\partial w_{j-1}^+}{\partial t} - \frac{1 - a_{2j-1}^2}{Ta_{2j-1}} w_{j-1}^+ + S^+ w_{j-1}^{\tan}, & \\ (1 + a_{2j-1})\frac{\partial w_{j-1}^-}{\partial t} + \frac{1 - a_{2j-1}^2}{Ta_{2j-1}} w_{j-1}^- + S^- w_{j-1}^{\tan} &= \\ (1 - a_{2j})\frac{\partial w_j^-}{\partial t} - \frac{1 - a_{2j}^2}{Ta_{2j}} w_j^- + S^- w_j^{\tan}, & \\ \frac{\partial w_j^{\tan}}{\partial t} + S^{\tan}\left(w_j^+, w_j^-\right) &= 0. \end{aligned}$$

Additionally we impose data from the interior related to w_0^+ and a termination condition on w_P^- - the precise choices of these may be implementation-dependent. The equations solved on faces with different normals are analogously defined. For our DG schemes we use $w_0^{+,-,\tan}$ as the outside states to define fluxes at the outer boundary of the mesh; full details will appear elsewhere.

For waveguide problems the auxiliary variables simply inherit the boundary conditions satisfied by the corresponding physical fields. For exterior problems we impose relations at edges and corners to close the system. These involve multiply-indexed auxiliary variables associated with the adjoining faces: P^2 at an edge and P^3 at a corner.

We note that a completely different approach to implementing the local boundary conditions is based on defining the auxiliary functions in a small layer. Termed the double absorbing boundary (DAB) formulation [9], the method has advantages for second order formulations of Maxwell's equations and for finite difference discretizations. In particular we have used to implement CRBC in conjunction with the Yee scheme [10], which we have made available at **www.rbcpack.org**. We have also used it for high order difference methods [11].

IV. Numerical Experiments

Here we demonstrate the accuracy of the method with DG discretizations of the TM system in two space dimensions. Further results, including computations in three space dimensions, will be presented in the talk. We consider initial value problems in a waveguide of width 1 and in free space. For the waveguide problem the computational domain is $(-1, 1) \times (0, 1)$ with PEC boundary conditions imposed at $x_2 = 0, 1$ and the CRBCs imposed at $x_1 = \pm 1$. Exact solutions are given by appropriate derivatives of solutions of the scalar wave equation produced by a point source centered near $(0, 0.1)$ with time amplitude $\exp\left(-125(t + .475)^2\right)$. For the free space problem the computational domain is $(-1, 1) \times (-1, 1)$ with CRBCs imposed at all four boundaries. This requires the corner closures alluded to above at the four corners of the domain. The exact solution is now produced, using the same prescription as above, with a free space solution of the wave equation produced by the point source centered near the origin.

We take $\epsilon = 0.8$, $\mu = 1.25$ and solve up to $T = 100$. Since the radiation boundaries are a distance 1 from the source we set $\eta = 100$. We use an upwind DG discretization with polynomial degree 9 and square elements of width $1/12$, time stepping using eighth order order Taylor series with $\Delta t = 1/600$. We compare results with $P = 5$ and $P = 9$. For these choices the *a priori* error bounds are 5.6×10^{-4} and 3.7×10^{-6} respectively. The actual maximum relative errors in the computations were approximately the same for each case and were below tolerance: 3.7×10^{-4} and 2.6×10^{-6}.

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