3D Diagonalization and Supplementation of Maxwell's Equations in Fully Bi-anisotropic and Inhomogeneous Media Part I: Proof of Existence by Construction

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Abstract – Consider the Maxwell's curl equations in fully bi-anisotropic and inhomogeneous media in three dimensional (x, y, z) spatial Cartesian coordinates. Let the media be characterized by 3×3 permittivity, electro-magnetic coupling, magneto-electric coupling, and permeability matrices $\boldsymbol{\varepsilon}^{3\times3}(x, y, z)$, $\boldsymbol{\xi}^{3\times3}(x, y, z)$, $\boldsymbol{\zeta}^{3\times3}(x, y, z)$, and $\boldsymbol{\mu}^{3\times3}(x, y, z)$, respectively. Assume a harmonic time-dependence according to $\exp(-j\omega t)$. The prime objective in this paper is to establish that the Maxwell's electrodynamic equations jointly with the constitutive equations can be diagonalized, leading to the $\mathcal{D}-$, and the associated supplementary $\mathcal{S}-$forms. A finitary algorithm involving "structural," "differential," and "material" matrices has been proposed, and the existence of the $\mathcal{D}-$ and $\mathcal{S}-$forms proved by construction. In the accompanying paper (Part II) the internal consistency of the $\mathcal{D}-$ and $\mathcal{S}-$forms has been shown, by proving their sharp equivalence with Maxwell's and constitutive equations.

Index Terms – Bi-anisotropic and inhomogeneous media, diagonalization, Maxwell's equations, supplementaion.

I. INTRODUCTION

A. Background and motivation

Recognizing patterns in the structure of fundamental equations in mathematical physics, identifying commonalities between seemingly different areas in physics, and the dream of unifying schemes have ever since been the driving force for many original work in science and engineering. In pursuit of casting light on fundamentals, it is also hoped that more systematized and efficient algorithms for the numerical solution of increasingly more challenging problems will emerge. This paper focuses on the two (governing) Maxwell's curl equations in fully bi-anisotropic and inhomogeneous media characterized by four 3×3 constitutive matrices, in three dimensional (3D) spatial Cartesian coordinates. The method is based on "unpacking" the governing and constitutive equations, and identifying irreducible "structural," "differential," and "material" matrices, which not entirely coincidentally exhibit intriguing properties. The occurrence of identity matrices of varying dimensions simplifies and systematizes the derivations and provides clues to the deeper understanding of the underlying equations in mathematical physics in general. In particular, the factorization of the identity matrix into structurally complex forms can be viewed as an enabling instrument to peer into the fabric of the governing- and constitutive equations in a myriad of ways. On the other hand, the appearance of the 2×2 null matrix, as the "complement" to the identity matrix simplifies the calculations.

The diagonalized $(\mathcal{D})$ and supplementary $(\mathcal{S})-$forms offer benefits from epistemological, theoretical, and numerical points of view. In the course of investigating other fundamental equations in mathematical physics, these features have been discussed in greater length and will not be repeated here [1]-[6]. It merely should be pointed out that the $\mathcal{D}-$form in spectral domain corresponds to an algebraic eigenvalue problem. The calculation of a given problem's eigenpairs is quintessential to obtaining Green's functions, and, as demonstrated in earlier works, [1] and [2], to developing regularization techniques for dealing with singularities of any order, and thus to analyzing the near- and far-fields. The existence of supplementary equations is a new result, which allows factorizing the fields and minimizing the complexity and cost of computations by relegating ex-

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pensive computations to post-processing processes. It turns out that the $\mathcal{D}-$ and $\mathcal{S}-$forms are both required for proving their sharp equivalence with the originating governing and constitutive equations, and thus their internal consistency. The accompanying paper (Part II, [7]) focuses on the details using the ‘‘Occam's razor.’’

Diagonalization can be carried out in a variety of ways, each aimed at illuminating certain desired features of the governing and constitutive equations, and of the resulting $\mathcal{D}-$ and $\mathcal{S}-$forms. The description of a strictly formal and ‘‘automatic’’ procedure would require considerably more space. (The stringent approach was detailed in the author's tutorial presentations at previous ACES conferences and recorded in the ACES proceedings.) The current self-sufficient presentation is compact and permits the reader to focus on the essentials and appreciate interrelationships.

B. Organization of the paper

The remaining sections are organized as follows. Section II ‘‘unpacks’’ Maxwell's curl equations in component form in the $(x, y, z)-$Cartesian coordinates and prepares them for further analysis by introducing normalized variables. Section III introduces the ‘‘essential,’’ ‘‘nonessential,’’ and ‘‘auxiliary’’ field variables. In the process three ‘‘structural,’’ and two ‘‘differential’’ matrices are identified, which play key roles in obtaining the $\mathcal{D}-$ and $\mathcal{S}-$forms. Section IV focuses on the factorization of the constitutive equations, identifying four characteristic ‘‘material’’ matrices. Using the developed preparatory tools, Section V constructs the $\mathcal{D}-$ and $\mathcal{S}-$forms. Section VI concludes the paper.

II. GOVERNING AND CONSTITUTIVE EQUATIONS

Consider Maxwell's electrodynamic equations with the field variables having the conventional meaning,

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}, \tag{1a}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{1b}$$

Consider the constitutive equation in fully bianisotropic and inhomogenous media characterized by 3×3 material matrices $\boldsymbol{\varepsilon}^{3\times3}(x, y, z)$, $\boldsymbol{\xi}^{3\times3}(x, y, z)$, $\boldsymbol{\zeta}^{3\times3}(x, y, z)$, and $\boldsymbol{\mu}^{3\times3}(x, y, z)$,

$$\mathbf{D} = \boldsymbol{\varepsilon}^{3\times3}(x, y, z)\mathbf{E} + \boldsymbol{\xi}^{3\times3}(x, y, z)\mathbf{H}, \tag{2a}$$

$$\mathbf{B} = \boldsymbol{\zeta}^{3\times3}(x, y, z)\mathbf{E} + \boldsymbol{\mu}^{3\times3}(x, y, z)\mathbf{H}. \tag{2b}$$

Assume a harmonic time-dependence according to $\exp(-j\omega t)$, and ‘‘unpack’’ the Eqs. (1):

$$\begin{bmatrix} 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} = -j\omega \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} + \begin{bmatrix} J_1 \\ J_2 \\ J_3 \end{bmatrix}, \tag{3a}$$

$$\begin{bmatrix} 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = j\omega \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}. \tag{3b}$$

For notational convenience, divide each of the above equations by $j\omega$ and introduce the ‘‘velocities’’ $\tilde{x} = j\omega x$, $\tilde{y} = j\omega y$, and $\tilde{z} = j\omega z$. Thus, $\partial_x/(j\omega) = \partial_{\tilde{x}}$, $\partial_y/(j\omega) = \partial_{\tilde{y}}$, $\partial_z/(j\omega) = \partial_{\tilde{z}}$. Furthermore,, introduce the ‘‘charges’’ $\tilde{J}_1 = J_1/(j\omega)$, $\tilde{J}_2 = J_2/(j\omega)$, and $\tilde{J}_3 = J_3/(j\omega)$. Write Eqs. (3) component-wise, and employ the introduced ‘‘velocities’’ and the ‘‘charges,’’

$$\begin{aligned} -\partial_{\tilde{z}} H_2 + \partial_{\tilde{y}} H_3 &= -D_1 + \tilde{J}_1, && (4a) \\ \partial_{\tilde{z}} H_1 - \partial_{\tilde{x}} H_3 &= -D_2 + \tilde{J}_2, && (4b) \\ -\partial_{\tilde{y}} H_1 + \partial_{\tilde{x}} H_2 &= -D_3 + \tilde{J}_3, && (4c) \\ -\partial_{\tilde{z}} E_2 + \partial_{\tilde{y}} E_3 &= B_1, && (4d) \\ \partial_{\tilde{z}} E_1 - \partial_{\tilde{x}} E_3 &= B_2, && (4e) \\ -\partial_{\tilde{y}} E_1 + \partial_{\tilde{x}} E_2 &= B_3. && (4f) \end{aligned}$$

III. ESSENTIAL-, NONESSENTIAL-, AND AUXILIARY VARIABLES

The main objective in this paper is the diagonalization and supplementation of the governing and constitutive equations with respect to the arbitrarily chosen $\tilde{z}-$axis. To this end, the first step consists of identifying the ‘‘essential-,’’ ‘‘nonessential-’’, and ‘‘auxiliary’’ field variables. When diagonalizing with respect to $\tilde{z}$, the field variables associated with $\partial_{\tilde{z}}$ are referred to as the ‘‘essential’’ field variables. Inspecting Eqs. (4) it is easily seen that the electromagnetic field components E_1, E_2, H_1, and H_2 are accompanied by $\partial_{\tilde{z}}$. The remaining components E_3 and H_3 are referred to as the ‘‘nonessential’’ field components. Equations involving $\partial_{\tilde{z}} E_1$, $\partial_{\tilde{z}} E_2$, $\partial_{\tilde{z}} H_1$, and $\partial_{\tilde{z}} H_2$, are, respectively, (4e), (4d), (4b), and (4a). The Eqs. (4c) and (4f) do not involve the partial derivative $\partial_{\tilde{z}}$. Next, reshuffle the order of the Eqs. (4) to reflect the successive appearances of the terms $\partial_{\tilde{z}} E_1$, $\partial_{\tilde{z}} E_2$, $\partial_{\tilde{z}} H_1$, and $\partial_{\tilde{z}} H_2$, and subdivide the Eqs. (4) into the systems of equations,

$$\begin{aligned} \partial_{\tilde{z}} E_1 - \partial_{\tilde{x}} E_3 &= B_2, && (5a) \\ -\partial_{\tilde{z}} E_2 + \partial_{\tilde{y}} E_3 &= B_1, && (5b) \\ \partial_{\tilde{z}} H_1 - \partial_{\tilde{x}} H_3 &= -D_2 + \tilde{J}_2, && (5c) \\ -\partial_{\tilde{z}} H_2 + \partial_{\tilde{y}} H_3 &= -D_1 + \tilde{J}_1, && (5d) \end{aligned}$$

$$
\begin{aligned}
-\partial_{\tilde{y}} H_1 + \partial_{\tilde{x}} H_2 &= -D_3 + \tilde{J}_3, && (6a)\\
-\partial_{\tilde{y}} E_1 + \partial_{\tilde{x}} E_2 &= B_3. && (6b)
\end{aligned}
$$

Bringing the terms accompanied by $\partial_{\tilde{z}}$ to the R.H.S. and rearranging,

$$
\begin{aligned}
& B_2 + \partial_{\tilde{x}} E_3 = \partial_{\tilde{z}} E_1, && (7a)\\
& -B_1 + \partial_{\tilde{y}} E_3 = \partial_{\tilde{z}} E_2, && (7b)\\
& -D_2 + \partial_{\tilde{x}} H_3 + \tilde{J}_2 = \partial_{\tilde{z}} H_1, && (7c)\\
& D_1 + \partial_{\tilde{y}} H_3 - \tilde{J}_1 = \partial_{\tilde{z}} H_2, && (7d)
\end{aligned}
$$

$$
\begin{aligned}
& D_3 = \partial_{\tilde{y}} H_1 - \partial_{\tilde{x}} H_2 + \tilde{J}_3, && (8a)\\
& B_3 = -\partial_{\tilde{y}} E_1 + \partial_{\tilde{x}} E_2. && (8b)
\end{aligned}
$$

Inspecting Eqs. (7) it is seen that the components of the fields $\mathbf{D}$ and $\mathbf{B}$ are not accompanied by any partial derivatives at all. The components of $\mathbf{D}$ and $\mathbf{B}$ are referred to as the ''auxiliary'' field components, and, in the framework of diagonalization and supplementation, they have to be eliminated from the equations. It should also be noted that Eqs. (7) containing the $\partial_{\tilde{z}}-$derivative, involve the components D_1, D_2, B_1, and B_2. On the other hand, Eqs. (8) not containing the $\partial_{\tilde{z}}-$derivative, involve the components D_3 and B_3. In virtue of these findings the following representations for (7) and (8), respectively, offer themselves:

$$
\underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{P}^{4\times4}} \underbrace{\begin{bmatrix} D_1 \\ D_2 \\ B_1 \\ B_2 \end{bmatrix}}_{\mathbf{\Phi}_c^{\parallel}} + \underbrace{\begin{bmatrix} \partial_{\tilde{x}} & 0 \\ \partial_{\tilde{y}} & 0 \\ 0 & \partial_{\tilde{x}} \\ 0 & \partial_{\tilde{y}} \end{bmatrix}}_{\mathbf{Q}_c^{4\times2}} \underbrace{\begin{bmatrix} E_3 \\ H_3 \end{bmatrix}}_{\mathbf{\Psi}_c^{\perp}}
+ \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\mathbf{P}^{4\times2}} \underbrace{\begin{bmatrix} \tilde{J}_1 \\ \tilde{J}_2 \end{bmatrix}}_{\tilde{\mathbf{J}}_c^{\parallel}} = \partial_{\tilde{z}} \underbrace{\begin{bmatrix} E_1 \\ E_2 \\ H_1 \\ H_2 \end{bmatrix}}_{\mathbf{\Psi}_c^{\parallel}}, \qquad (9a)
$$

$$
\underbrace{\begin{bmatrix} D_3 \\ B_3 \end{bmatrix}}_{\mathbf{\Phi}_c^{\perp}} = \underbrace{\begin{bmatrix} 0 & 0 & \partial_{\tilde{y}} & -\partial_{\tilde{x}} \\ -\partial_{\tilde{y}} & \partial_{\tilde{x}} & 0 & 0 \end{bmatrix}}_{\mathbf{Q}_c^{2\times4}} \underbrace{\begin{bmatrix} E_1 \\ E_2 \\ H_1 \\ H_2 \end{bmatrix}}_{\mathbf{\Psi}_c^{\parallel}}
+ \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\mathbf{P}^{2\times1}} \underbrace{\tilde{J}_3}_{\tilde{J}_c^{\perp}}. \qquad (9b)
$$

Using the introduced matrices and vectors, as indicated in Eqs. (9), the following compact forms can be established, which build the basis for our analysis,

$$
\begin{aligned}
& \mathbf{P}^{4\times4}\mathbf{\Phi}_c^{\parallel} + \mathbf{Q}_c^{4\times2}\mathbf{\Psi}_c^{\perp} + \mathbf{P}^{4\times2}\tilde{\mathbf{J}}_c^{\parallel} = \partial_{\tilde{z}}\mathbf{\Psi}_c^{\parallel}, && (10a)\\
& \mathbf{\Phi}_c^{\perp} = \mathbf{Q}_c^{2\times4}\mathbf{\Psi}_c^{\parallel} + \mathbf{P}^{2\times1}\tilde{J}_c^{\perp}. && (10b)
\end{aligned}
$$

Comments: The matrices $\mathbf{P}^{4\times4}$, $\mathbf{P}^{4\times2}$, and $\mathbf{P}^{2\times1}$ in (9) are ''universal structural'' matrices, in the sense that their structures are independent of the diagonalization with respect to $\tilde{x}$, $\tilde{y}$, or $\tilde{z}$. The matrices $\mathbf{Q}_c^{4\times2}$ and $\mathbf{Q}_c^{2\times4}$ (and their counterparts $\mathbf{Q}_a^{4\times2}$ and $\mathbf{Q}_a^{2\times4}$, and $\mathbf{Q}_b^{4\times2}$ and $\mathbf{Q}_b^{2\times4}$) are matrix differential operators, possessing an ''intriguing'' property, as will be shown in Part II, [7]. The field vector $\mathbf{\Psi}_c^{\parallel}$ (along with its counterparts $\mathbf{\Psi}_a^{\parallel}$ and $\mathbf{\Psi}_b^{\parallel}$) is an ''essential'' field vector. The essential field vectors $\mathbf{\Psi}_c^{\parallel}$, $\mathbf{\Psi}_a^{\parallel}$, and $\mathbf{\Psi}_b^{\parallel}$, respectively, arise in the $\mathcal{D}_c-$, $\mathcal{D}_a-$, and $\mathcal{D}_b-$forms. Their associated field vectors $\mathbf{\Psi}_c^{\perp}$, $\mathbf{\Psi}_a^{\perp}$, and $\mathbf{\Psi}_b^{\perp}$, are ''nonessential'' field vectors. The nonessential field vectors $\mathbf{\Psi}_c^{\perp}$, $\mathbf{\Psi}_a^{\perp}$, and $\mathbf{\Psi}_b^{\perp}$, respectively, arise in the $\mathcal{S}_c-$, $\mathcal{S}_a-$, and $\mathcal{S}_b-$forms. Furthermore, $\mathbf{\Phi}_c^{\parallel}$ and $\mathbf{\Phi}_c^{\perp}$ (along with their counterparts $\mathbf{\Phi}_a^{\parallel}$ and $\mathbf{\Phi}_a^{\perp}$, and $\mathbf{\Phi}_b^{\parallel}$ and $\mathbf{\Phi}_b^{\perp}$) are auxiliary field vectors. Auxiliary vectors must be eliminated from the $(\mathcal{D}_c, \mathcal{S}_c)-$, $(\mathcal{D}_a, \mathcal{S}_a)-$, and $(\mathcal{D}_b, \mathcal{S}_b)-$forms.

IV. CONSTITUTIVE EQUATIONS

''Unpacking'' the constitutive Eqs. (2a) and (2b), leads to:

$$
\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} + \begin{bmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ \xi_{31} & \xi_{32} & \xi_{33} \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix}, \qquad (11)
$$

$$
\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} \zeta_{11} & \zeta_{12} & \zeta_{13} \\ \zeta_{21} & \zeta_{22} & \zeta_{23} \\ \zeta_{31} & \zeta_{32} & \zeta_{33} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} + \begin{bmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix}, \qquad (12)
$$

respectively. The aforementioned categorization (grouping) of the field vectors suggests partitioning

Eqs. (11) and (12), respectively, into:

$$\begin{aligned}\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} &= \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{bmatrix}\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_{13} \\ \varepsilon_{23} \end{bmatrix} E_3 \\ &+ \begin{bmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{bmatrix}\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} + \begin{bmatrix} \xi_{13} \\ \xi_{23} \end{bmatrix} H_3, \quad \text{(13a)}\end{aligned}$$

$$\begin{aligned}D_3 &= \begin{bmatrix} \varepsilon_{31} & \varepsilon_{32} \end{bmatrix}\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} + \varepsilon_{33} E_3 \\ &+ \begin{bmatrix} \xi_{31} & \xi_{32} \end{bmatrix}\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} + \xi_{33} H_3, \qquad \text{(13b)}\end{aligned}$$

$$\begin{aligned}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} &= \begin{bmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \end{bmatrix}\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} + \begin{bmatrix} \zeta_{13} \\ \zeta_{23} \end{bmatrix} E_3 \\ &+ \begin{bmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{bmatrix}\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} + \begin{bmatrix} \mu_{13} \\ \mu_{23} \end{bmatrix} H_3, \quad \text{(14a)}\end{aligned}$$

$$\begin{aligned}B_3 &= \begin{bmatrix} \zeta_{31} & \zeta_{32} \end{bmatrix}\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} + \zeta_{33} E_3 \\ &+ \begin{bmatrix} \mu_{31} & \mu_{32} \end{bmatrix}\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} + \mu_{33} H_3. \qquad \text{(14b)}\end{aligned}$$

Equations (13) and (14) can be written more compactly in the forms:

$$\begin{aligned}\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} &= \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \xi_{11} & \xi_{12} \\ \varepsilon_{21} & \varepsilon_{22} & \xi_{21} & \xi_{22} \end{bmatrix}\begin{bmatrix} E_1 \\ E_2 \\ H_1 \\ H_2 \end{bmatrix} \\ &+ \begin{bmatrix} \varepsilon_{13} & \xi_{13} \\ \varepsilon_{23} & \xi_{23} \end{bmatrix}\begin{bmatrix} E_3 \\ H_3 \end{bmatrix}, \qquad \text{(15a)}\end{aligned}$$

$$\begin{aligned}D_3 &= \begin{bmatrix} \varepsilon_{31} & \varepsilon_{32} & \xi_{31} & \xi_{32} \end{bmatrix}\begin{bmatrix} E_1 \\ E_2 \\ H_1 \\ H_2 \end{bmatrix} \\ &+ \begin{bmatrix} \varepsilon_{33} & \xi_{33} \end{bmatrix}\begin{bmatrix} E_3 \\ H_3 \end{bmatrix}, \qquad \text{(15b)}\end{aligned}$$

and

$$\begin{aligned}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} &= \begin{bmatrix} \zeta_{11} & \zeta_{12} & \mu_{11} & \mu_{12} \\ \zeta_{21} & \zeta_{22} & \mu_{21} & \mu_{22} \end{bmatrix}\begin{bmatrix} E_1 \\ E_2 \\ H_1 \\ H_2 \end{bmatrix} \\ &+ \begin{bmatrix} \zeta_{13} & \mu_{13} \\ \zeta_{23} & \mu_{23} \end{bmatrix}\begin{bmatrix} E_3 \\ H_3 \end{bmatrix}, \qquad \text{(16a)}\end{aligned}$$

$$\begin{aligned}B_3 &= \begin{bmatrix} \zeta_{31} & \zeta_{32} & \mu_{31} & \mu_{32} \end{bmatrix}\begin{bmatrix} E_1 \\ E_2 \\ H_1 \\ H_2 \end{bmatrix} \\ &+ \begin{bmatrix} \zeta_{33} & \mu_{33} \end{bmatrix}\begin{bmatrix} E_3 \\ H_3 \end{bmatrix}. \qquad \text{(16b)}\end{aligned}$$

Combining (15a) with (16a),

$$\begin{aligned}\underbrace{\begin{bmatrix} D_1 \\ D_2 \\ B_1 \\ B_2 \end{bmatrix}}_{\mathbf{\Phi}_c^{\parallel}} &= \underbrace{\begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \xi_{11} & \xi_{12} \\ \varepsilon_{21} & \varepsilon_{22} & \xi_{21} & \xi_{22} \\ \zeta_{11} & \zeta_{12} & \mu_{11} & \mu_{12} \\ \zeta_{21} & \zeta_{22} & \mu_{21} & \mu_{22} \end{bmatrix}}_{\mathbf{M}_c^{4\times 4}}\underbrace{\begin{bmatrix} E_1 \\ E_2 \\ H_1 \\ H_2 \end{bmatrix}}_{\mathbf{\Psi}_c^{\parallel}} \\ &+ \underbrace{\begin{bmatrix} \varepsilon_{13} & \xi_{13} \\ \varepsilon_{23} & \xi_{23} \\ \zeta_{13} & \mu_{13} \\ \zeta_{23} & \mu_{23} \end{bmatrix}}_{\mathbf{M}_c^{4\times 2}}\underbrace{\begin{bmatrix} E_3 \\ H_3 \end{bmatrix}}_{\mathbf{\Psi}_c^{\perp}}. \qquad \text{(17)}\end{aligned}$$

Employing the "material" matrices $\mathbf{M}_c^{4\times 4}$ and $\mathbf{M}_c^{4\times 2}$, and the fields $\mathbf{\Phi}_c^{\parallel}$, $\mathbf{\Psi}_c^{\parallel}$, and $\mathbf{\Psi}_c^{\perp}$, as indicated in (17),

$$\mathbf{\Phi}_c^{\parallel} = \mathbf{M}_c^{4\times 4}\boldsymbol{\psi}_c^{\parallel} + \mathbf{M}_c^{4\times 2}\mathbf{\Psi}_c^{\perp}. \qquad (18)$$

Similarly, combining (15b) with (16b),

$$\begin{aligned}\underbrace{\begin{bmatrix} D_3 \\ B_3 \end{bmatrix}}_{\mathbf{\Phi}_c^{\perp}} &= \underbrace{\begin{bmatrix} \varepsilon_{31} & \varepsilon_{32} & \xi_{31} & \xi_{32} \\ \zeta_{31} & \zeta_{32} & \mu_{31} & \mu_{32} \end{bmatrix}}_{\mathbf{M}_c^{2\times 4}}\underbrace{\begin{bmatrix} E_1 \\ E_2 \\ H_1 \\ H_2 \end{bmatrix}}_{\mathbf{\Psi}_c^{\parallel}} \\ &+ \underbrace{\begin{bmatrix} \varepsilon_{33} & \xi_{33} \\ \zeta_{33} & \mu_{33} \end{bmatrix}}_{\mathbf{M}_c^{2\times 2}}\underbrace{\begin{bmatrix} E_3 \\ H_3 \end{bmatrix}}_{\mathbf{\Psi}_c^{\perp}}. \qquad \text{(19)}\end{aligned}$$

Employing the "material" matrices $\mathbf{M}_c^{2\times 4}$ and $\mathbf{M}_c^{2\times 2}$, and the fields $\mathbf{\Phi}_c^{\perp}$, $\mathbf{\Psi}_c^{\parallel}$, and $\mathbf{\Psi}_c^{\perp}$, as indicated in (19),

$$\mathbf{\Phi}_c^{\perp} = \mathbf{M}_c^{2\times 4}\mathbf{\Psi}_c^{\parallel} + \mathbf{M}_c^{2\times 2}\mathbf{\Psi}_c^{\perp}. \qquad (20)$$

Preliminary summary: For easy reference, the governing equation dressed in the forms in (10), and the material equations formulated in the forms (18) and (20), are summarized below:

$$\mathbf{P}^{4\times4}\mathbf{\Phi}_c^{\parallel} + \mathbf{Q}_c^{4\times2}\mathbf{\Psi}_c^{\perp} + \mathbf{P}^{4\times2}\tilde{\mathbf{J}}_c^{\parallel} = \partial_{\tilde{z}}\mathbf{\Psi}_c^{\parallel}, \tag{21a}$$
$$\mathbf{\Phi}_c^{\perp} = \mathbf{Q}_c^{2\times4}\mathbf{\Psi}_c^{\parallel} + \mathbf{P}^{2\times1}\tilde{J}_c^{\perp}, \tag{21b}$$
$$\mathbf{\Phi}_c^{\parallel} = \mathbf{M}_c^{4\times4}\mathbf{\Psi}_c^{\parallel} + \mathbf{M}_c^{4\times2}\mathbf{\Psi}_c^{\perp}, \tag{21c}$$
$$\mathbf{\Phi}_c^{\perp} = \mathbf{M}_c^{2\times4}\mathbf{\Psi}_c^{\parallel} + \mathbf{M}_c^{2\times2}\mathbf{\Psi}_c^{\perp}. \tag{21d}$$

The governing Eqs. (21a) and (21b), and the constitutive Eqs. (21c) and (21d), express the set of electromagnetic field components $(E_1, E_2, E_3, H_1, H_2, H_3)$ in terms of the essential $\mathbf{\Psi}_c^{\parallel} = [E_1, E_2, H_1, H_2]^T$ and nonessential field components $\mathbf{\Psi}_c^{\perp} = [E_3, H_3]^T$. Furthermore, the differential operators are factorized into the $(\partial_{\tilde{x}}, \partial_{\tilde{y}})-$dependent operators $\mathbf{Q}_c^{2\times4}$ and $\mathbf{Q}_c^{4\times2}$ and the partial derivative $\partial_{\tilde{z}}$. It should also be noted that in addition to the essential- and nonessential field components, the auxiliary field components $\mathbf{\Phi}_c^{\parallel} = [D_1, D_2, B_1, B_2]^T$ and $\mathbf{\Phi}_c^{\perp} = [D_3, B_3]^T$ appear in the equations. Within the developed framework, auxiliary field variables must be eliminated from the Eqs. (21). The rationale for this stems from the fact that the auxiliary field components are, by definition, not accompanied by any spatial partial derivatives. The elimination of the auxiliary field components leads to the desired diagonalized $\mathcal{D}_c-$ and the associated supplementary $\mathcal{S}_c-$forms. The next section presents the details involved in the manipulations. Thereby, the individual steps will be explicated and the results interpreted.

V. THE $\mathcal{D}_c-$ AND THE ASSOCIATED $\mathcal{S}_c-$FORMS

Substituting (21c) into (21a), and equating the terms at the R.H.S. of (21d) and (21b),

$$\begin{aligned}&\mathbf{P}^{4\times4}\left\{\mathbf{M}_c^{4\times4}\mathbf{\Psi}_c^{\parallel} + \mathbf{M}_c^{4\times2}\mathbf{\Psi}_c^{\perp}\right\} + \mathbf{Q}_c^{4\times2}\mathbf{\Psi}_c^{\perp}\\&+\mathbf{P}^{4\times2}\tilde{\mathbf{J}}_c^{\parallel} = \partial_{\tilde{z}}\mathbf{\Psi}_c^{\parallel},\end{aligned} \tag{22a}$$

$$\mathbf{M}_c^{2\times4}\mathbf{\Psi}_c^{\parallel} + \mathbf{M}_c^{2\times2}\mathbf{\Psi}_c^{\perp} = \mathbf{Q}_c^{2\times4}\mathbf{\Psi}_c^{\parallel} + \mathbf{P}^{2\times1}\tilde{J}_c^{\perp}. \tag{22b}$$

The nonessential field vector $\mathbf{\Psi}_c^{\perp}$ must be eliminated from (22a) to obtain the $\mathcal{D}_c-$form. To this effect (22b) is employed, which is preferably rewritten in the form,

$$\mathbf{M}_c^{2\times2}\mathbf{\Psi}_c^{\perp} = \left(-\mathbf{M}_c^{2\times4} + \mathbf{Q}_c^{2\times4}\right)\mathbf{\Psi}_c^{\parallel} + \mathbf{P}^{2\times1}\tilde{J}_c^{\perp}. \tag{23}$$

A. Physical realizability condition

In view of (23) it is immediate that for expressing $\mathbf{\Psi}_c^{\perp}$ in terms of $\mathbf{\Psi}_c^{\parallel}$ and the source term $\tilde{J}_c^{\perp}$, the existence of the inverse of $\mathbf{M}_c^{2\times2}$ must be assumed. Consequently,

$$\begin{aligned}\det\left\{\mathbf{M}_c^{2\times2}\right\} &= \det\begin{bmatrix}\varepsilon_{33} & \xi_{33}\\ \zeta_{33} & \mu_{33}\end{bmatrix}, &(24a)\\ &= \varepsilon_{33}\mu_{33} - \zeta_{33}\xi_{33} \neq 0. &(24b)\end{aligned}$$

To obtain the $\mathcal{S}_c-$ and $\mathcal{D}_c-$forms it is merely required that the condition stipulated in (24b) is valid.

B. The $\mathcal{S}_c-$form

Multiplying (23) by $\left[\mathbf{M}_c^{2\times2}\right]^{-1}$ from the L.H.S.,

$$\begin{aligned}\mathbf{\Psi}_c^{\perp} &= \left[\mathbf{M}_c^{2\times2}\right]^{-1}\left(-\mathbf{M}_c^{2\times4} + \mathbf{Q}_c^{2\times4}\right)\mathbf{\Psi}_c^{\parallel}\\ &+ \left[\mathbf{M}_c^{2\times2}\right]^{-1}\mathbf{P}^{2\times1}\tilde{J}_c^{\perp}.\end{aligned} \tag{25}$$

This is the desired $\mathcal{S}_c-$form. It expresses the nonessential field components $\mathbf{\Psi}_c^{\perp}$ in terms of the essential field components $\mathbf{\Psi}_c^{\parallel}$ and the source term $\tilde{J}_c^{\perp}$: given $\tilde{J}_c^{\perp}$, once $\mathbf{\Psi}_c^{\parallel}$ has been determined, the nonessential field vector $\mathbf{\Psi}_c^{\perp}$ can be obtained according to the inexpensive post-processing step expressed in (25).

C. The $\mathcal{D}_c-$form

Focus on (22a) and pull the $\mathbf{\Psi}_c^{\perp}-$terms together,

$$\begin{aligned}&\mathbf{P}^{4\times4}\mathbf{M}_c^{4\times4}\mathbf{\Psi}_c^{\parallel} + \left(\mathbf{P}^{4\times4}\mathbf{M}_c^{4\times2} + \mathbf{Q}_c^{4\times2}\right)\mathbf{\Psi}_c^{\perp}\\&+\mathbf{P}^{4\times2}\tilde{\mathbf{J}}_c^{\parallel} = \partial_{\tilde{z}}\mathbf{\Psi}_c^{\parallel}.\end{aligned} \tag{26}$$

Substituting (25) into (26) allows the elimination of the nonessential field vector $\mathbf{\Psi}_c^{\perp}$,

$$\begin{aligned}&\mathbf{P}^{4\times4}\mathbf{M}_c^{4\times4}\mathbf{\Psi}_c^{\parallel} + \left(\mathbf{P}^{4\times4}\mathbf{M}_c^{4\times2} + \mathbf{Q}_c^{4\times2}\right)\Big\{\left[\mathbf{M}_c^{2\times2}\right]^{-1}\\&\times\left(-\mathbf{M}_c^{2\times4} + \mathbf{Q}_c^{2\times4}\right)\mathbf{\Psi}_c^{\parallel} + \left[\mathbf{M}_c^{2\times2}\right]^{-1}\mathbf{P}^{2\times1}\tilde{J}_c^{\perp}\Big\}\\&+\mathbf{P}^{4\times2}\tilde{\mathbf{J}}_c^{\parallel} = \partial_{\tilde{z}}\mathbf{\Psi}_c^{\parallel}.\end{aligned} \tag{27}$$

Factoring out the $\mathbf{\Psi}_c^{\parallel}-$terms and rearranging lead to the $\mathcal{D}_c-$form,

$$\begin{aligned}&\Big\{\mathbf{P}^{4\times4}\mathbf{M}_c^{4\times4} + \left(\mathbf{P}^{4\times4}\mathbf{M}_c^{4\times2} + \mathbf{Q}_c^{4\times2}\right)\\&\times\left[\mathbf{M}_c^{2\times2}\right]^{-1}\left(-\mathbf{M}_c^{2\times4} + \mathbf{Q}_c^{2\times4}\right)\Big\}\mathbf{\Psi}_c^{\parallel}\\&+\left(\mathbf{P}^{4\times4}\mathbf{M}_c^{4\times2} + \mathbf{Q}_c^{4\times2}\right)\left[\mathbf{M}_c^{2\times2}\right]^{-1}\mathbf{P}^{2\times1}\tilde{J}_c^{\perp}\\&+\mathbf{P}^{4\times2}\tilde{\mathbf{J}}_c^{\parallel} = \partial_{\tilde{z}}\mathbf{\Psi}_c^{\parallel}.\end{aligned} \tag{28}$$

This is the desired $\mathcal{D}_c-$form, involving the essential field vector $\mathbf{\Psi}_c^{\parallel}$ and the sources terms $\tilde{J}_c^{\perp}$ and $\tilde{\mathbf{J}}_c^{\parallel}$.

VI. CONCLUSION

Maxwell's electrodynamic equations were considered in fully bi-anisotropic and inhomogeneous media in three spatial dimensions. A harmonic time dependence according to $\exp(-j\omega t)$ and an $(x, y, z)-$Cartesian coordinate system were assumed. It was rigorously established that the Maxwell's equations along with the constitutive equations can be transformed into the diagonalized $(\mathcal{D}-)$ and the associated supplementary $(\mathcal{S}-)$forms. To this end, three "structural," two "differential," and four "material," matrices were introduced and their properties discussed. It was shown that the conditions $\varepsilon_{ii}\mu_{ii} - \xi_{ii}\zeta_{ii} \neq 0$ $(i = 1, 2, 3)$ suffice to construct the $\mathcal{D}-$ and $\mathcal{S}-$forms. It was alluded to the fact that the derived forms play eminent roles in developing regularization techniques for dealing with infinities arising in computational electromagnetics and calculating the near- and far-fields. The utilitarian properties of the $\mathcal{D}-$ and $\mathcal{S}-$forms have been detailed in the author's earlier works [1]-[6]. The proof of their sharp equivalence with the combined Maxwell's and constitutive equations, and, thus, their internal consistency, has been provided in the accompanying paper, [7].

REFERENCES

[1] A. R. Baghai-Wadji, "3-D electrostatic charge distribution on finitely thick busbars in micro acoustic devices: Combined regularization in the near- and far-field," IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control (UFFC), vol. 62, no. 6, June 2015, pp. 1132-1144.

[2] A. R. Baghai-Wadji, "Dyadic universal functions and simultaneous near-field/far-field regularization of elasto-dynamic dyadic Green's functions for 3D mass-loading analysis in micro-acoustic devices," IEEE Transactions on UFFC, vol. 63, no. 10, October 2016, pp. 1563-1574.

[3] A. R. Baghai-Wadji, "3D Thermo-acousto-electric diagonalized and supplementary equations in fully tri-anisotropic and inhomogeneous media, Part I: Proof of existence by construction," IEEE Journal on Multiscale and Multiphysics Computational Techniques (JMMCT), vol. 3, 2018, pp.149-158.

[4] A. R. Baghai-Wadji, "3D Thermo-acousto-electric diagonalized and supplementary equations in fully tri-anisotropic and inhomogeneous media, Part II: Relative proof of consistency," IEEE Journal on Multiscale and Multiphysics Computational Techniques (JMMCT), vol. 3, 2018, pp. 159-166.

[5] A. R. Baghai-Wadji, "Diagonalizability of Thermo Electromagnetic Equations in Inhomogeneous and Fully Tri-anisotropic Media," Proc. Applied Computational Electromagnetics Society Symposium (ACES), 2018, pp.1-2.

[6] A. R. Baghai-Wadji, "The Path from Monadic to Tetradic Green's Functions," Proc. International Conference on Electromagnetics in Advanced Applications (ICEAA), 2018, pp.1-4.

[7] A. R. Baghai-Wadji, "3D diagonalization and supplementation of Maxwell's equations in fully bi-anisotropic and inhomogeneous media, Part II: Relative proof of consistency," ACES Journal, this issue.

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