3D Diagonalization and Supplementation of Electrostatic Field Equations in Fully Anisotropic and Inhomogeneous Media Proof of Existence and Consistency

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Abstract − Consider Maxwell's homogeneous curl equation $\nabla \times \mathbf{E} = 0$ for the electric field vector $\mathbf{E}$ and the inhomogeneous divergence equation $\nabla \cdot \mathbf{D} = \rho$ for the dielectric displacement vector $\mathbf{D}$ and the charge density function ρ in the static limit. Assume an $(x, y, z)-$Cartesian coordinate system. Consider the constitutive equation $\mathbf{D} = \boldsymbol{\varepsilon}\ \mathbf{E}$, with the 3×3 position-dependent positive-definite permittivity matrix $\boldsymbol{\varepsilon}(x, y, z)$ modeling fully anisotropic and inhomogeneous dielectric media. This paper proves that $\nabla \times \mathbf{E} = 0$ and $\nabla \cdot \mathbf{D} = \rho$ along with $\mathbf{D} = \boldsymbol{\varepsilon}\ \mathbf{E}$ are diagonalizable with respect to the arbitrarily chosen $z-$axis leading to the $\mathcal{D}_c-$form. The existence of an associated supplementary equation, the $\mathcal{S}_c-$form, has also been demonstrated. Finally, it is shown that the constructed $(\mathcal{D}_c, \mathcal{S}_c)-$forms are sharply equivalent with the originating set of equations $\nabla \times \mathbf{E} = 0$, $\nabla \cdot \mathbf{D} = \rho$, and $\mathbf{D} = \boldsymbol{\varepsilon}\ \mathbf{E}$, and, thus, internally consistent. The proof scheme is relative in the sense that its validity hinges on the consistency of Maxwell's equations in the static limit and the material realizability conditions.

Index Terms − Anisotropic and inhomogeneous dielectric media, diagonalization, electrostatic field, supplementation.

I. INTRODUCTION

In the accompanying paper, [1], Maxwell's electrodynamic equations in linear fully bi-anisotropic and inhomogeneous media were analyzed. It was demonstrated that the governing $(\mathcal{G})$ and constitutive $(\mathcal{C})$ equations can be diagonalized $(\mathcal{D}-$form) with reference to any arbitrary direction in space. Furthermore, it was shown that there exists a unique supplementary equation $(\mathcal{S}-$form) associated with the $\mathcal{D}-$form. The existence of $(\mathcal{D}, \mathcal{S})-$forms was established by construction. In [2], it was rigorously proved that the constructed $(\mathcal{D}, \mathcal{S})-$forms are sharply equivalent with the originating set of equations $(\mathcal{G}, \mathcal{C})$. Assuming the consistency of $(\mathcal{G}, \mathcal{C})$, it was then concluded that the $(\mathcal{D}, \mathcal{S})-$forms are internally consistent. The present paper considers Maxwell's $\nabla \times \mathbf{E} = 0$ and $\nabla \cdot \mathbf{D} = \rho$ equations in the electrostatic limit. Fully anisotropic and inhomogeneous dielectric media characterized by the 3×3 position-dependent positive-definite permittivity matrix $\boldsymbol{\varepsilon}(x, y, z)$ have been assumed. The constitutive equation $\mathbf{D} = \boldsymbol{\varepsilon}\mathbf{E}$ relates the dielectric displacement vector $\mathbf{D}$ to the electric field vector $\mathbf{E}$. Rigorous proofs of existence and internal consistency of the $(\mathcal{D}, \mathcal{S})-$forms substantially refine the related discussion in [3]. The merits of the $(\mathcal{D}, \mathcal{S})-$forms have been alluded to in [1] and [2], and the references therein, and will not be repeated here. Nonetheless one outstanding feature of the $\mathcal{D}-$form deserves mentioning: the $\mathcal{D}-$form automatically gives rise to the "interface conditions." This fundamental property has been detailed in [3]. Despite the fact that the paper is self-sufficient, the reader would most likely benefit from acquainting themselves with the contents of [1] and [2].

The paper has been organized as follows. The brief Section II fixes the overall notation. Section III explains the construction of the $(\mathcal{D}, \mathcal{S})-$forms, by stating and proving a theorem. Section IV is devoted to the consistency analysis of the constructed $(\mathcal{D}, \mathcal{S})-$forms. Thereby, a second theorem will be stated and a relative proof will be provided. Sections III and IV consider the $z-$axis as the direction along which the diagonalization is performed. The letter "c," used as a suffix or superscript, signifies the fact that the chosen direction has been the $z-$axis. In the Appendix the formulae corresponding to the $x-$, $y-$, and the $z-$axes have been

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summarized. Correspondingly, the formulae have been equipped with the letters "a," "b," and "c." The "a-" and "b-" related formulae have been obtained from the formulae derived in Sections III and IV, by cyclic permutations $1 \to 2$, $2 \to 3$, $3 \to 1$, $a \to b$, $b \to c$, $c \to a$. Section V concludes the paper.

II. MAXWELL'S EQUATIONS IN THE STATIC LIMIT

Consider the set of equations $(\mathcal{G}, \mathcal{C})$,

$$\nabla \times \mathbf{E} = 0, \tag{1a}$$

$$\nabla \cdot \mathbf{D} = \rho, \tag{1b}$$

$$\mathbf{D} = \boldsymbol{\varepsilon}(x, y, z)\mathbf{E}. \tag{1c}$$

A sufficient condition for $\nabla \times \mathbf{E} = 0$ to be valid is:

$$\mathbf{E} = -\nabla\varphi, \tag{2}$$

with $\varphi(x, y, z)$ being the electric potential function. Utilizing matrix notation and "unpacking" the Eqs. (2), (1b) and (1c), respectively,

$$E_1 = -\partial_x\varphi, \tag{3a}$$

$$E_2 = -\partial_y\varphi, \tag{3b}$$

$$E_3 = -\partial_z\varphi, \tag{3c}$$

$$\partial_x D_1 + \partial_y D_2 + \partial_z D_3 = \rho, \tag{4}$$

$$D_1 = \varepsilon_{11}E_1 + \varepsilon_{12}E_2 + \varepsilon_{13}E_3, \tag{5a}$$

$$D_2 = \varepsilon_{21}E_1 + \varepsilon_{22}E_2 + \varepsilon_{23}E_3, \tag{5b}$$

$$D_3 = \varepsilon_{31}E_1 + \varepsilon_{32}E_2 + \varepsilon_{33}E_3. \tag{5c}$$

III. 3D DIAGONALIZATION AND SUPPLEMENTATION ALONG THE Z−AXIS

Consider the governing Eqs. (3) and (4) along with the constitutive Eqs. (5). Focus on the z−axis. The objective in this section is to prove, by construction, the existence of the diagonalized $(\mathcal{D}_c)$ and the associated supplementary $(\mathcal{S}_c)$ forms, with $(\mathcal{D}_c, \mathcal{S}_c)$ being sharply equivalent with the originating set of equations $(\mathcal{G}, \mathcal{C})$. Recall that the suffix "c" signifies the z−axis. Not surprisingly, the partial derivative ∂_z shall play a significant role in the presented arguments and derivations. Diagonalization with respect to the x− and y−axes are specified by the suffixes "a" and "b," respectively, with ∂_x and ∂_y being the main players.

Theorem: The set of equations $(\mathcal{G}, \mathcal{C})$ can uniquely be transformed into the $(\mathcal{D}_c, \mathcal{S}_c)$−forms.

Proof: The strategy for proofing the existence of the $\mathcal{D}_c$− and $\mathcal{S}_c$−forms amounts to constructing them. The first step in obtaining the $\mathcal{D}_c$− and $\mathcal{S}_c$−forms is to identify field components which are accompanied by ∂_z. Simple inspection shows that these are φ, Eq. (3c), and D_3, Eq. (4). The proof requires partitioning the field variables into certain categories, introduced next.

Essential field variables: Field variables accompanied by ∂_z are referred to as the "essential" field variables. In the current context, these are φ and D_3.

Nonessential field variables: The remaining two components of $\mathbf{D}$; i.e., D_1, and D_2, are referred to as the "nonessential" field variables.

Auxiliary field variables: Equations (5) do not involve any spatial derivatives. More importantly, the spatial derivatives of the components of $\mathbf{E}$ do not arise in neither the governing nor the constitutive equations. Fields of this kind are referred to as the "auxiliary" field variables.

In the $(\mathcal{D}, \mathcal{S})$−framework, auxiliary field variables must be eliminated at once. This requirement prompts substituting (3) into (5),

$$D_1 = -\varepsilon_{11}\partial_x\varphi - \varepsilon_{12}\partial_y\varphi - \varepsilon_{13}\partial_z\varphi, \tag{6a}$$

$$D_2 = -\varepsilon_{21}\partial_x\varphi - \varepsilon_{22}\partial_y\varphi - \varepsilon_{23}\partial_z\varphi, \tag{6b}$$

$$D_3 = -\varepsilon_{31}\partial_x\varphi - \varepsilon_{32}\partial_y\varphi - \varepsilon_{33}\partial_z\varphi. \tag{6c}$$

In virtue of the positive-definiteness of $\boldsymbol{\varepsilon}$, $\varepsilon_{ii} \neq 0$ $(i = 1, 2, 3)$. Dividing (6c) by ε_{33} and rearranging,

$$\left(-\frac{\varepsilon_{31}}{\varepsilon_{33}}\partial_x - \frac{\varepsilon_{32}}{\varepsilon_{33}}\partial_y\right)\varphi - \frac{1}{\varepsilon_{33}}D_3 = \partial_z\varphi. \tag{7}$$

Introducing the operators,

$$\mathcal{L}^c_{11} = -\frac{\varepsilon_{31}}{\varepsilon_{33}}\partial_x - \frac{\varepsilon_{32}}{\varepsilon_{33}}\partial_y, \tag{8a}$$

$$\mathcal{L}^c_{12} = -\frac{1}{\varepsilon_{33}}, \tag{8b}$$

equation (7) can be written compactly,

$$\mathcal{L}^c_{11}\varphi + \mathcal{L}^c_{12}D_3 = \partial_z\varphi. \tag{9}$$

This completes the construction of the first of the two equations which constitute the $\mathcal{D}_c$−form.

To obtain the missing counterpart, rewrite (4),

$$\begin{bmatrix} -\partial_x & -\partial_y \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} + \rho = \partial_z D_3. \tag{10}$$

For this equation to qualify as a $\mathcal{D}_c-$form, the nonessential field variables D_1 and D_2 must be expressed in terms of the essential field variables φ and D_3. To this end, consider (6a) and (6b) and eliminate the term $\partial_z\varphi$ from their R.H.S. Fortunately, an expression for $\partial_z\varphi$ in terms of φ and D_3 is already available, Eq. (9). Substituting for $\partial_z\varphi$ from (9) into (6a) and (6b),

$$\begin{aligned} D_1 &= -\varepsilon_{11}\partial_x\varphi - \varepsilon_{12}\partial_y\varphi \\ &\quad -\varepsilon_{13}\left(\mathcal{L}^c_{11}\varphi + \mathcal{L}^c_{12}D_3\right), \end{aligned} \tag{11a}$$

$$\begin{aligned} D_2 &= -\varepsilon_{21}\partial_x\varphi - \varepsilon_{22}\partial_y\varphi \\ &\quad -\varepsilon_{23}\left(\mathcal{L}^c_{11}\varphi + \mathcal{L}^c_{12}D_3\right). \end{aligned} \tag{11b}$$

Or, equivalently,

$$\begin{aligned} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} &= \begin{bmatrix} -\varepsilon_{11}\partial_x - \varepsilon_{12}\partial_y - \varepsilon_{13}\mathcal{L}^c_{11} \\ -\varepsilon_{21}\partial_x - \varepsilon_{22}\partial_y - \varepsilon_{23}\mathcal{L}^c_{11} \end{bmatrix}\varphi \\ &+ \begin{bmatrix} -\varepsilon_{13}\mathcal{L}^c_{12} \\ -\varepsilon_{23}\mathcal{L}^c_{12} \end{bmatrix} D_3. \end{aligned} \tag{12}$$

Employing the explicit expressions for $\mathcal{L}^c_{11}$ and $\mathcal{L}^c_{12}$, Eqs. (8),

$$\begin{aligned} &\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \\ &= \begin{bmatrix} -\left(\varepsilon_{11} - \frac{\varepsilon_{13}\varepsilon_{31}}{\varepsilon_{33}}\right)\partial_x - \left(\varepsilon_{12} - \frac{\varepsilon_{13}\varepsilon_{32}}{\varepsilon_{33}}\right)\partial_y \\ -\left(\varepsilon_{21} - \frac{\varepsilon_{23}\varepsilon_{31}}{\varepsilon_{33}}\right)\partial_x - \left(\varepsilon_{22} - \frac{\varepsilon_{23}\varepsilon_{32}}{\varepsilon_{33}}\right)\partial_y \end{bmatrix}\varphi \\ &+ \begin{bmatrix} \frac{\varepsilon_{13}}{\varepsilon_{33}} \\ \frac{\varepsilon_{23}}{\varepsilon_{33}} \end{bmatrix} D_3. \end{aligned} \tag{13}$$

Introducing operators $\mathcal{A}^c_{ij}$ $(i, j = 1, 2)$,

$$\mathcal{A}^c_{11} = -\left(\varepsilon_{11} - \frac{\varepsilon_{13}\varepsilon_{31}}{\varepsilon_{33}}\right)\partial_x - \left(\varepsilon_{12} - \frac{\varepsilon_{13}\varepsilon_{32}}{\varepsilon_{33}}\right)\partial_y, \tag{14a}$$

$$\mathcal{A}^c_{12} = \frac{\varepsilon_{13}}{\varepsilon_{33}}, \tag{14b}$$

$$\mathcal{A}^c_{21} = -\left(\varepsilon_{21} - \frac{\varepsilon_{23}\varepsilon_{31}}{\varepsilon_{33}}\right)\partial_x - \left(\varepsilon_{22} - \frac{\varepsilon_{23}\varepsilon_{32}}{\varepsilon_{33}}\right)\partial_y, \tag{14c}$$

$$\mathcal{A}^c_{22} = \frac{\varepsilon_{23}}{\varepsilon_{33}}, \tag{14d}$$

equation (13) can be written compactly,

$$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} \mathcal{A}^c_{11} & \mathcal{A}^c_{12} \\ \mathcal{A}^c_{21} & \mathcal{A}^c_{22} \end{bmatrix} \begin{bmatrix} \varphi \\ D_3 \end{bmatrix}. \tag{15}$$

This concludes the construction of the supplementary equations, the $\mathcal{S}_c-$form, expressing the nonessential components D_1 and D_2 in terms of the essential variables φ and D_3.

Substitute (15) into (10) to eliminate the nonessential field variables D_1 and D_2,

$$\begin{aligned} &\begin{bmatrix} -\partial_x & -\partial_y \end{bmatrix} \begin{bmatrix} \mathcal{A}^c_{11} & \mathcal{A}^c_{12} \\ \mathcal{A}^c_{21} & \mathcal{A}^c_{22} \end{bmatrix} \begin{bmatrix} \varphi \\ D_3 \end{bmatrix} \\ &\quad + \rho = \partial_z D_3. \end{aligned} \tag{16}$$

Defining the composite operators $\mathcal{L}^c_{21}$ and $\mathcal{L}^c_{22}$,

$$\mathcal{L}^c_{21} = -\partial_x\mathcal{A}^c_{11} - \partial_y\mathcal{A}^c_{21}, \tag{17a}$$

$$\mathcal{L}^c_{22} = -\partial_x\mathcal{A}^c_{12} - \partial_y\mathcal{A}^c_{22}, \tag{17b}$$

(16) leads to the second equation of the $\mathcal{D}_c-$form,

$$\begin{bmatrix} \mathcal{L}^c_{21} & \mathcal{L}^c_{22} \end{bmatrix} \begin{bmatrix} \varphi \\ D_3 \end{bmatrix} + \rho = \partial_z D_3. \tag{18}$$

Combining (9) with (18),

$$\begin{bmatrix} \mathcal{L}^c_{11} & \mathcal{L}^c_{12} \\ \mathcal{L}^c_{21} & \mathcal{L}^c_{22} \end{bmatrix} \begin{bmatrix} \varphi \\ D_3 \end{bmatrix} + \begin{bmatrix} 0 \\ \rho \end{bmatrix} = \partial_z \begin{bmatrix} \varphi \\ D_3 \end{bmatrix}. \tag{19}$$

This completes the construction of the $\mathcal{D}_c-$form.

IV. THE INTERNAL CONSISTENNCY OF THE $\mathcal{D}_c-$ AND $\mathcal{S}_c-$FORMS

Theorem: The constructed $\mathcal{D}_c-$ and $\mathcal{S}_c-$forms, Eqs. (19) and (15), respectively, are internally consistent.

Proof: The proof strategy consists of demonstrating that $(\mathcal{D}_c, \mathcal{S}_c)$ is sharply equivalent with $(\mathcal{G}, \mathcal{C})$: The derivation of $(\mathcal{D}_c, \mathcal{S}_c)$ exclusively requires the entirety of $(\mathcal{G}, \mathcal{C})$. Conversely, $(\mathcal{G}, \mathcal{C})$ can exclusively be obtained from the entirety of $(\mathcal{D}_c, \mathcal{S}_c)$.

Consider the $\mathcal{S}_c-$form, Eq. (15). Apply the operator $[\partial_x \;\; \partial_y]$ from the L.H.S.,

$$\begin{bmatrix} \partial_x & \partial_y \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

$$= \begin{bmatrix} \partial_x & \partial_y \end{bmatrix} \begin{bmatrix} \mathcal{A}^c_{11} & \mathcal{A}^c_{12} \\ \mathcal{A}^c_{21} & \mathcal{A}^c_{22} \end{bmatrix} \begin{bmatrix} \varphi \\ D_3 \end{bmatrix}, \tag{20a}$$

$$= \begin{bmatrix} \underbrace{\partial_x \mathcal{A}^c_{11} + \partial_y \mathcal{A}^c_{21}}_{-\mathcal{L}^c_{21}} & \underbrace{\partial_x \mathcal{A}^c_{12} + \partial_y \mathcal{A}^c_{22}}_{-\mathcal{L}^c_{22}} \end{bmatrix} \begin{bmatrix} \varphi \\ D_3 \end{bmatrix}. \tag{20b}$$

In view of definition equations (17), and recognizing the appearance of the operators $-\mathcal{L}^c_{21}$ and $-\mathcal{L}^c_{22}$ in (20b), as indicated,

$$\begin{bmatrix} \partial_x & \partial_y \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = - \begin{bmatrix} \mathcal{L}^c_{21} & \mathcal{L}^c_{22} \end{bmatrix} \begin{bmatrix} \varphi \\ D_3 \end{bmatrix}. \tag{21}$$

Considering (18); i.e., the second equation of the $\mathcal{D}_c$−form, the R.H.S. of (21) equals $\rho - \partial_z D_3$. Thus, (21) results in,

$$\partial_x D_1 + \partial_y D_2 = \rho - \partial_z D_3. \tag{22}$$

Or, written more compactly,

$$\nabla \cdot \mathbf{D} = \rho. \tag{23}$$

Fact 1: *The second equation of the $\mathcal{D}_c$−form and both equations of the $\mathcal{S}_c$−form allow reproducing the Maxwell's divergence equation for* $\mathbf{D}$.

Next consider the first equation of the $\mathcal{D}_c$−form, (9), which in view of the definition equations of $\mathcal{L}^c_{11}$ and $\mathcal{L}^c_{12}$, given in Eqs. (8), leads to,

$$\left(-\frac{\varepsilon_{31}}{\varepsilon_{33}}\partial_x - \frac{\varepsilon_{32}}{\varepsilon_{33}}\partial_y\right)\varphi - \frac{1}{\varepsilon_{33}} D_3 = \partial_z \varphi. \tag{24}$$

Multiply both sides of (24) by ε_{33}, and rearrange,

$$D_3 = -\varepsilon_{31}\partial_x\varphi - \varepsilon_{32}\partial_y\varphi - \varepsilon_{33}\partial_z\varphi. \tag{25}$$

Define the electric field components E_i $(i = 1, 2, 3)$,

$$E_1 = -\partial_x \varphi, \tag{26a}$$
$$E_2 = -\partial_y \varphi, \tag{26b}$$
$$E_3 = -\partial_z \varphi, \tag{26c}$$

which are sufficient for the validity of $\nabla \times \mathbf{E} = 0$. Then, (25) leads to,

$$D_3 = \varepsilon_{31} E_1 + \varepsilon_{32} E_2 + \varepsilon_{33} E_3. \tag{27}$$

Fact 2: *The first equation of the $\mathcal{D}_c$−form together with the sufficient conditions for the validity of Maxwell's curl equation for the electric field ($\nabla \times \mathbf{E} = 0$) allow reproducing the third constitutive equation.*

Comment: Thus far Maxwell's equations $\nabla \times \mathbf{E} = 0$ and $\nabla \cdot \mathbf{D} = \rho$ and the third of the constitution equations have been reconstructed. The reconstruction of the first- and the second constitutive equations will complete the proof of the sharp equivalence of the $(\mathcal{D}_c, \mathcal{S}_c)$−forms with the originating governing and constitutive equations $(\mathcal{G}, \mathcal{C})$.

Consider the $\mathcal{S}_c$−form, Eq. (15). Consider the explicit expressions for the operators $\mathcal{A}^c_{ij}$ $(i, j = 1, 2)$, (14). "Unpacking" $\mathcal{A}^c_{ij}$, leads to,

$$\begin{aligned} D_1 \; = \; & -\left(\varepsilon_{11} - \frac{\varepsilon_{13}\varepsilon_{31}}{\varepsilon_{33}}\right)\partial_x\varphi \\ & -\left(\varepsilon_{12} - \frac{\varepsilon_{13}\varepsilon_{32}}{\varepsilon_{33}}\right)\partial_y\varphi + \frac{\varepsilon_{13}}{\varepsilon_{33}} D_3, \end{aligned} \tag{28}$$

$$\begin{aligned} D_2 \; = \; & -\left(\varepsilon_{21} - \frac{\varepsilon_{23}\varepsilon_{31}}{\varepsilon_{33}}\right)\partial_x\varphi \\ & -\left(\varepsilon_{22} - \frac{\varepsilon_{23}\varepsilon_{32}}{\varepsilon_{33}}\right)\partial_y\varphi + \frac{\varepsilon_{23}}{\varepsilon_{33}} D_3. \end{aligned} \tag{29}$$

Rearranging (28) and (29),

$$\begin{aligned} D_1 = & -\varepsilon_{11}\partial_x\varphi - \varepsilon_{12}\partial_y\varphi \\ & + \varepsilon_{13}\left\{ \underbrace{\left[\frac{\varepsilon_{31}}{\varepsilon_{33}}\partial_x + \frac{\varepsilon_{32}}{\varepsilon_{33}}\partial_y\right]}_{=-\mathcal{L}^c_{11}}\varphi + \underbrace{\left[\frac{1}{\varepsilon_{33}}\right]}_{=-\mathcal{L}^c_{12}} D_3 \right\}, \end{aligned} \tag{30}$$

$$\begin{aligned} D_2 = & -\varepsilon_{21}\partial_x\varphi - \varepsilon_{22}\partial_y\varphi \\ & + \varepsilon_{23}\left\{ \underbrace{\left[\frac{\varepsilon_{31}}{\varepsilon_{33}}\partial_x + \frac{\varepsilon_{32}}{\varepsilon_{33}}\partial_y\right]}_{=-\mathcal{L}^c_{11}}\varphi + \underbrace{\left[\frac{1}{\varepsilon_{33}}\right]}_{=-\mathcal{L}^c_{12}} D_3 \right\}, \end{aligned} \tag{31}$$

where the operators $-\mathcal{L}^c_{11}$ and $-\mathcal{L}^c_{12}$ have been identified, as indicated. Simplifying,

$$\begin{aligned} D_1 = & -\varepsilon_{11}\partial_x\varphi - \varepsilon_{12}\partial_y\varphi \\ & - \varepsilon_{13}\underbrace{[\mathcal{L}^c_{11}\varphi + \mathcal{L}^c_{12} D_3]}_{=\partial_z\varphi}, \end{aligned} \tag{32}$$

$$\begin{aligned} D_2 = & -\varepsilon_{21}\partial_x\varphi - \varepsilon_{22}\partial_y\varphi \\ & - \varepsilon_{23}\underbrace{[\mathcal{L}^c_{11}\varphi + \mathcal{L}^c_{12} D_3]}_{=\partial_z\varphi}. \end{aligned} \tag{33}$$

Recognizing the first equation of the $\mathcal{D}_c-$form, Eq. (9), as indicated in (32) and (33),

$$D_1 = -\varepsilon_{11}\partial_x\varphi - \varepsilon_{12}\partial_y\varphi - \varepsilon_{13}\partial_z\varphi, \quad (34a)$$
$$D_2 = -\varepsilon_{21}\partial_x\varphi - \varepsilon_{22}\partial_y\varphi - \varepsilon_{23}\partial_z\varphi. \quad (34b)$$

Together with the stipulated conditions (26),

$$D_1 = \varepsilon_{11}E_1 + \varepsilon_{12}E_2 + \varepsilon_{13}E_3, \quad (35a)$$
$$D_2 = \varepsilon_{21}E_1 + \varepsilon_{22}E_2 + \varepsilon_{23}E_3. \quad (35b)$$

Fact 3: *The two equations of the $\mathcal{S}_c-$form, the first equation of the $\mathcal{D}_c-$form, and the sufficient conditions for the validity of $\nabla\times\mathbf{E}=0$; i.e., $E_1=-\partial_x\varphi$, $E_2=-\partial_y\varphi$, and $E_3=-\partial_z\varphi$, were used to reconstruct the second- and the third constitutive equations.*

Consequently, in virtue of (35) and (27) the complete set of constitutive equations has been reconstructed,

$$\mathbf{D} = \boldsymbol{\varepsilon}\mathbf{E}. \quad (36)$$

The analyses in Sections III and IV complete the proof of the sharp equivalence of the constructed $(\mathcal{D}_c, \mathcal{S}_c)-$forms with the Maxwell's electrostatic equations and the constitutive equations in fully anisotropic and inhomogeneous media. Consequently, postulating the consistency of the Maxwell's equations in the electrostatic limit along with the material realizability conditions, it can be inferred that the constructed $(\mathcal{D}_c, \mathcal{S}_c)-$forms are internally consistent. This completes the proof of existence of the $(\mathcal{D}_c, \mathcal{S}_c)-$forms and the relative proof of their internal consistency.

V. CONCLUSION

Governing $(\mathcal{G})$ and constitutive $(\mathcal{C})$ equations in mathematical physics can be diagonalized ($\mathcal{D}-$form) with respect to any chosen direction in space. The fact that there also exists an accompanying dual supplementary equation ($\mathcal{S}-$form) associated with the $\mathcal{D}-$form is a recent result. This paper focused on the important electrostatic limit, by considering fully-anisotropic and inhomogeneous dielectrics in three spatial dimensions. It completes the $(\mathcal{D}, \mathcal{S})-$treatment of the Maxwell's electrodynamic equations presented in [1] and [2]. Furthermore, it substantiates the discussion in [3] by rigorously discussing the consistency of the derived $(\mathcal{D}, \mathcal{S})-$forms. Selecting the $z-$axis, it was rigorously established that the system of $(\mathcal{G}, \mathcal{C})$ equations can be partitioned into the $(\mathcal{D}, \mathcal{S})-$forms. It was also demonstrated that the $(\mathcal{D}, \mathcal{S})-$forms are sharply equivalent with the $(\mathcal{G}, \mathcal{C})$ equations and thus internally consistent. In carrying out the proofs it was required that diagonal entries of the permittivity matrix $\boldsymbol{\varepsilon}$ must be non-zero, a condition which in virtue of the positive-definiteness of $\boldsymbol{\varepsilon}$ is satisfied. The diagonalization and supplementation of magneto-static fields will be rigorously treated in an upcoming work.

VI. APPENDIX

Diagonalization and Supplementation with respect to the $x-$axis:

A. Operator matrix entries for the $\mathcal{S}_a-$form

$$\mathcal{A}^a_{11} = -\left(\varepsilon_{22} - \frac{\varepsilon_{21}\varepsilon_{12}}{\varepsilon_{11}}\right)\partial_y - \left(\varepsilon_{23} - \frac{\varepsilon_{21}\varepsilon_{13}}{\varepsilon_{11}}\right)\partial_z, \quad (37a)$$

$$\mathcal{A}^a_{12} = \frac{\varepsilon_{21}}{\varepsilon_{11}}, \quad (37b)$$

$$\mathcal{A}^a_{21} = -\left(\varepsilon_{32} - \frac{\varepsilon_{31}\varepsilon_{12}}{\varepsilon_{11}}\right)\partial_y - \left(\varepsilon_{33} - \frac{\varepsilon_{31}\varepsilon_{13}}{\varepsilon_{11}}\right)\partial_z, \quad (37c)$$

$$\mathcal{A}^a_{22} = \frac{\varepsilon_{31}}{\varepsilon_{11}}, \quad (37d)$$

$$\begin{bmatrix} D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} \mathcal{A}^a_{11} & \mathcal{A}^a_{12} \\ \mathcal{A}^a_{21} & \mathcal{A}^a_{22} \end{bmatrix} \begin{bmatrix} \varphi \\ D_1 \end{bmatrix}. \quad (38)$$

B. Operator matrix entries for the $\mathcal{D}_a-$form

$$\mathcal{L}^a_{11} = -\frac{\varepsilon_{12}}{\varepsilon_{11}}\partial_y - \frac{\varepsilon_{13}}{\varepsilon_{11}}\partial_z, \quad (39a)$$

$$\mathcal{L}^a_{12} = -\frac{1}{\varepsilon_{11}}, \quad (39b)$$

$$\mathcal{L}^a_{21} = -\partial_y\mathcal{A}^a_{11} - \partial_z\mathcal{A}^a_{21}, \quad (39c)$$

$$\mathcal{L}^a_{22} = -\partial_y\mathcal{A}^a_{12} - \partial_z\mathcal{A}^a_{22}, \quad (39d)$$

$$\begin{bmatrix} \mathcal{L}^a_{11} & \mathcal{L}^a_{12} \\ \mathcal{L}^a_{21} & \mathcal{L}^a_{22} \end{bmatrix} \begin{bmatrix} \varphi \\ D_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \rho \end{bmatrix} = \partial_x \begin{bmatrix} \varphi \\ D_1 \end{bmatrix}. \quad (40)$$

Diagonalization and Supplementation with respect to the $y-$axis:

C. Operator matrix entries for the $\mathcal{S}_b-$form

$$\mathcal{A}_{11}^b = -\left(\varepsilon_{33} - \frac{\varepsilon_{32}\varepsilon_{23}}{\varepsilon_{22}}\right)\partial_z - \left(\varepsilon_{31} - \frac{\varepsilon_{32}\varepsilon_{21}}{\varepsilon_{22}}\right)\partial_x, \quad (41a)$$

$$\mathcal{A}_{12}^b = \frac{\varepsilon_{32}}{\varepsilon_{22}}, \quad (41b)$$

$$\mathcal{A}_{21}^b = -\left(\varepsilon_{13} - \frac{\varepsilon_{12}\varepsilon_{23}}{\varepsilon_{22}}\right)\partial_z - \left(\varepsilon_{11} - \frac{\varepsilon_{12}\varepsilon_{21}}{\varepsilon_{22}}\right)\partial_x, \quad (41c)$$

$$\mathcal{A}_{22}^b = \frac{\varepsilon_{12}}{\varepsilon_{22}}, \quad (41d)$$

$$\begin{bmatrix} D_3 \\ D_1 \end{bmatrix} = \begin{bmatrix} \mathcal{A}_{11}^b & \mathcal{A}_{12}^b \\ \mathcal{A}_{21}^b & \mathcal{A}_{22}^b \end{bmatrix} \begin{bmatrix} \varphi \\ D_2 \end{bmatrix}. \quad (42)$$

D. Operator matrix entries for the $\mathcal{D}_b-$form

$$\mathcal{L}_{11}^b = -\frac{\varepsilon_{23}}{\varepsilon_{22}}\partial_z - \frac{\varepsilon_{21}}{\varepsilon_{22}}\partial_x, \quad (43a)$$

$$\mathcal{L}_{12}^b = -\frac{1}{\varepsilon_{22}}, \quad (43b)$$

$$\mathcal{L}_{21}^b = -\partial_z\mathcal{A}_{11}^b - \partial_x\mathcal{A}_{21}^b, \quad (43c)$$

$$\mathcal{L}_{22}^b = -\partial_z\mathcal{A}_{12}^b - \partial_x\mathcal{A}_{22}^b, \quad (43d)$$

$$\begin{bmatrix} \mathcal{L}_{11}^b & \mathcal{L}_{12}^b \\ \mathcal{L}_{21}^b & \mathcal{L}_{22}^b \end{bmatrix} \begin{bmatrix} \varphi \\ D_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \rho \end{bmatrix} = \partial_y \begin{bmatrix} \varphi \\ D_2 \end{bmatrix}. \quad (44)$$

Diagonalization and Supplementation with respect to the $z-$axis:

E. Operator matrix entries for the $\mathcal{S}_c-$form

$$\mathcal{A}_{11}^c = -\left(\varepsilon_{11} - c\frac{\varepsilon_{13}\varepsilon_{31}}{\varepsilon_{33}}\right)\partial_x - \left(\varepsilon_{12} - \frac{\varepsilon_{13}\varepsilon_{32}}{\varepsilon_{33}}\right)\partial_y, \quad (45a)$$

$$\mathcal{A}_{12}^c = \frac{\varepsilon_{13}}{\varepsilon_{33}}, \quad (45b)$$

$$\mathcal{A}_{21}^c = -\left(\varepsilon_{21} - \frac{\varepsilon_{23}\varepsilon_{31}}{\varepsilon_{33}}\right)\partial_x - \left(\varepsilon_{22} - \frac{\varepsilon_{23}\varepsilon_{32}}{\varepsilon_{33}}\right)\partial_y, \quad (45c)$$

$$\mathcal{A}_{22}^c = \frac{\varepsilon_{23}}{\varepsilon_{33}}, \quad (45d)$$

$$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} \mathcal{A}_{11}^c & \mathcal{A}_{12}^c \\ \mathcal{A}_{21}^c & \mathcal{A}_{22}^c \end{bmatrix} \begin{bmatrix} \varphi \\ D_3 \end{bmatrix}. \quad (46)$$

F. Operator matrix entries for the $\mathcal{S}_c-$form

$$\mathcal{L}_{11}^c = -\frac{\varepsilon_{31}}{\varepsilon_{33}}\partial_x - \frac{\varepsilon_{32}}{\varepsilon_{33}}\partial_y, \quad (47a)$$

$$\mathcal{L}_{12}^c = -\frac{1}{\varepsilon_{33}}, \quad (47b)$$

$$\mathcal{L}_{21}^c = -\partial_x\mathcal{A}_{11}^c - \partial_y\mathcal{A}_{21}^c, \quad (47c)$$

$$\mathcal{L}_{22}^c = -\partial_x\mathcal{A}_{12}^c - \partial_y\mathcal{A}_{22}^c, \quad (47d)$$

$$\begin{bmatrix} \mathcal{L}_{11}^c & \mathcal{L}_{12}^c \\ \mathcal{L}_{21}^c & \mathcal{L}_{22}^c \end{bmatrix} \begin{bmatrix} \varphi \\ D_3 \end{bmatrix} + \begin{bmatrix} 0 \\ \rho \end{bmatrix} = \partial_z \begin{bmatrix} \varphi \\ D_3 \end{bmatrix}. \quad (48)$$

REFERENCES

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[3] A. R. Baghai-Wadji, ‘‘3-D electrostatic charge distribution on finitely thick busbars in micro acoustic devices: Combined regularization in the near- and far-field,’’ IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control (UFFC), vol. 62, no. 6, June 2015, pp. 1132-1144.

Alireza Baghai-Wadji: For a biography, please refer to the accompanying paper, [1], in this issue.