A Novel AGEI Solution of Parabolic Equation for EM Scattering Problems

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Abstract **─** Parabolic equation (PE) has been widely used for EM propagating and scattering problems for its high efficiency. By using the finite differential (FD) method, the calculation can be taken in a series of transverse planes in a marching manner. In this paper, the alternating group explicit iterative (AGEI) method is applied to solve the alternating direction implicit based parabolic equation (ADI-PE). As a result, the CPU time can be further saved when compared with the CN-PE and ADI-PE methods. Numerical results are shown for demonstrating the accuracy and efficiency.

Index Terms **─** Alternating group explicit iterative (AGEI) method, electromagnetic scattering, parabolic equation.

I. INTRODUCTION

The rigorous numerical methods, such as the Finite Difference Time Domain (FDTD), the Method of Moment (MoM) and the Finite Element Method (FEM) are widely used for electromagnetic analysis. However, a huge number of computational resources are needed with the number of unknowns increasing, thus the efficiency will become low. On the other hand, the high frequency methods have low accuracy with few computational resources. The parabolic equation (PE) [1-11] is an approximation of the wave equation, which can give encouraging accuracy with limited computational resources. Therefore, the PE method takes a bridge between rigorous numerical methods and high frequency methods.

By using the finite differential (FD) method along the paraxial direction, the calculation can be taken plane by plane. As a result, the computational resources can be saved largely. There are several methods that have been used to solve the parabolic equation, such as the Split-Step Fourier Transform (SSFT) [1], the Crank-Nicolson (CN) [2-4], the Alternate Direction Implicit (ADI) [5-7], and the Alternate Group Explicit (AGE) [8-9, 20]. Moreover, several kinds of high-order approximations have been introduced to get the wide-angle prosperities [10-11, 19]. Furthermore, some other numerical algorithms, including the Method of Moment (MoM) [12-13], the Geometrical Theory of Diffraction (GTD) [14], and other techniques [15-16] are combined with the PE method, which broaden the application of the PE method. It should be noted here that the PE can only model the object does not undergo large changes in direction. Moreover, the objects, which are small compared to the wavelength, cannot be simulated by PE method. Since the creeping waves cannot be captured by the PE method.

The implicit FD methods are widely used for their simplicity, stability and efficiency [2-7]. The CN scheme is one of the most popular implicit FD methods [2-4]. Nevertheless, a huge computer resource is required with the electrical size of the targets increasing. Then the ADI method is proposed to accelerate the calculation of the PE method [8-9]. By using the ADI scheme, the fields in any transverse plane can be calculated line by line, which reduce the computation complexity by solving the unknowns in one dimension. On the other hand, the explicit FD methods can achieve high computational efficiency, but may result in instability. Therefore, the development of methods with both the high efficiency and stability has a practical significance.

In this paper, the ADI-based parabolic equations are derived firstly. In this way, a series of tridiagonal matrix equations are needed to be solved in each transverse plane. Then the alternating group explicit iterative (AGEI) method [17-18] is used to solve these equations. As a result, all the unknown fields in each transverse plane can be computed explicitly without solving any matrix equation. Therefore, the CPU time can be saved significantly than the traditional ADI-PE method. Several numerical examples are given to demonstrate the accuracy and efficiency of the proposed AGEI-PE method.

II. THEORY AND FORMULATIONS

A. ADI-PE method

Considering a PEC object illuminated by a plane wave in free space, a reduced function associated with a field component is introduced as follows:

$$u(x,y,z) = e^{-ikx}E(x,y,z)\,, \tag{1}$$

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where k is the wave number.

The standard forward parabolic equation can be obtained via substituting Equation (1) into the wave equation and factorization,

$$\frac{\partial u}{\partial x} = -ik(1-\sqrt{Q})u\ , \tag{2}$$

where Q is the pseudo-differential operator, which can be expressed as:

$$Q = \frac{1}{k^2}\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)+n^2\ . \tag{3}$$

Equation (2) is one order differential along the x axis. The FD scheme along the paraxial direction can be easily applied, and the calculation can be taken plane by plane.

The ADI-PE can be derived directly from the CN-PE as indicated in [5-6]. The matrix form of the ADI-PE is:

$$\begin{bmatrix} -\frac{ir_y}{4k} & 1+\frac{ir_y}{2k} & -\frac{ir_y}{4k}\end{bmatrix}\begin{bmatrix} u_{j-1,k}^{n+1/2} \\ u_{j,k}^{n+1/2} \\ u_{j+1,k}^{n+1/2}\end{bmatrix} = \begin{bmatrix} \frac{ir_z}{4k} & 1-\frac{ir_z}{2k} & \frac{ir_z}{4k}\end{bmatrix}\begin{bmatrix} u_{j,k-1}^{n} \\ u_{j,k}^{n} \\ u_{j,k+1}^{n}\end{bmatrix}, \tag{4}$$

$$\begin{bmatrix} -\frac{ir_z}{4k} & 1+\frac{ir_z}{2k} & -\frac{ir_z}{4k}\end{bmatrix}\begin{bmatrix} u_{j,k-1}^{n+1} \\ u_{j,k}^{n+1} \\ u_{j,k+1}^{n+1}\end{bmatrix} = \begin{bmatrix} \frac{ir_y}{4k} & 1-\frac{ir_y}{2k} & \frac{ir_y}{4k}\end{bmatrix}\begin{bmatrix} u_{j-1,k}^{n+1/2} \\ u_{j,k}^{n+1/2} \\ u_{j+1,k}^{n+1/2}\end{bmatrix}, \tag{5}$$

where $r_y = \Delta x / \Delta y^2$, $r_z = \Delta x / \Delta z^2$, $u_{j,k}^n$ denotes the reduced scattered field at the point of $(n\Delta x, j\Delta y, k\Delta z)$.

As a result, there is an intermediate plane introduced between the n *th* and n+1 *th* planes with less unknowns. Moreover the scattered fields can be calculated line by line. Finally, a series of tridiagonal matrices are to be solved by the ADI-PE method in each transverse plane.

B. AGEI solution of ADI-PE

Both the Equations (4) and (5) are tridiagonal matrix equations. Therefore, they can be solved by using the AGEI scheme. Suppose the impedance matrix can be written as:

$$A = \begin{bmatrix} a & b & & & & \\ b & a & b & & & \\ & b & a & b & & \\ & & & \ddots & & \\ & & & b & a & b \\ & & & & b & a \end{bmatrix}. \tag{6}$$

It should be noted that $a=1+\frac{ir_y}{2k}$, $b=-\frac{ir_y}{4k}$ for Equation (4), and $a=1+\frac{ir_z}{2k}$, $b=-\frac{ir_z}{4k}$ for Equation (5). Then the impedance matrix is split into two parts, which can be expressed as:

$$A = G_1 + G_2$$
$$= \begin{bmatrix} \frac{a}{2} & & & & & \\ & \frac{a}{2} & b & & & \\ & b & \frac{a}{2} & & & \\ & & & \ddots & & \\ & & & & \frac{a}{2} & b \\ & & & & b & \frac{a}{2} \end{bmatrix} + \begin{bmatrix} \frac{a}{2} & b & & & & \\ b & \frac{a}{2} & & & & \\ & & \ddots & & & \\ & & & \frac{a}{2} & b & \\ & & & b & \frac{a}{2} & \\ & & & & & \frac{a}{2} \end{bmatrix}, \tag{7}$$

Then the impedance matrix equation to be solved can be simplified as:

$$Au = (G_1+G_2)u = f\ . \tag{8}$$

Furthermore, the following equivalent matrix equations can be obtained:

$$\left(\lambda I + G_1\right)u = \left(\lambda I - G_2\right)u + f\ , \tag{9}$$

or

$$\left(\lambda I + G_2\right)u = \left(\lambda I - G_1\right)u + f\ . \tag{10}$$

At last, an alternating group explicit iterative (AGEI) method is applied to (8),

$$\begin{cases} \left(\lambda I + G_1\right)v^{k+1} = \left(\lambda I - G_2\right)u^k + f \\ \left(\lambda I + G_2\right)u^{k+1} = \left(\lambda I - G_1\right)v^{k+1} + f \end{cases}, \tag{11}$$

where $k = 0,1,2...$, λ is the Peaceman-Rachford constant.

More specifically, the iterative method can be rewritten as:

$$\begin{cases} v^{k+1} = \left(\lambda I + G_1\right)^{-1}\left[\left(\lambda I - G_2\right)u^k + f\right] \\ u^{k+1} = \left(\lambda I + G_2\right)^{-1}\left[\left(\lambda I - G_1\right)v^{k+1} + f\right] \end{cases}. \tag{12}$$

Substitute v^{k+1} into the second matrix equation, then the following result can be obtained:

$$\begin{aligned} u^{k+1} &= \\ &(\lambda I+G_2)^{-1}\left(\lambda I-G_1\right)(\lambda I+G_1)^{-1}\left(\lambda I-G_2\right)u^k+R \\ &=(\lambda I+G_2)^{-1}\left(\lambda I-G_1\right)(\lambda I+G_1)^{-1}\left(\lambda I-G_2\right)u^k+ \\ &[(\lambda I+G_2)^{-1}\left(\lambda I-G_1\right)(\lambda I+G_1)^{-1}+(\lambda I+G_2)^{-1}]f \end{aligned} \quad (13)$$

For the first line of u^{k+1}, the calculation can be taken as:

$$\begin{aligned} u_1^{k+1} = & t^2\left[(\lambda^2-\frac{a^2}{4})^2+b^2(b^2+a\lambda-\frac{a^2}{2})\right]u_1^k \\ & +t^2\left(-2\lambda b(\lambda^2-\frac{a^2}{4})\right)u_2^k+2t^2\lambda b^2(\lambda-\frac{a}{2})u_3^k \\ & -2t^2\lambda b^3u_4^k+r_1 \end{aligned} \quad (14)$$

where

$$\begin{aligned} r_1 = & (\lambda-\frac{a}{2})[\left(\lambda+\frac{a}{2}\right)^2-b^2]tf_1 \\ & -b(\lambda^2-\frac{a^2}{4}+\mathrm{b}^2)tf_2+2\lambda b^2tf_3 \end{aligned}, \quad (15)$$

$$t=\frac{1}{\left(\lambda+\frac{a}{2}\right)^2-b^2}. \quad (16)$$

For the second line of u^{k+1}, the calculation can be taken as:

$$\begin{aligned} u_2^{k+1} = & (-b)\frac{2\lambda(\lambda^2-\frac{a^2}{4})(\lambda+\frac{a}{2})+2\,\mathrm{ab}^2\,\lambda}{\lambda+\frac{a}{2}}t^2u_1^k \\ & +\frac{(\lambda-\frac{a}{2})[(\lambda+\frac{a}{2})^2[\lambda^2-\frac{a^2}{4}+2\,\mathrm{b}^2]-\mathrm{b}^2]}{\lambda+\frac{a}{2}}t^2u_2^k \\ & -2\lambda b\left(\lambda^2-\frac{a^2}{4}\right)t^2u_3^k\,2\lambda b^2\left(\lambda+\frac{a}{2}\right)t^2u_4^k+r_2 \end{aligned}, \quad (17)$$

where

$$\begin{aligned} r_2 = & \left[(\lambda-\frac{a}{2})\frac{\left(\lambda+\frac{a}{2}\right)^2-b^2}{\lambda+\frac{a}{2}}(-bt)-(\lambda-\frac{a}{2})\right]f_1 \\ & +\left[(\lambda+\frac{a}{2})(\lambda^2-\frac{a^2}{4}+b^2)-b\right]tf_2 \\ & +(\lambda+\frac{a}{2})(-2\lambda b)tf_3 \end{aligned}. \quad (18)$$

Similarly, for i=3, 5, 7, 9, … , the solution can be expressed as:

$$\begin{aligned} u_i^{k+1} = & 2\lambda b^2t^2\left(\lambda+\frac{a}{2}\right)u_{i-2}^k-2\lambda bt^2\left(\lambda^2-\frac{a^2}{4}\right)u_{i-1}^k \\ & +t^2(\lambda^2-\frac{a^2}{4}+b^2)^2u_i^k-2\lambda bt^2(\lambda^2-\frac{a^2}{4}+b^2)u_{i+1}^k \\ & +2\lambda b^2t^2(\lambda-\frac{a}{2})u_{i+2}^k+r_i \end{aligned}, \quad (19)$$

$$\begin{aligned} u_{i+1}^{k+1} = & -2\lambda b^3t^2u_{i-2}^k-2\lambda bt^2(\lambda^2-\frac{a^2}{4}+b^2)u_{i-1}^k \\ & +(\lambda^2-\frac{a^2}{4}+b^2)^2t^2u_i^k-2t^2\lambda b\left(\lambda^2-\frac{a^2}{4}\right)u_{i+1}^k \\ & +2\lambda b^2t^2\left(\lambda+\frac{a}{2}\right)u_{i+2}^k+r_{i+1} \end{aligned}, \quad (20)$$

where

$$\begin{aligned} r_i = & (\lambda+\frac{a}{2})(-2\lambda bt)\,f_{i-2}+\left[(\lambda+\frac{a}{2})(\lambda^2-\frac{a^2}{4}+\mathrm{b}^2)\right]tf_{i-1} \\ & -\left(b(\lambda^2-\frac{a^2}{4}+\mathrm{b}^2)-\lambda-\frac{a}{2}\right)tf_i+\left(2\lambda b^2-b\right)tf_{i+1} \end{aligned}, \quad (21)$$

$$\begin{aligned} r_{i+1} = & 2\lambda b^2tf_{i-2}-\left[b(\lambda^2-\frac{a^2}{4}+\mathrm{b}^2)+(\lambda-\frac{a}{2})\right]tf_{i-1} \\ & +\left[(\lambda+\frac{a}{2})(\lambda^2-\frac{a^2}{4}+\mathrm{b}^2)-\mathrm{b}\right]tf_i \\ & +(\lambda+\frac{a}{2})(-2\lambda b)tf_{i+1} \end{aligned}. \quad (22)$$

It should be noted that the Peaceman-Rachford constant is set to be 0.5 for all the numerical results.

C. Implementation aspects

The three scalar parabolic equations of x, y, z, directions are coupled through the inhomogeneous boundary conditions. For the conducting targets, the tangential component of the total field equals zero on the surface of the scattering target. Moreover, the divergence-free condition is used for the unicity [2]. In each transverse plane, the perfectly matched layer (PML) is introduced to truncate the computational domain. The computation begins before the scattering target and stops beyond it. Finally, the scattering properties can be obtained by applying the near-far field conversion. Furthermore, the full bistatic RCS result are calculated by several rotating PE runs.

Then the RCS in direction (θ,ϕ) along polarization $\boldsymbol{t}$ can be written as:

$$\begin{aligned} & \sigma_t(\theta,\phi)= \\ & \frac{k^2\cos^2\theta}{\pi}\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\boldsymbol{u}e^{ikx_0}\cdot\boldsymbol{t}e^{-ik\sin\theta(y\cos\phi+z\sin\phi)}dydz\right|^2 \end{aligned}, \quad (23)$$

where $\boldsymbol{u}$ is the reduced scattered fields in the last transverse plane for a specified frequency.

III. NUMERICAL RESULTS

At first, the electromagnetic scattering from a PEC cylinder with 5 m radius and 6 m height is considered at the frequency of 300 MHz. The paraxial direction is along the x axis and the incident angle is fixed at $\theta_{inc} = 90°$, $\phi_{inc} = 0°$. There are totally 60 transverse planes to be calculated with 150×150 nodes in each transverse plane. The range steps are set to be 0.1 m. As shown in Fig. 1, the bistatic RCS results are compared between the MoM accelerated by the multilevel fast multipole method (MLFMM) and the proposed AGEI-PE method. There is a good agreement between them. Moreover, as shown in Table 1, both the comparisons of both the memory requirement and the CPU time are made among the MoM, CN-PE, ADI-PE and AGEI-PE methods. It can be seen that higher efficiency can be achieved for the proposed AGEI-PE method when compared with other methods.

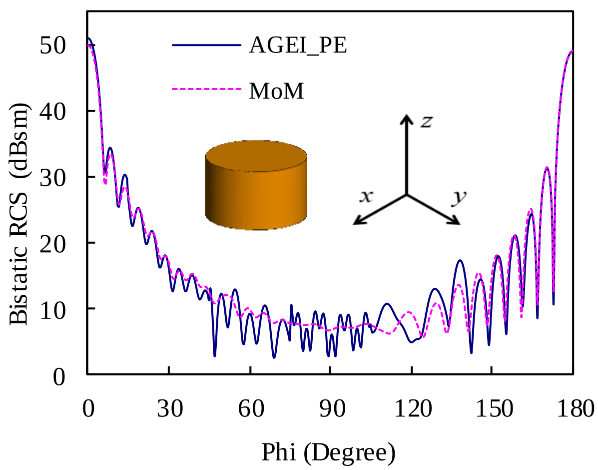

Fig. 1. Bistatic RCS result for the PEC cylinder.

Table 1: Comparisons of computational resources among the MoM, CN-PE, ADI-PE and AGEI-PE methods for the PEC cylinder

	Memory (MB)	CPU Time (s)
MoM	562	3958
CN-PE	515	475
ADI-PE	105	273
AGEI-PE	91	120

Secondly, the analysis is taken for a PEC block at the frequency of 300 MHz with the length of 8 m. The incident angle is fixed at $\theta_{inc} = 90°$, $\phi_{inc} = 0°$. In this example, the range steps are chosen to be 0.1 m. As a result, there are 40 transverse planes to be calculated with 150×150 nodes in each transverse plane. As shown in Fig. 2, the bistatic RCS curves of the PEC block are compared between the MoM accelerated by MLFMM and the proposed AGEI-PE method. There is a good agreement between them. Additionally, as shown in Table 2, the computational resources are compared among the MoM, CN-PE, ADI-PE and AGEI-PE methods.

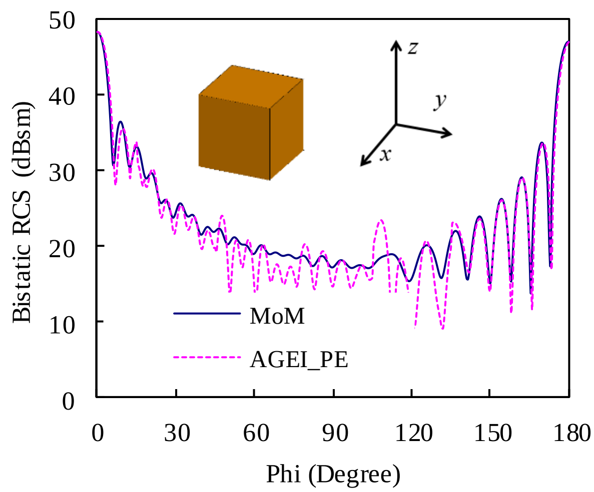

Fig. 2. Bistatic RCS result for the PEC block.

Table 2: Comparisons of computational resources among the MoM, CN-PE, ADI-PE and AGEI-PE methods for the PEC clock

	Memory (MB)	CPU Time (s)
MoM	667	7648
CN	559	586
ADI	127	348
AGEI	119	147

At last, a complicated model is considered, an aircraft at the frequency of 5 GHz. The incident angle is fixed at $\theta_{inc} = 90°$, $\phi_{inc} = 0°$. There are 167 transverse planes to be calculated with the range steps of 0.06 m and 100×100 nodes in each transverse plane. As shown in Fig. 3, the full bistatic RCS results are given. It can be found that the proposed AGEI-PE method can be used as an efficient tool to analyze the electromagnetic scattering from arbitrary structures.

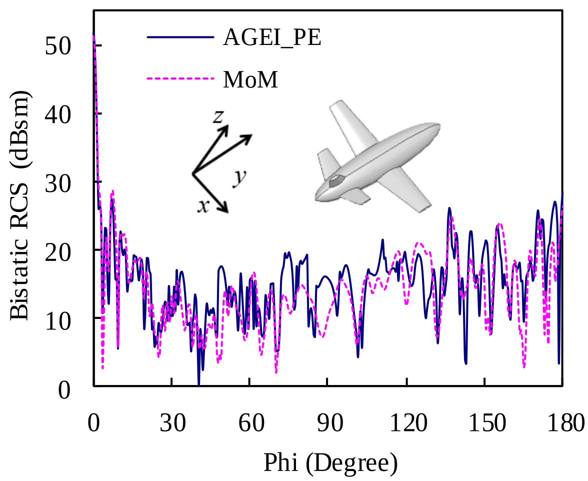

Fig. 3. Bistatic RCS result for the PEC aircraft.

IV. CONCLUSION

An AGEI solution of ADI-PE is proposed in the paper. By splitting the tridiagonal matrix into two parts and proper transformation, the matrix equations of ADI-PE method can be solved explicitly. Moreover, the proposed AGEI-PE method is easily to be paralleled. In this way, high computational efficiency can be achieved with encouraging accuracy. Numerical results are given to demonstrate the accuracy and efficiency of the proposed AGEI-PE scheme.

REFERENCES

[1] R. Janaswamy, "A curvilinear coordinate-based split-step parabolic equation method for propagation predictions over terrain," *IEEE Trans. Antennas and Propagation*, vol. 46, no. 7, pp. 1089-1098, 1998.

[2] A. A. Zaporozhets and M. F. Levy, "Bistatic RCS calculations with the vector parabolic equation method," *IEEE Trans. Antennas and Propagation*, vol. 47, no. 11, 1999.

[3] Z. He and R. S. Chen, "Fast analysis of wide-band scattering from electrically large targets with time-domain parabolic equation method," *Computer Physics Communication*, vol. 200, pp. 139-146, 2016.

[4] M. F. Levy, "Parabolic equation methods for electromagnetic wave propagation," London: *The Institution of Electrical Engineers*, 2000.

[5] R. Martelly and R. Janaswamy, "Modeling radio transmission loss in curved, branched and rough-walled tunnels with the ADI-PE method," *IEEE Trans. Antennas and Propagation*, vol. 58, no. 6, pp. 2037-2045, 2010.

[6] Z. He, Z. H. Fan, D. Z. Ding, and R. S. Chen, "Efficient radar cross-section computation of electrically large targets with ADI-PE method," *Electronics Letters*, vol. 51, no. 4, pp. 360-362, 2015.

[7] Z. He, Z. H. Fan, D. Z. Ding, and R. S. Chen, "GPU-accelerated ADI-PE method for the analysis of EM scatterings," *Electronics Letters*, vol. 51, no. 21, pp. 1652-1654, 2015.

[8] D. J. Evans and A. R. B. Abdullah, "Group explicit method parabolic equations," *Inter. J. Computer Math*., vol. 14, pp. 73-105, 1983.

[9] D. J. Evans and M. S. Sahimi, "The solution of nonlinear parabolic partial differential equations by the alternating group explicit (AGE) method," *Computer Methods in Applied Mechanics and Engineering*, vol. 84, no. 1, pp. 15-42, 2009.

[10] Z. He and R. S. Chen, "A wide-angle ADI-PE method for EM scattering from electrically large targets," *Electromagnetics*, vol. 36, iss. 2, pp. 94-101, 2016.

[11] M. D. Collins, "A split-step pad solution for the parabolic equation method," *J. Acoust. Soc. Am*., vol. 93, iss. 4, pp. 1736-1742, 1993.

[12] Z. He, Z. H. Fan, D. Z. Ding, and R. S. Chen, "A vector parabolic equation method combined with MLFMM for scattering from a cavity," *Applied Computational Electromagnetics Society Journal*, vol. 30, no. 5, pp. 496-502, 2015.

[13] Z. He, T. Su, and R. S. Chen, "Vector parabolic equation method for the scattering from PEC objects in half-space," *Applied Computational Electromagnetics Society Journal*, vol. 30, no. 8, pp. 877-883, 2015.

[14] R. Janaswamy, "Path loss predictions in the presence of buildings on flat terrain: A 3D vector parabolic equation approach," *IEEE Trans. Antennas Propag*., vol. 51, no. 8, pp. 1716-1729, 2003.

[15] Z. He and R. S. Chen, "A vector meshless parabolic equation method for three-dimensional electromagnetic scatterings," *IEEE Transactions on Antennas and Propagation*, vol. 63, no. 6, pp. 2595-2603, 2015.

[16] Z. He, Z. H. Fan, and R. S. Chen, "Spectral element method based parabolic equation for EM scattering problems," *Waves in Random and Complex Media*, vol. 26, iss. 1, pp. 80-88, 2016.

[17] N. Bildik and G. Avci, "On the solution of Poisson equation by alternating group explicit iterative method," *Applied Mathematics and Computation*, vol. 166, no. 2, pp. 411-425, 2005.

[18] R. B. Kellogg, "An alternating direction method for operator equations," *SIAM J. Soc. Indust. Appl. Math*., vol. 12, no. 4, pp. 848-854, 1964.

[19] Z. He and R. S. Chen, "A wide-angle ADI-PE method for EM scattering from electrically large targets," *Electromagnetics*, vol. 36, iss. 2, pp. 94-101, 2016.

[20] Z. He and R. S. Chen, “A novel parallel parabolic equation method for electromagnetic scatterings,” *IEEE Transactions on Antennas and Propagation*, vol. 64, no. 11, pp. 4777-4784, 2016.

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