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# Thermoelastic interaction in a viscoelastic functionally graded half-space under three-phase-lag model

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This paper aims at studying the thermo-viscoelastic interaction in a functionally graded, infinite, Kelvin–Voigt-type viscoelastic, thermally conducting medium due to the presence of periodically varying heat sources. Three-phase-lag thermoelastic model, Green–Naghdi model II (i.e. the model which predicts thermoelasticity without energy dissipation) and Green–Naghdi model III (i.e. the model which predicts thermoelasticity with energy dissipation) are employed to study thermomechanical coupling, thermal and mechanical relaxation effects. In the absence of mechanical relaxations (viscous effect), the results for various generalised theories of thermoelasticity may be obtained as particular cases. The governing equations are expressed in Laplace–Fourier double transform domain and are solved in that domain. The inversion of the Fourier transform is carried out using residual calculus, where the poles of the integrand are obtained numerically in the complex domain by using Laguerre's method and the inversion of the Laplace transform is done numerically using a method based on the Fourier series expansion technique. The numerical estimates of the thermal displacement, temperature, stress and strain are obtained for a hypothetical material. A comparison of the results for different theories is presented and the effect of viscosity is also shown and the effect of non-homogeneity is also seen for different values of the non-homogeneity parameter.

**Keywords:** generalised thermoelasticity; three-phase-lag thermoelastic model; functionally graded materials; Kelvin–Voigt model; periodically varying heat sources

## 1. Introduction

Linear viscoelasticity has been an important area of research since the period of Maxwell, Boltzman, Voigt and Kelvin. Valuable information regarding linear viscoelasticity theory may be obtained in the books of Gross (1953), Staverman and Schwarzl (1956), Alfery and Gurnee (1956), Ferry (1970), Bland (1960) and Lakes (1998). Many researchers like Biot (1954, 1955), Gurtin and Sternberg (1962), Iiioushin and Pobedria (1970), Tanner (1988) have contributed notably on thermoviscoelasticity. Freudenthal (1954) has pointed out that most of the solids, when subjected to dynamic loading, exhibit viscous effects.

The Kelvin–Voigt model is one of the macroscopic mechanical models often used to describe the viscoelastic behaviour of a material. The model represents the delayed elastic response subjected to stress when the deformation is time-dependent but recoverable. The dynamic interaction of thermal and mechanical fields in solids has great

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practical applications in modern aeronautics, astronautics, nuclear reactors and high-energy accelerators, for example.

Biot (1956) formulated the coupled thermoelasticity theory to eliminate the paradox inherent in the classical uncoupled theory that elastic deformation has no effect on the temperature. The field equations for both the theories are of a mixed parabolic-hyperbolic type, which predict infinite speeds for thermoelastic signals, contrary to physical observations. Hetnarski and Ignaczak (1999) examined five generalisations to the coupled theory of thermoelasticity. The first generalisation is due to Lord and Shulman (1967) who formulated the generalised thermoelasticity theory involving one relaxation time. This theory is referred to as L–S theory or extended thermoelasticity theory (ETE) in which the Maxwell–Cattaneo law replaces the Fourier law of heat conduction by introducing a single parameter that acts as the relaxation time. The second generalisation to the coupled theory of thermoelasticity is due to Green and Lindsay (1972), called G–L theory or the temperature-rate dependent theory (TRDTE), which involves two relaxation times. Problems concerning these generalised theories such as ETE and TRDTE have been studied by Chandrasekharaiah (1986), Ignaczak (1989). Muller (1971) proposed an entropy production inequality that led to restrictions on a class of constitutive equations. A generalisation of this inequality was developed by Green and Laws (1972). Green and Lindsay obtained a modified version of the constitutive equations. These equations were independently obtained by Suhubi (1975). For a review, works of Ignaczak (1989) may be mentioned where presentation of the two theories and some important results are achieved in this field.

The third generalisation to the coupled theory of thermoelasticity is known as low-temperature thermoelasticity, introduced by Hetnarski and Ignaczak (1996), called the H–I theory. This model is characterised by a system of nonlinear field equations. Low-temperature nonlinear models of heat conduction that predict wave-like thermal signals and which are supposed to hold at low temperatures have also been proposed and studied in some works by Kosinski (1989) and Kosinski and Cimmelli (1997).

The fourth generalisation to the coupled theory is concerned with the thermoelasticity theory without energy dissipation (TEWOED) introduced by Green and Naghdi (1991, 1993), referred to as G–N theory of type II in which the classical Fourier law is replaced by a heat flux rate-temperature gradient relation. The heat transport equation does not involve a temperature-rate term and as such this model admits undamped thermoelastic waves in thermoelastic material. In the context of linearised version of this theory (1993), the theorem on uniqueness of solutions has been established by Chandrasekharaiah (1996a, 1996b). The fourth generalisation of the thermoelasticity theory developed by Green and Naghdi also involves a heat conduction law, which includes the conventional law and one that involves the thermal displacement gradient among the constitutive variables. This model is referred to as the GN model III which involves dissipation of energy in general and admits damped thermoelastic waves. Taheri et al. (2005) have employed Green–Naghdi theories of type II and type III to study the thermal and mechanical waves in an annulus domain. Mallik and Kanoria (2007) have studied one-dimensional thermoelastic disturbances in an isotropic functionally graded medium in the context of generalised thermoelasticity without energy dissipation (TEWOED). Problems concerning these theory have been studied by many authors, such as Bandyopadhyay and Roychoudhuri (2005), Roychoudhuri and Bandyopadhyay (2005), Kar and Kanoria (2009b), Banik, Mallik, and Kanoria (2007, 2009), Roychoudhuri and Dutta (2005) and Mallik and Kanoria (2008, 2009).

The fifth generalisation to the thermoelasticity theory is known as the dual phase-lag model developed by Tzou (1995) and Chandrasekharaiah (1998). Tzou (1995) considered microstructural effects into the delayed response in time in the macroscopic formulation by taking into account that the increase of the lattice temperature is delayed due to phonon-electron interactions on the macroscopic level. A macroscopic lagging (or delayed) response between the temperature gradient and the heat flux vector seems to be a possible outcome due to such progressive interactions. Tzou (1995) introduced two-phase lags to both the heat flux vector and the temperature gradient and considered a constitutive equation to describe the lagging behaviour in the heat conduction in solids. Here, the classical Fourier law is replaced by an approximation to modification of the law with two different translations for the heat flux vector and the temperature gradient.

Recently, Roychoudhuri (2007) had established a generalised mathematical model of a coupled thermoelasticity theory that includes three-phase lags in the heat flux vector, the temperature gradient and in the thermal displacement gradient. The more general model established reduces to the previous models as special cases. According to this model, the heat flux has been modified as $\vec{q}(P, t+\tau_q) = -[K\vec{\nabla}\theta(P, t+\tau_T) + K^*\vec{\nabla}v(P, t+\tau_v)]$, where $\vec{\nabla}v\,(\dot{v}=\theta)$ is the thermal displacement gradient and $K^*$ is the additional material constant. To study some practical relevant problems, particularly in heat transfer problems involving very short time intervals and in the problems of very high heat fluxes, the hyperbolic equation gives significantly different results than the parabolic equation. According to this phenomenon, the lagging behaviour in the heat conduction in solids should not be ignored, particularly when the elapsed times during a transient process are very small, say, about $10^{-7}$ s or if the heat flux is very much high. The three-phase-lag model is very useful in the problems of nuclear boiling, exothermic catalytic reactions, phonon-electron interactions, phonon scattering etc., where the delay time $\tau_q$ captures the thermal wave behaviour (a small-scale response in time), the phase-lag $\tau_T$ captures the effect of phonon-electron interactions (a microscopic response in space), the other delay time $\tau_v$ is effective since, in the three-phase-lag model, the thermal displacement gradient is considered as a constitutive variable, whereas in the conventional thermoelasticity theory, temperature gradient is considered as a constitutive variable.

However, over the last few decades various problems in solid mechanics are being studied where the elastic coefficients are no longer constants but are a function of position. The investigations result from the fact that the idea of nonhomogeneity in elastic coefficients is not all hypothetical, but more realistic. Elastic properties in soil may vary considerably with positions. The Earth's crust itself is nonhomogeneous. Beside these, some structural materials such as functionally graded materials (FGMs) have distinct nonhomogeneous character. For example, in graded composite materials, graded regions are treated as a series of perfectly bonded composite layers, each layer being assigned slightly different properties. In FGMs, the material properties vary gradually with location within the body. In many applications, FGMs are found to be better substitutes for conventional homogeneous materials. Among several uses of FGMs, one such is the use of FGMs in automotive brakes and clutches (Lee & Barber, 1993) where the effect of frictional heat generation is the subject of concern to the scientists. When brakes are applied to a moving system, the kinetic energy produced at the wheel is transformed into heat energy, which does not dissipate fast enough into the air stream from the brake surface into the brake disc and as a result, the high temperatures and thermal stresses that accompany them produce a number of disadvantageous effects, such as surface cracks or permanent distortions. The thermal effect also affects the contact pressure between the surfaces. In order to avoid such types of damage, FGMs have been considered as protecting coatings between the contact surfaces.

The use of FGMs can eliminate or control thermal stresses in structural components. Wang and Mai (2005) analysed the transient one-dimensional thermal stresses in non-homogeneous materials such as plates, cylinders and spheres using a finite element method. Ootao and Tanigawa (2006) studied a one-dimensional transient thermoelastic problem of an FGM hollow cylinder whose thermal and thermoelastic constants were assumed to vary with the power product form of a radial coordinate variable. Shao, Wang, and Ang (2007) solved a thermomechanical problem of an FGM hollow circular cylinder whose material properties were assumed to be temperature-independent and vary continuously in the radial direction. Noda and Guo (2008) had solved a thermal shock problem for an FGM plate with a surface crack where the thermomechanical properties of the plate were assumed to vary along the thickness direction. Ghosh and Kanoria studied the thermoelastic response in an FGM spherically isotropic infinite elastic medium having a spherical cavity (2008) and in an FGM spherically isotropic hollow sphere (2009) in the context of the linear theory of generalised thermoelasticity with two relaxation time parameters (Green and Lindsay theory). Barik, Kanoria, and Chaudhuri (2008) had studied a contact problem in FGM. The thermoelastic interactions in a functionally graded isotropic unbounded medium varying heat source have been studied by Banik and Kanoria (2011). In addition to these reports, thermoelastic analysis in FGMs has been studied by a number of different researchers.

The objective of the present contribution is to consider one-dimensional thermoelastic disturbances in an infinite, isotropic, functionally graded thermo-viscoelastic medium in the context of three-phase-lag thermoelastic model, GN model II (TEWOED) and GN model III (thermoelasticity with energy dissipation [TEWED]), in the presence of the distributed periodically varying heat sources. All the thermophysical properties of the FGM under consideration are assumed to vary as an exponential power of the space coordinate. The governing equations for this problem are taken into Laplace–Fourier transform domain. The solutions for the displacement, temperature, thermal stress and strain in Laplace transform domain are obtained by Fourier inversion, which is carried out by using the residual calculus, where the poles of the integrand are obtained numerically in the complex domain by using Laguerre's method. The numerical inversion of the Laplace transform are carried out by using a method based on Fourier series expansion technique (Honig & Hirdes, 1984). Numerical results for the thermophysial quantities have been obtained for a copper-like material and have been presented graphically to study the effect of nonhomogeneity. The effect of viscosity is also shown.

## 2. Basic equations

The stress-strain-temperature relation is

$$\tau_{ij} = 2\mu^* e_{ij} + [\lambda^* \Delta - \gamma^*(\theta - \theta_0)]\delta_{ij}, \quad i,j = 1,2,3 \tag{2.1}$$

where the parameters $\lambda^*$, $\mu^*$ and $\gamma^*$ are defined as

$$\lambda^* = \lambda_e\left(1 + \alpha_0 \frac{\partial}{\partial t}\right), \quad \mu^* = \mu_e\left(1 + \alpha_1 \frac{\partial}{\partial t}\right), \quad \gamma^* = \gamma_e\left(1 + \gamma_0 \frac{\partial}{\partial t}\right),$$

where $\gamma_e = (3\lambda_e + 2\mu_e)\alpha_t$, $\gamma_0 = (3\lambda_e\alpha_0 + 2\mu_e\alpha_1)\alpha_t/\gamma_e$; $\lambda_e$ and $\mu_e$ being Lamés constants, $\alpha_0$, $\alpha_1$ the visco-thermoelastic relaxation times, $\alpha_t$ the coefficient of linear thermal expansion, $\tau_{ij}$ the stress tensor, $\theta_0$ the reference temperature, $\theta$ the temperature field, the cubical dilation $\Delta = e_{ii}$ and $e_{ij}$ the strain tensor given by

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{2.2}$$

where $u_i$ ($i$ = 1, 2, 3) are the displacement components.

Stress equation of motion in the absence of body force is

$$\tau_{ij,j} = \rho\ddot{u}_i, \quad i,j = 1,2,3 \tag{2.3}$$

where $\rho$ is the density of the medium.

The heat equation for the dynamic coupled generalised visco-thermoelasticity based on the three-phase-lag thermoelasticity model is given by

$$\left(1 + \tau_q\frac{\partial}{\partial t} + \frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\left(\rho c_v\ddot{\theta} + \gamma^*\theta_0 div\ddot{\vec{u}} - \rho\dot{Q}\right) = \left[K^*\theta_{,i}\right]_{,i} + \tau_T\left[K\ddot{\theta}_{,i}\right]_{,i} + \left[\tau_v^*\dot{\theta}_{,i}\right]_{,i}, \tag{2.4}$$

where $c_v$ is the specific heat at constant strain, $K^*$ is an additional material constant, $K$ is the thermal conductivity, $Q$ is the rate of internal heat generation per unit mass, $\dot{v} = \theta$; $v$ being the thermal displacement, $\tau_v^* = K + K^*\tau_v$; delay time $\tau_v$ is called the phase-lag of the thermal displacement gradient and $\tau_q$ is called the phase-lag of the heat flux. Here, the dot denotes derivative with respect to time.

GN theory type III and GN theory type II can be recovered from Equation (2.4) by taking $\tau_q = \tau_T = \tau_v = 0$ and $\tau_q = \tau_T = \tau_v = 0$, $K \ll K^*$.

## 3. Formulation of the problem

We now consider a functionally graded infinite isotropic thermo-viscoelastic body at a uniform reference temperature $\theta_0$ in the presence of periodically varying heat sources distributed over a plane area. We shall consider one-dimensional disturbances of the medium, so that the thermal displacement vector $\vec{u}$ and temperature field $\theta$ can be expressed in the following form

$$\begin{aligned}\vec{u} &= (u(x,t),0,0),\\ \theta &= \theta(x,t).\end{aligned} \tag{3.1}$$

For a functionally graded solid, the parameters $\lambda_e$, $\mu_e$, $K$, $K^*$, $\gamma_e$ and $\rho$ are no longer constant but become space-dependent. Thus, we replace $\lambda_e$, $\mu_e$, $K$, $K^*$, $\gamma_e$ and $\rho$ by $\lambda'_e f(\vec{x})$, $\mu'_e f(\vec{x})$, $K_0 f(\vec{x})$, $K_0^* f(\vec{x})$, $\gamma'_e f(\vec{x})$ and $\rho_0 f(\vec{x})$, respectively, where $\lambda'_e$, $\mu'_e$, $K_0$, $K_0^*$, $\gamma'_e$ and $\rho_0$ are assumed to be constants. $f(\vec{x})$ is a given nondimensional function of the space variable $\vec{x} = (x,y,z)$. Further, it is assumed that material properties depend only on the $x$ coordinate. So, we can take $f(\vec{x})$ as $f(x)$. In the context of the linear theory of generalised thermoelasticity in the absence of body forces based on the three-phase-lag thermoelasticity model (Roychoudhuri, 2007) the constitutive equation, strain component, the equation of motion and the heat equation can be written as follows

$$\tau_{xx} = f(x)\left[2\mu'_e\left(1+\alpha_1\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial x} + \left\{\lambda'_e\left(1+\alpha_0\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial x} - \gamma'_e\left(1+\gamma_0\frac{\partial}{\partial t}\right)(\theta-\theta_0)\right\}\right], \tag{3.2}$$

$$e_{xx} = \frac{\partial u}{\partial x}, \tag{3.3}$$

$$\begin{aligned}\rho_0 f(x)\frac{\partial^2 u}{\partial t^2} = {} & f(x)\left[2\mu'_e\left(1+\alpha_1\frac{\partial}{\partial t}\right)\frac{\partial^2 u}{\partial x^2} + \left\{\lambda'_e\left(1+\alpha_0\frac{\partial}{\partial t}\right)\frac{\partial^2 u}{\partial x^2} - \gamma'_e\left(1+\gamma_0\frac{\partial}{\partial t}\right)\frac{\partial \theta}{\partial x}\right\}\right] \\ & + \left[2\mu'_e\left(1+\alpha_1\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial x} + \left\{\lambda'_e\left(1+\alpha_0\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial x} - \gamma'_e\left(1+\gamma_0\frac{\partial}{\partial t}\right)(\theta-\theta_0)\right\}\right]\frac{\partial f(x)}{\partial x}\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}&\frac{\partial}{\partial x}\left[K_0^* f(x)\frac{\partial \theta}{\partial x}\right] + \frac{\partial}{\partial x}\left[K_0\tau_T f(x)\frac{\partial \ddot{\theta}}{\partial x}\right] + \frac{\partial}{\partial x}\left[\left(K_0^* f(x)\tau_v + K_0 f(x)\right)\frac{\partial \dot{\theta}}{\partial x}\right] \\ &= \left(1+\tau_q\frac{\partial}{\partial t} + \frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\left[\rho_0 c_v f(x) + \gamma'_e\left(1+\gamma_0\frac{\partial}{\partial t}\right)\theta_0\frac{\partial^3 u}{\partial t^2 \partial x} f(x) - \rho_0 f(x)\dot{Q}\right],\end{aligned} \tag{3.5}$$

Introducing the following nondimensional variables

$$x' = \frac{x}{l}, \quad t' = \frac{vt}{l}, \quad \theta' = \frac{\theta-\theta_0}{\theta_0}, \quad u' = \frac{\lambda'_e + 2\mu'_e}{l\gamma'_e\theta_0}u,$$

$$\tau'_{x'x'} = \frac{\tau_{xx}}{\gamma'_e\theta_0}, \quad e'_{x'x'} = e_{xx}, \quad \tau'_q = \frac{\tau_q v}{l},$$

$$\tau'_T = \frac{\tau_T v}{l}, \quad \tau'_v = \frac{\tau_v v}{l}, \quad f'(x') = f(x), \quad \alpha'_0 = \frac{\alpha_0 v}{l}, \quad \alpha'_1 = \frac{\alpha_1 v}{l}, \quad \gamma'_0 = \frac{\gamma_0 v}{l}, \tag{3.6}$$

where $l$ is a standard length and $v$ is a standard speed. Then, after removing primes, Equations (3.2)–(3.5) can be written in non-dimensional form as follows

$$\tau_{xx} = f(x)\left[1 + \left\{\alpha_0 + (\alpha_1 - \alpha_0)\frac{2C_S^2}{C_P^2}\right\}\frac{\partial}{\partial t}\right]\frac{\partial u}{\partial x} - f(x)\left(1+\gamma_0\frac{\partial}{\partial t}\right)\theta, \tag{3.7}$$

$$e_{xx} = \frac{\gamma'_e\theta_0}{\lambda'_e + 2\mu'_e}\frac{\partial u}{\partial x} = \beta_1\frac{\partial u}{\partial x}, \tag{3.8}$$

$$\begin{aligned}f(x)\frac{\partial^2 u}{\partial t^2} = {} & \left[C_P^2 + \left\{\alpha_0(C_P^2 - 2C_S^2) + 2\alpha_1 C_S^2\right\}\frac{\partial}{\partial t}\right]\frac{\partial^2 u}{\partial x^2} f(x) - C_P^2 f(x)\left(1+\gamma_0\frac{\partial}{\partial t}\right)\frac{\partial \theta}{\theta x} \\ & + \left(\left[C_P^2 + \left\{\alpha_0(C_P^2 - 2C_S^2) + 2\alpha_1 C_S^2\right\}\frac{\partial}{\partial t}\right]\frac{\partial u}{\partial x} - C_P^2\left(1+\gamma_0\frac{\partial}{\partial t}\right)\theta\right)\frac{\partial f(x)}{\partial x},\end{aligned} \tag{3.9}$$

$$\left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\left[\ddot{\theta}+\varepsilon\left(1+\gamma_0\frac{\partial}{\partial t}\right)\frac{\partial \ddot{u}}{\partial x}-Q_0\right]f(x)$$
$$=\left[C_T^2\frac{\partial^2\theta}{\partial x^2}+\tau_T C_K^2\frac{\partial^2\ddot{\theta}}{\partial x^2}+(C_K^2+\tau_v C_T^2)\frac{\partial^2\dot{\theta}}{\partial x^2}\right]f(x)$$
$$+\left[C_T^2\frac{\partial\theta}{\partial x}+\tau_T C_K^2\frac{\partial\ddot{\theta}}{\partial x}+(C_K^2+\tau_v C_T^2)\frac{\partial\dot{\theta}}{\partial x}\right]\frac{\partial f(x)}{\partial x}. \tag{3.10}$$

where

$$C_P^2=\frac{\lambda_e'+2\mu_e'}{\rho_0 v^2},\quad C_S^2=\frac{\mu_e'}{\rho_0 v^2},\quad C_T^2=\frac{K_0^*}{\rho_0 c_v v^2},\quad C_K^2=\frac{K_0}{\rho_0 c_v l v},\quad Q_0=\frac{\dot{Q}l}{c_v\theta_0 v},$$
$$\varepsilon=\frac{\gamma_e'^2\theta_0}{\rho_0 c_v(\lambda_e'+2\mu_e')},$$

and it is to be noted that GN model III and GN model II can be recovered from Equation (3.10) by taking $\tau_T=\tau_q=\tau_v=0$ and $\tau_T=\tau_q=\tau_v=0$, $K\ll K^*$, respectively.

In the previous expressions, $C_P$, $C_S$ and $C_T$ represent nondimensional dilatational, shear and thermal wave velocities, respectively, $C_K$ is the damping coefficient and $\varepsilon$ is the thermoelastic coupling constant.

We assume that the medium is initially at rest. The undisturbed state is maintained at reference temperature. Then we have

$$u(x,0)=\dot{u}(x,0)=\theta(x,0)=\dot{\theta}(x,0)=0.$$

### 3.1 Periodically varying heat source

Now let us assume that heat sources are distributed over the plane $x = 0$ in the following form

$$\begin{aligned} Q_0 &= Q_0^*\delta(x)\sin\left(\frac{\pi t}{\tau}\right) && \text{for}\quad 0\le t\le\tau,\\ &= 0 && \text{for}\quad t>\tau, \end{aligned} \tag{3.11}$$

### 3.2 Exponential variation of nonhomogeneity

We take $f(x) = e^{-kx}$, where $k$ is a dimensionless constant. Then the corresponding equation reduce to

$$\tau_{xx}(x,t)=e^{-kx}\left\{\left[1+\left\{\alpha_0+(\alpha_1-\alpha_0)\frac{2C_S^2}{C_P^2}\right\}\frac{\partial}{\partial t}\right]\frac{\partial u}{\partial x}-\left(1+\gamma_0\frac{\partial}{\partial t}\right)\theta\right\}, \tag{3.12}$$

$$e_{xx}(x,t)=\beta_1\frac{\partial u}{\partial x},\quad \text{where}\quad \beta_1=\frac{\gamma_e'\theta_0}{\lambda_e'+2\mu_e'}. \tag{3.13}$$

$$\frac{\partial^2 u}{\partial t^2} = \left[C_P^2 + \left\{\alpha_0(C_P^2 - 2C_S^2) + 2\alpha_1 C_S^2\right\}\frac{\partial}{\partial t}\right]\frac{\partial^2 u}{\partial x^2} - C_P^2\left(1+\gamma_0\frac{\partial}{\partial t}\right)\frac{\partial\theta}{\partial x}$$
$$- k\left[C_P^2 + \left\{\alpha_0(C_P^2 - 2C_S^2) + 2\alpha_1 C_S^2\right\}\frac{\partial}{\partial t}\right]\frac{\partial u}{\partial x} + kC_P^2\left(1+\gamma_0\frac{\partial}{\partial t}\right)\theta, \qquad (3.14)$$

$$\left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right)\left[\frac{\partial^2\theta}{\partial t^2}+\varepsilon\left(1+\gamma_0\frac{\partial}{\partial t}\right)\frac{\partial^3 u}{\partial t^2\partial x}-Q_0\right]$$
$$= C_T^2\left(\frac{\partial^2\theta}{\partial x^2}-k\frac{\partial\theta}{\partial x}\right)+\tau_T C_K^2\left(\frac{\partial^2\ddot{\theta}}{\partial x^2}-k\frac{\partial\ddot{\theta}}{\partial x}\right)+(C_K^2+\tau_v C_T^2)\left(\frac{\partial^2\dot{\theta}}{\partial x^2}-k\frac{\partial\dot{\theta}}{\partial x}\right). \qquad (3.15)$$

Let us define the Laplace–Fourier double transform of a function $f(x, t)$ by

$$\overline{f}(x,p) = \int_0^\infty e^{-pt} f(x,t)dt, \quad \mathrm{Re}(p) > 0,$$
$$\widehat{\overline{f}}(\xi,p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\xi x}\overline{f}(x,p)dx. \qquad (3.16)$$

Applying the Laplace–Fourier double integral transform to Equations (3.12)–(3.15), we get

$$\widehat{\overline{\tau}}_{xx}(\xi,p) = -i(\xi+ik)\left\{\left[1+\left\{\alpha_0+(\alpha_1-\alpha_0)\frac{2C_S^2}{C_P^2}\right\}p\right]\widehat{\overline{u}}(\xi+ik,p)-(1+\gamma_0 p)\widehat{\overline{\theta}}(\xi+ik,p)\right\}, \qquad (3.17)$$

$$\widehat{\overline{e}}_{xx}(\xi,p) = -i\beta_1\xi\widehat{\overline{u}}(\xi,p), \qquad (3.18)$$

$$\widehat{\overline{u}}(\xi,p) = \frac{(k+i\xi)C_P^2(1+\gamma_0 p)}{\left[p^2+\left\{C_P^2+\left\{\alpha_0(C_P^2-2C_S^2)+2\alpha_1 C_S^2\right\}p\right\}(\xi^2-i\xi k)\right]}\widehat{\overline{\theta}}(\xi,p), \qquad (3.19)$$

$$\left[\left(1+p\tau_q+\frac{p^2\tau_q^2}{2}\right)p^2\left\{C_T^2+\tau_T C_K^2+p\left(C_K^2+\tau_v C_T^2\right)\right\}\xi^2\right.$$
$$\left.-ik\xi\left\{C_T^2+p^2\tau_T C_K^2+p\left(C_K^2+\tau_v C_T^2\right)\right\}\right]\widehat{\overline{\theta}}(\xi,p)$$
$$-i\varepsilon p^2\xi(1+\gamma_0 p)\left(1+p\tau_q+\frac{p^2\tau_q^2}{2}\right)\widehat{\overline{u}}(\xi,p)$$
$$=\left(1+p\tau_q+\frac{p^2\tau_q^2}{2}\right)\widehat{\overline{Q}}_0, \qquad (3.20)$$

Solving Equations (3.19) and (3.20) for $\widehat{\overline{u}}(\xi,p)$ and $\widehat{\overline{\theta}}(\xi,p)$, we obtain

$$\widehat{\overline{u}}(\xi,p) = \frac{C_P^2(k+i\xi)(1+\gamma_0 p)\left(1+p\tau_q+\frac{p^2\tau_q^2}{2}\right)\widehat{\overline{Q}}_0}{M^*(\xi)}, \qquad (3.21)$$

$$\widehat{\overline{\theta}}(\xi,p) = \frac{\widehat{\overline{Q}}_0\left[p^2+\left\{C_P^2+\left\{\alpha_0(C_P^2-2C_S^2)+2\alpha_1 C_S^2\right\}p\right\}\left(\xi^2-i\xi k\right)\right]\left(1+p\tau_q+\frac{p^2\tau_q^2}{2}\right)}{M^*(\xi)}, \qquad (3.22)$$

where

$$\begin{aligned} M^*(\xi) &= M_1(p)\xi^4 + M_2(p)\xi^2 + M_3(p) \\ &= M_1(p)(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4), \end{aligned} \tag{3.23}$$

and $M_1(p)$, $M_2(p)$ and $M_3(p)$ are given by

$$M_1(p) = \left\{(C_K^2 + \tau_v C_T^2)p + (C_T^2 + \tau_T C_K^2)\right\}\left\{C_P^2 + \left\{\alpha_0(C_P^2 - 2C_S^2) + 2\alpha_1 C_S^2\right\}p\right\}, \tag{3.24}$$

$$\begin{aligned} M_2(p) = {} & p^2\varepsilon(1+\gamma_0 p)^2 C_P^2\left(1 + p\tau_q + \frac{p^2\tau_q^2}{2}\right) - k^2\left\{C_T^2 + p^2\tau_T C_K^2 + p(C_T^2 + \tau_T C_K^2)\right\} \\ & \times \left\{C_P^2 + \left\{\alpha_0(C_P^2 - 2C_S^2) + 2\alpha_1 C_S^2\right\}p\right\} + p^2\left\{C_T^2 + \tau_T C_K^2 + p(C_K^2 + \tau_v C_T^2)\right\} \\ & + \left(1 + p\tau_q + \frac{p^2\tau_q^2}{2}\right)p^2\left\{C_P^2 + \left\{\alpha_0(C_P^2 - 2C_S^2) + 2\alpha_1 C_S^2\right\}p\right\}, \end{aligned} \tag{3.25}$$

$$M_3(p) = p^4\left(1 + p\tau_q + \frac{p^2\tau_q^2}{2}\right). \tag{3.26}$$

The expression for strain and stress in the Laplace–Fourier transform domain can be obtained from Equations (3.17) and (3.18) using (3.21) and (3.22) as follows

$$\hat{\bar{e}}_{xx}(\xi,p) = \frac{(\xi^2 - i\xi k)\beta_1 C_P^2 \hat{\bar{Q}}_0 (1+\gamma_0 p)\left(1 + p\tau_q + \frac{p^2\tau_q^2}{2}\right)}{M^*(\xi)}, \tag{3.27}$$

$$\begin{aligned} \hat{\bar{\tau}}_{xx}(\xi,p) = {} & \frac{(1+\gamma_0 p)\left(1 + p\tau_q + \frac{p^2\tau_q^2}{2}\right)\hat{\bar{Q}}_0(\xi^2 + i\xi k)}{M^*(\xi + ik)}\left[\left\{\alpha_0 C_P^2(\alpha_1 - \alpha_0)2C_S^2\right\}pC_P^2 \right. \\ & \left. - \left\{p^2 + \alpha_0(C_P^2 - 2C_S^2)p + 2\alpha_1 C_S^2 p\right\}\right]. \end{aligned} \tag{3.28}$$

Inverse Fourier transforms of Equations (3.21), (3.22), (3.27) and (3.28) give the following solutions for displacement, temperature, strain and stress in the Laplace transform domain

$$\bar{u}(x,p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{C_P^2(k + i\xi)(1+\gamma_0 p)\left(1 + p\tau_q + \frac{p^2\tau_q^2}{2}\right)\hat{\bar{Q}}_0}{M^*(\xi)} e^{-i\xi x} d\xi, \tag{3.29}$$

$$\begin{aligned} & \bar{\theta}(x,p) \\ & = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\hat{\bar{Q}}_0\left[p^2 + \left\{C_P^2 + \left\{\alpha_0(C_P^2 - 2C_S^2) + 2\alpha_1 C_S^2\right\}p\right\}(\xi^2 - i\xi k)\right]\left(1 + p\tau_q + \frac{p^2\tau_q^2}{2}\right)}{M^*(\xi)} e^{-i\xi x} d\xi, \end{aligned} \tag{3.30}$$

$$\bar{e}_{xx}(x,p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{(\xi^2 - i\xi k)\beta_1 C_P^2 \hat{\bar{Q}}_0 (1+\gamma_0 p)\left(1 + p\tau_q + \frac{p^2\tau_q^2}{2}\right)}{M^*(\xi)} e^{-i\xi x} d\xi, \tag{3.31}$$

$$\overline{\tau}_{xx}(x,p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{(1+\gamma_0 p)\left(1+p\tau_q+\frac{p^2\tau_q^2}{2}\right)\widehat{\overline{Q}}_0\left(\xi^2+i\xi k\right)}{M^*(\xi+ik)} \tag{3.32}$$
$$\left[\left\{\alpha_0 C_P^2(\alpha_1-\alpha_0)2C_S^2\right\}pC_P^2 - \left\{p^2+\alpha_0(C_P^2-2C_S^2)p+2\alpha_1 C_S^2 p\right\}\right]e^{-i\xi x}d\xi,$$

where

$$\begin{aligned} M(\xi+ik) &= M_1(p)\xi^4 - M_2(p)\xi^3 + M_3(p)\xi^2 - M_4(p)\xi + M_5(p) \\ &= M_1(p)(\xi-l_1)(\xi-l_2)(\xi-l_3)(\xi-l_4). \end{aligned}$$

Since the heat sources are distributed over the plane $x = 0$ in the following form

$$\begin{aligned} Q_0 &= Q_0^*\delta(x)\sin\left(\tfrac{\pi t}{\tau}\right) \quad \text{for} \quad 0 \le t \le \tau, \\ &= 0 \qquad\qquad\qquad \text{for} \quad t > \tau, \end{aligned} \tag{3.33}$$

then

$$\widehat{\overline{Q}}_0 = \frac{Q_0^*\pi\tau(1+e^{-p\tau})}{\sqrt{2\pi}(\pi^2+p^2\tau^2)}. \tag{3.34}$$

Thus, the expressions for thermal displacement, temperature, thermal stress and strain in Laplace transform domain take the following form

$$\bar{u}(x,p) = \int_{-\infty}^{\infty} \frac{Q_0^*\tau(1+e^{-p\tau})C_P^2(k+i\xi)(1+\gamma_0 p)\left(1+p\tau_q+\frac{p^2\tau_q^2}{2}\right)}{2(\pi^2+p^2\tau^2)M^*(\xi)} e^{-i\xi x}d\xi, \tag{3.35}$$

$$\overline{\theta}(x,p) = \int_{-\infty}^{\infty} \frac{Q_0^*\tau(1+e^{-p\tau})\left(1+p\tau_q+\frac{p^2\tau_q^2}{2}\right)}{2(\pi^2+p^2\tau^2)M^*(\xi)} \tag{3.36}$$
$$\left[p^2+\{C_P^2+\left\{\alpha_0\left(C_P^2-2C_S^2\right)+2\alpha_1 C_S^2\right\}p\}\left(\xi^2-i\xi k\right)\right]e^{-i\xi x}d\xi,$$

$$\overline{\tau}_{xx}(x,p) = \int_{-\infty}^{\infty} \frac{Q_0^*\tau(1+e^{-p\tau})(1+\gamma_0 p)\left(\xi^2+i\xi k\right)\left(1+p\tau_q+\frac{p^2\tau_q^2}{2}\right)}{2(\pi^2+p^2\tau^2)M^*(\xi+ik)}$$
$$\left[\{\alpha_0 C_P^2+(\alpha_1-\alpha_0)C_S^2\}pC_P^2 - \left\{p^2+\alpha_0\left(C_P^2-2C_S^2\right)p+2\alpha_1 C_S^2 p\right\}\right]e^{-i\xi x}d\xi, \tag{3.37}$$

$$\bar{e}_{xx}(x,p) = \int_{-\infty}^{\infty} \frac{Q_0^*\beta_1 C_P^2\tau(1+e^{-p\tau})(1+\gamma_0 p)\left(\xi^2-i\xi k\right)\left(1+p\tau_q+\frac{p^2\tau_q^2}{2}\right)}{2(\pi^2+p^2\tau^2)M^*(\xi)} e^{-i\xi x}d\xi, \tag{3.38}$$

Applying contour integration to Equations (3.35)–(3.38) we obtain

$$\bar{u}(x,p) = -iQ_0^*\pi\tau N(p) \sum_{\substack{j=1 \\ \mathrm{Im}(\xi_j)<0}}^{4} A_j(i\xi_j + k)e^{-i\xi_j x} \quad \text{for} \quad x > 0$$
$$= iQ_0^*\pi\tau N(p) \sum_{\substack{j=1 \\ \mathrm{Im}(\xi_j)>0}}^{4} A_j(i\xi_j + k)e^{-i\xi_j x} \quad \text{for} \quad x < 0, \tag{3.39}$$

$$\overline{\theta}(x,p) = -\frac{iQ_0^*\pi\tau N(p)}{(1+\gamma_0 p)C_P^2} \sum_{\substack{j=1 \\ \mathrm{Im}(\xi_j)<0}}^{4} A_j\Im(\xi_j)e^{-i\xi_j x} \quad \text{for} \quad x > 0$$
$$= \frac{iQ_0^*\pi\tau N(p)}{(1+\gamma_0 p)C_P^2} \sum_{\substack{j=1 \\ \mathrm{Im}(\xi_j)>0}}^{4} A_j\Im(\xi_j)e^{-i\xi_j x} \quad \text{for} \quad x < 0, \tag{3.40}$$

$$\overline{\tau}_{xx}(x,p) = \frac{i\beta_1 Q_0^*\pi\tau N(p)}{C_P^2} \sum_{\substack{j=1 \\ \mathrm{Im}(l_j)<0}}^{4} B_j(l_j^2 + ikl_j)\wp(l_j)e^{-il_j x} \quad \text{for} \quad x > 0$$
$$= -\frac{i\beta_1 Q_0^*\pi\tau N(p)}{C_P^2} \sum_{\substack{j=1 \\ \mathrm{Im}(l_j)>0}}^{4} B_j(l_j^2 + ikl_j)\wp(l_j)e^{-il_j x} \quad \text{for} \quad x < 0, \tag{3.41}$$

$$\bar{e}_{xx}(x,p) = iQ_0^*\pi\tau N(p)p^2 \sum_{\substack{j=1 \\ \mathrm{Im}(\xi_j)<0}}^{4} A_j(\xi_j^2 - i\xi_j k)e^{-i\xi_j x} \quad \text{for} \quad x > 0$$
$$= -iQ_0^*\pi\tau N(p)p^2 \sum_{\substack{j=1 \\ \mathrm{Im}(\xi_j)>0}}^{4} A_j(\xi_j^2 - i\xi_j k)e^{-i\xi_j x} \quad \text{for} \quad x < 0, \tag{3.42}$$

where $A_j$'s and $B_j$'s are given by

$$A_j = \prod_{\substack{n=1 \\ n\neq j}}^{4} \frac{1}{(\xi_j - \xi_n)},$$

$$B_j = \prod_{\substack{n=1 \\ n\neq j}}^{4} \frac{1}{(l_j - l_n)}, \quad j = 1,2,3,4, \tag{3.43}$$

$$N(p) = \frac{C_P^2(1+e^{-p\tau})(1+\gamma_0 p)\left(1+p\tau_q+\frac{p^2\tau_q^2}{2}\right)}{M_1(p)(\pi^2+p^2\tau^2)}. \tag{3.44}$$

$$\Im(\xi_j) = \left[p^2 + \left\{C_P^2 + \left\{\alpha_0\left(C_P^2 - 2C_S^2\right) + 2\alpha_1 C_S^2\right\}p\right\}\left(\xi_j^2 - i\xi_j k\right)\right], \tag{3.45}$$

and

$$\wp(l_j) = \left(l_j^2 + il_j k\right)\left[\left\{\alpha_0 C_P^2 + (\alpha_1-\alpha_0)C_S^2\right\}pC_P^2 - \left\{p^2 + \alpha_0\left(C_P^2 - 2C_S^2\right)p + 2\alpha_1 C_S^2 p\right\}\right]. \tag{3.46}$$

## 4 Numerical inversion of Laplace transform

Let $\bar{f}(x,p)$ be the Laplace transform of a function $f(x, t)$. Then the inversion formula for Laplace transform can be written as

$$f(x,t) = \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} e^{pt}\bar{f}(x,p)dp, \tag{4.1}$$

where $d$ is an arbitrary real number greater than the real part of all the singularities of $\bar{f}(x,p)$. Taking $p = d + iw$, the preceding integral takes the form

$$f(x,t) = \frac{e^{dt}}{2\pi}\int_{-\infty}^{\infty} e^{itw}\bar{f}(x,d+iw)dw. \tag{4.2}$$

Expanding the function $h(x, t) = e^{-dt}f(x, t)$ in a Fourier series in the interval $[0, 2T]$, we obtain the approximate formula (Honig & Hirdes, 1984),

$$f(x,t) = f_\infty(x,t) + E_D, \tag{4.3}$$

where

$$f_\infty(x,t) = \frac{1}{2}c_0 + \sum_{k=1}^{\infty} c_k \quad \text{for} \quad 0 \le t \le 2T \tag{4.4}$$

and

$$c_k = \frac{e^{dt}}{T}\left[e^{\frac{ik\pi t}{T}}\bar{f}\left(x, d+\frac{ik\pi t}{T}\right)\right]. \tag{4.5}$$

The discretisation error $E_D$ can be made arbitrary small by choosing $d$ large enough (Honig & Hirdes, 1984). Since the infinite series in Equation (4.4) can be summed up to a finite number $N$ of terms, the approximate value of $f(x, t)$ becomes

$$f_N(x,t) = \frac{1}{2}c_0 + \sum_{k=1}^{N} c_k \quad \text{for} \quad 0 \le t \le 2T \tag{4.6}$$

Using the preceding formula to evaluate $f(x, t)$ we introduce a truncation error $E_T$ that must be added to the discretisation error to produce the total approximation error. Two methods are used to reduce the total error. First, the 'Korrecktur' method to reduce the discretization error. Next, the $\varepsilon$-algorithm is used to accelerate convergence (Honig & Hirdes, 1984). The Korrecktur method uses the following formula to evaluate the function $f(x, t)$

$$f(x,t) = f_\infty(x,t) - e^{-2dT} f_\infty(x, 2T + t) + E'_D, \tag{4.7}$$

where the discretisation error $|E'_D| \ll |E_D|$. Thus, the approximate value of $f(x, t)$ becomes

$$f_{NK}(x,t) = f_N(x,t) - e^{-2dT} f_{N'}(x, 2T + t), \tag{4.8}$$

where $N'$ is an integer such that $N' < N$. We shall now describe the $\varepsilon$-algorithm that is used to accelerate the convergence of the series in Equation (4.6). Let $N = 2q + 1$, where $q$ is a natural number and let $s_m = \sum_{k=1}^{m} c_k$ be the sequence of the partial sum of series in (4.6). We define the $\varepsilon$-sequence by

$$\varepsilon_{0,m} = 0, \quad \varepsilon_{1,m} = s_m$$

and

$$\varepsilon_{r+1,m} = \varepsilon_{r-1,m+1} + \frac{1}{\varepsilon_{r,m+1} - \varepsilon_{r,m}}, \quad r = 1, 2, 3, \ldots \quad .$$

It can be shown that the sequence $\varepsilon_{1,1}, \varepsilon_{3,1}, \varepsilon_{5,1}, \ldots, \varepsilon_{N,1}$ converges to $f(x,t) + E_D - \frac{c_0}{2}$ faster than the sequence of partial sums $s_m$, $m = 1, 2, 3, \ldots$. The actual procedure used to invert the Laplace transform consists of using Equation (4.8) together with the $\varepsilon$-algorithm. The values of $d$ and $T$ are chosen according to the criterion outlined in (Honig & Hirdes, 1984).

## 5. Numerical results and discussions

To get the solution for thermal displacement, temperature and thermal stress in space-time domain, we have to apply the Laplace inversion formula to Equations (3.39)–(3.42), respectively. This has been done numerically using a method based on Fourier series expansion technique. To get the roots of the polynomials $M^*(\xi)$ and $M^*(\xi + ik)$ in the complex domain, we have used Laguerre's method. For computational purposes, a copper-like material has been taken into consideration. The values of the material constants are taken as follows (Roychoudhuri & Dutta, 2005)

$$\varepsilon = 0.0168, \quad \lambda_e = 1.387 \times 10^{11}\,\mathrm{N/m^2},$$

$$\mu_e = 0.448 \times 10^{11}\mathrm{N/m^2}, \quad \alpha_t = 1.67 \times 10^{-8}/\mathrm{K}, \quad \theta_0 = 1\,\mathrm{K}$$

and the hypothetical values of phase-lag parameters are taken as

$$\alpha_0 = 0.05s, \quad \alpha_1 = 0.1s, \quad \tau_q = 0.001s, \quad \tau_T = 0.05s, \quad \tau_v = 0.05s,$$

which agrees with the stability condition of Quintanilla and Racke (2008) that under three-phase-lag heat conduction, if $K^*\tau_q < \tau_v^* < \frac{2K\tau_T}{\tau_q}$, where $\tau_v = K + K^*\tau_v$, the solutions are always exponentially stable. Also, we have taken $Q_0^* = 1$, $\tau = 1$, $C_p = 1$, $C_T = 2$ and $C_K = 0.6$, so the faster wave is the thermal wave.

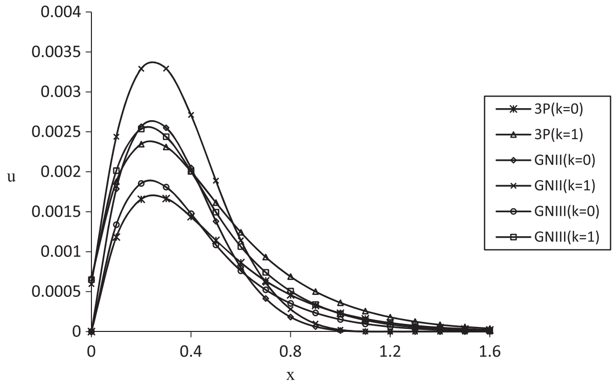


Figure 1. Variation of $u$ vs. $x$ for $t = 0.6$ and $k = 0, 1$ for viscous material.

In order to study the effect of nonhomogeneity on thermal displacement, temperature and stress for a viscous material, we now present our results in their graphical representations (Figures 1–3). Figures 1–3 show the variation of thermal displacement, temperature and stress for three models (GN II, GN III and 3P lag model) for time $t = 0.6$ and for nonhomogeneity parameter $k = 0$ and 1.

Figure 1 depicts the variation of thermal displacement $u$ versus $x$ for $t = 0.6$ when the nonhomogeneity parameter is taken to be $k = 0$, 1 for the three different models to study the effect of nonhomogeneity in the interval $0 < x < 1.6$. From the figure, it is seen that the displacement will increase to reach its maximum near $x = 0.3$ and then it falls to zero for all models. It is also observed that in each of the models, as the nonhomogeneity parameter increases, the magnitude of the peak of the displacement component will also increase. Also, it is seen that for different nonhomogeneity parameters, the decay of the magnitude of the thermal displacement is faster for GN II model than that of GN III model which is again faster than 3P lag model.

Figure 2 is plotted to study the effect of nonhomogeneity on the temperature $\theta$ with the distance $x$. It is observed that as the nonhomogeneity parameter increases, the magnitude of $\theta$ will also increase for a fixed $x$ and beyond this, the magnitude of $\theta$ diminishes to zero. It can also be verified from the expression of $\overline{\theta}$ given in Equation (3.40)

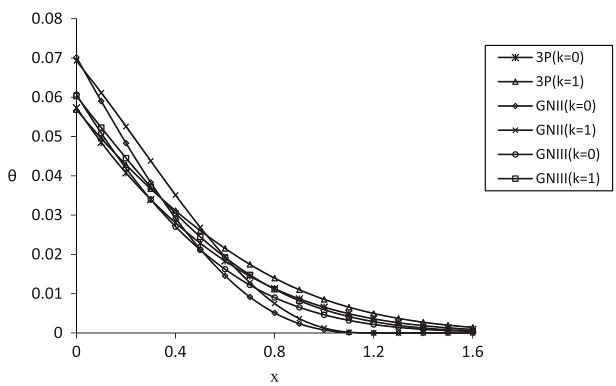


Figure 2. Variation of $\theta$ vs. $x$ for $t = 0.6$ and $k = 0, 1$ for viscous material.

involving $e^{-i\xi_j x}$ $Im(\xi_j) < 0$ for $x > 0$. Also, it is observed that for different $k$, the decay of $\theta$ for GN II model is faster than GN III model which is again faster than 3P lag model.

Figure 3 shows the variation of the thermal stress $\tau_{xx}$ vs. distance $x$ for $t = 0.6$. It is seen than near the application of the heat source, the stress is compressive in nature, which is physically plausible. The magnitude of the thermal stress is greater for GN II model compared to that of GN III model which is again greater for 3P lag model. After that the magnitude of $\tau_{xx}$ decays sharply for GN II model compared to GN III model, which decays sharply than that for 3P lag model.

In order to study the effect of viscosity on the thermophysical quantities, Figures 4–6 are plotted for $k = 1$ and $t = 0.6$. Figure 4 depicts the variation of displacement $u$ versus $x$ when $t = 0.6$ for both viscous and non-viscous materials for the different models (GN II, GN III, 3P). It is observed that due to the presence of the viscosity term, the peak of the displacement is larger for non-viscous material than that of the viscous material for all the models and the rate of decay becomes slower for the viscous material also. Similar qualitative behaviour is seen for all the three models.

Figure 5 depicts the variation of temperature $\theta$ versus $x$ for both viscous and non-viscous materials. Here also, the presence of viscosity term, the rate of the decay

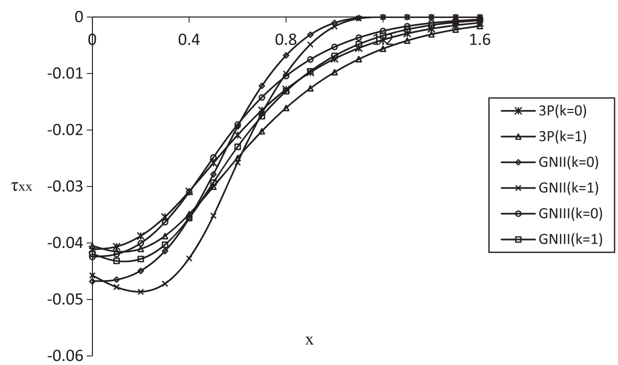


Figure 3. Variation of $\tau_{xx}$ vs. $x$ for $t = 0.6$ and $k = 0, 1$ for viscous material.

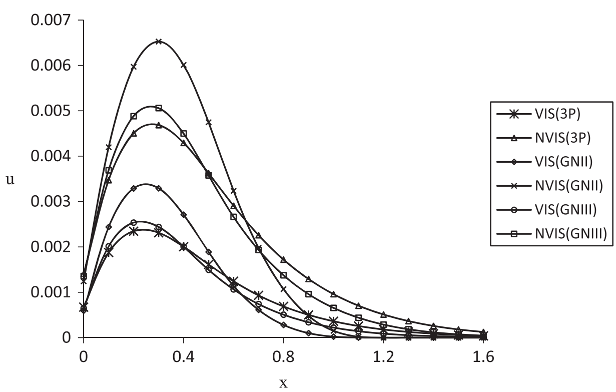


Figure 4. Variation of $u$ vs. $x$ for $t = 0.6$ and $k = 1$.

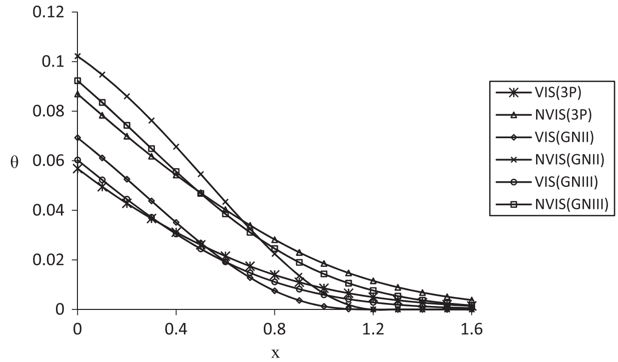


Figure 5. Variation of $\theta$ vs. $x$ for $t = 0.6$ and $k = 1$.

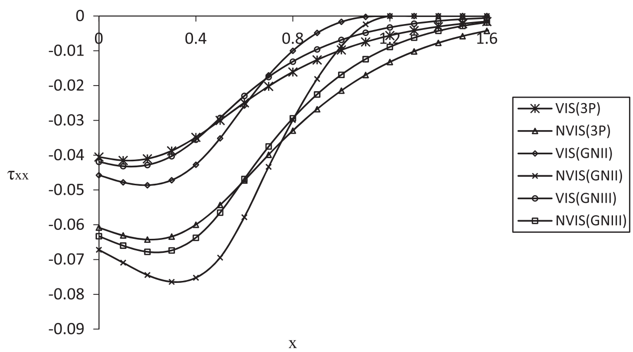


Figure 6. Variation of $\tau_{xx}$ vs. $x$ for $t = 0.6$ and $k = 1$.

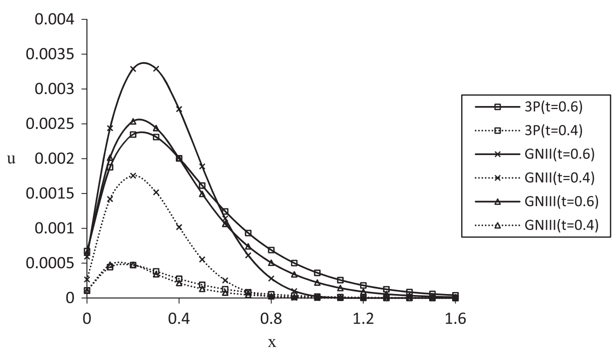


Figure 7. Variation of $u$ vs. $x$ for $t = 0.6, 0.4$ and $k = 1$ for viscous material.

becomes slower. A similar qualitative behaviour of the temperature is seen for different models.

Figure 6 is plotted to study the effect of viscosity on the stress component $\tau_{xx}$ for different models for nonhomogeneous material. Due to the presence of the

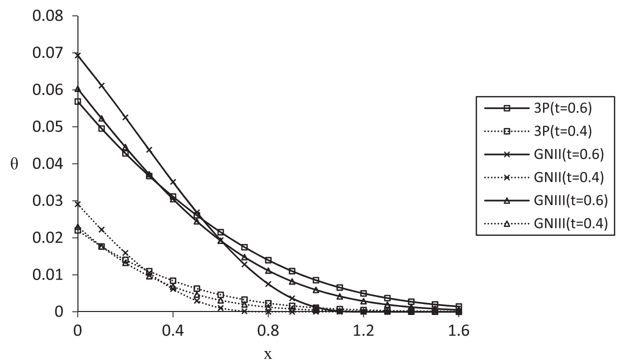


Figure 8. Variation of $\theta$ vs. $x$ for $t = 0.6, 0.4$ and $k = 1$ for viscous material.

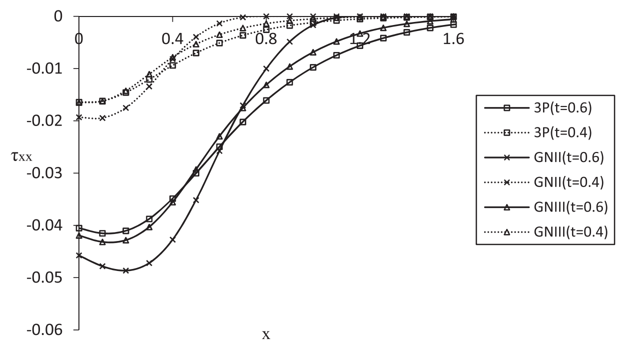


Figure 9. Variation of $\tau_{xx}$ vs. $x$ for $t = 0.6, 0.4$ and $k = 1$ for viscous material.

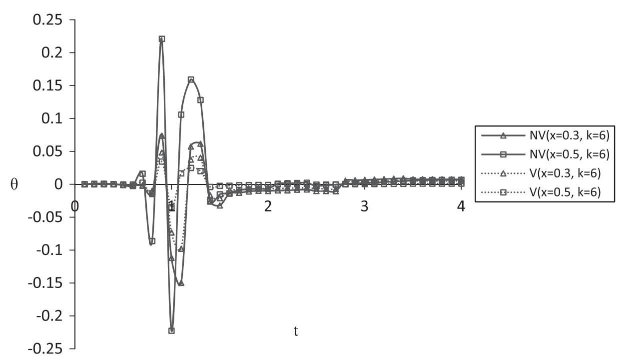


Figure 10. Variation of $\theta$ vs. $t$ for $x = 0.3, 0.5$, $C_T = 6$ and $k = 6$.

viscosity term, the rate of decay of $\tau_{xx}$ becomes slower. Also, it is observed that for the GN II model, the stress component has a larger value than for the GN III model and the 3P model, which is physically plausible since there is no dissipative term present in GN II model.

Figures 7–9 are plotted to study the variation of thermophysical quantities for viscous material when $t = 0.4, 0.6$ for three models. From these figures, it is seen that in earlier situations, the magnitude of the thermophysical quantities are lesser than the magnitude of the same for later on.

Figure 10 depicts the variation temperature ($\theta$) versus $t$ for both viscous and non-viscous materials for $C_T = 6$ and nonhomogeneous material ($k = 6$) for $x = 0.3, 0.5$. It is evident from the figure that the oscillatory behaviour of $\theta$ is seen for $0.6 < t < 1.8$ and after that, the temperature almost disappears inside the body i.e. the thermal wave is propagating with time and with increase of time, it reaches to a steady state.

## 6. Conclusions

The present problem of investigating the thermophysical quantities in an isotropic functionally graded Kelvin–Voigt viscoelastic material subjected to a periodically varying heat source is studied in the light of generalised thermoelasticity theories with three different models (GN II, GN III and 3P lag model). The material properties are assumed to vary as exponentially with distance. The analysis of the results permits some concluding remarks:

(1) The effect of nonhomogeneity on all the thermophysical quantities is seen. It is prominent that the increase of the nonhomogeneity parameter will increase the magnitudes of the thermophysical quantities. So, while designing any FGM, the effect of nonhomogeneity should be taken into consideration.
(2) The presence of the thermo-viscoelastic relaxation parameters will decrease the magnitude of the thermophyscial quantities and the decay of the physical quantities becomes slower due to the presence of the viscosity term.
(3) Here, all the results for nonhomogeneity parameter $k = 0$ and three-phase-lag model, agree with the existing literature (Kanoria & Mallik, 2010).

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