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Effect of harmonically varying heat on FG nanobeams in the context of a nonlocal two-temperature thermoelasticity theory

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The effect of two temperatures on functionally graded nanobeams due to harmonically varying heat is investigated. Material properties of the nanobeam are assumed to be graded in the thickness direction according to a novel power-law distribution in terms of the volume fractions of the metal and ceramic constituents. The generalised thermoelasticity model based upon Green and Naghdi's theory as well as the nonlocal thin beam theory is used to solve this problem. The governing equations are expressed in Laplace transform domain. Based on Fourier series expansion technique, the inversion of Laplace transform is made numerically. Some comparisons have been shown to present the effect of the nonlocal parameter, the temperature discrepancy parameter and the angular frequency of thermal vibration on all the studied field quantities. Additional results across the thickness of the nanobeam are presented graphically.

Keywords: thermoelasticity without energy dissipation; two temperatures; FG nanobeam; harmonically varying heat; a nonlocal beam theory

1. Introduction

The generalised theory of thermoelasticity has been developed to overcome the physically unrealistic prediction of the coupled dynamical theory of thermoelasticity that thermal signals propagate with infinite speed. Lord and Shulman theory (Lord & Shulman, 1967) and Green and Lindsay temperature-rate dependent theory (Green & Lindsay, 1972) are two well-established theories of generalised thermoelasticity. They introduce the thermal relaxation parameters in the basic equations of coupled dynamical thermoelasticity theory and admit the finite value of heat propagation speed. The finiteness of the speed of the thermal signal has been found to have experiment evidence too. The generalised thermoelasticity theories are therefore more realistic, and have found much interest in recent research. Green and Naghdi (GN) (Green & Naghdi, 1993) have formulated a new model of thermoelasticity for the homogeneous and isotropic materials. An important characteristic feature of this model, which is not present in other thermoelastic theories, is that theory does not accommodate dissipation of thermal energy.

Thermoelasticity with two temperatures is one of the nonclassical theories of elastic solids. The main difference of this theory with respect to the classical one is in thermal dependence. Chen and Gurtin (1968) and Chen, Gurtin, and Willams (1969) have

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formulated a theory of heat conduction in deformable bodies. The theory depends on two distinct temperatures: the conductive temperature φ and the thermodynamic temperature θ. For time-independent situations, the difference between these two temperatures is proportional to the heat supply and in the absence of any heat supply, the two temperatures are identical. For time-dependent problems, however, and for wave propagation problems in particular, the two temperatures are in general different, regardless of the presence of heat supply. The two temperatures and the strain are found to have representation in the form of a travelling wave plus throughout the body (see Boley, 1956). The internal energy, entropy, stress, heat flux and thermodynamic temperature at a given material point and time are deduced using the conductive temperature and its two gradients. Warren and Chen (1973) have investigated the wave propagation in the two-temperature theory of thermoelasticity. Quintanilla (2004) has presented some phenomena in thermoelasticity with two temperatures. Zenkour and Abouelregal (2014) have used the state-space approach for an infinite medium with a spherical cavity based upon two-temperature generalised thermoelasticity theory and fractional heat conduction.

Micro-mechanical resonators and nanomechanical resonators have attracted considerable attention recently due to their many important technological applications. Accurate analysis of various effects on the characteristics of resonators, such as resonant frequencies and quality factors, is crucial for designing high-performance components. Many authors have studied the vibration and heat transfer process of beams. The beam may be thermally induced (see Al-Huniti, Al-Nimr, & Naij, 2001; Kidawa-Kukla, 2003; Manolis & Beskos, 1980) or subjected to a suddenly applied heat input distributed along its span (see Boley, 1972). Additional treatments of FG beams under a moving heat source have been presented by Ching and Yen (2006), Malekzadeh and Shojaee (in press) and Mareishi, Mohammadi, and Rafiee (2013).

The nonlocal elasticity theory initiated by Eringen (1972, 1983) and Eringen and Edelen (1972) is widely used. It is to be noted that the local theories assume that the stress at a point is a function of strain at that point. However, the nonlocal elasticity theory assumes that the stress at a point is a function of strains at all points in the continuum. In this work, the governing equations of the two-temperature Green and Naghdi's thermoelasticity theory as well as the nonlocal beam theory of FG nanobeams are given. The present nanobeam is made from a FG ceramic-metal material. The upper surface of the FG nanobeam is fully ceramic whereas the lower surface is fully metal. The effect in heat conduction and the coupling effect between the temperature and strain rate is studied. The solution for the generalised thermoelastic vibration of the FG nanobeam due to a harmonically varying heat is developed. The effects of the nonlocal parameter, the angular frequency of thermal vibration and the two-temperature parameter on the physical field quantities are also studied.

2. Basic equations and formulation of the problem

The heat conduction equation of Green and Naghdi's two-temperature theory takes the form (Quintanilla, 2004):

$$\nabla(K^{*} \cdot \nabla\varphi) = \frac{\partial}{\partial t}\left(\rho C^{e}\frac{\partial\theta}{\partial t} + \gamma T_{0}\frac{\partial e_{kk}}{\partial t} - Q\right). \tag{1}$$

The conduction-dynamical heat equation is given by:

$$\theta = \varphi - a\varphi_{,kk}, \tag{2}$$

where a is a nonnegative two-temperature parameter. The equations of motion without body forces take the forms:

$$\sigma_{ji,j} = \rho \ddot{u}_i. \tag{3}$$

The constitutive equations for different local theories are given by:

$$\sigma_{ij} = 2\mu e_{ij} + (\lambda e_{kk} - \gamma\theta)\delta_{ij}. \tag{4}$$

In these relations, K^* is a material constant characteristic of the theory of Green and Naghdi, ρ is the density, C^e is the specific heat at constant strain, $\theta = T - T_0$ denotes the thermodynamical temperature, T_0 is the reference temperature, Q is the heat supplied per unit volume from the external work, e_{ij} is the strain tensor, $\gamma = (3\lambda + 2\mu)\alpha_t$ is the coupling parameters, in which, λ and μ being Lame's coefficients and α_t being the coefficient of linear thermal expansion, φ is the conductive temperature measured from the temperature φ_0, σ_{ij} are the components of the stress tensor, u_i are the components of the displacement vector and δ_{ij} is Kronecker's delta function.

Let us consider a FG thermoelastic solid beam in Cartesian coordinate systems *Oxyz*. The x-axis is drawn along the axial direction of the beam and the y- and z-axes correspond to the width and thickness, respectively (see Figure 1). The small flexural deflections of the nanobeam with dimensions of length $L(0 \le x \le L)$, width $b(-b/2 \le y \le +b/2)$ and thickness $h(-h/2 \le z \le +h/2)$ are considered. A new model of FGMs is presented to treat the governing equations of the thermoelastic nanobeam that subjected to a harmonically varying heat. Based on this model, the effective material property $P(z)$ gradation through the thickness direction is presented by (Zenkour, 2006, 2014):

$$P(z) = P_m e^{n_P(2z-h)/h}, \quad n_P = \ln\sqrt{P_m/P_c}. \tag{5}$$

where P_m and P_c represent the metal and ceramic properties, respectively. This study assumes, instead of Lame's coefficients, that Young's modulus E, material density ρ, the material constant K^* and the stress-temperature modulus γ of the FGM change continuously through the thickness direction of the beam according to the above gradation relation. It is to be noted that the material properties of the considered beam

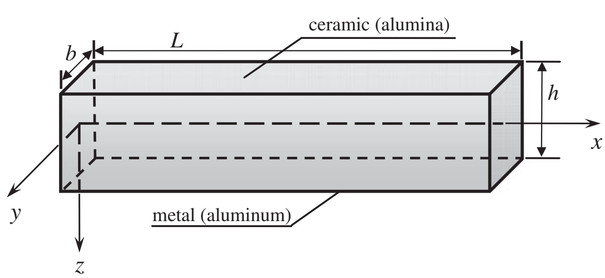

Figure 1. Schematic diagram for the FG nanobeam.

are metal-rich near the lower surface ($z=+h/2$) and ceramic-rich near the upper surface ($z=-h/2$) of the beam.

In the present study, the classical beam theory based upon Euler-Bernoulli assumption (Mareishi et al., 2013) is adopted. This means that any plane cross-section, initially perpendicular to the axis of the beam, remains plane and perpendicular to the neutral surface during bending. Thus, the displacements are given by:

$$u = -z\frac{\partial w}{\partial x},\ v = 0,\ w(x,y,z,t) = w(x,t), \tag{6}$$

where w is the lateral deflection. Substituting this Euler-Bernoulli assumption into Equation (1), with the aid of Equation (2) and Equation (5), gives the heat conduction equation given in Equation (1) for the beam without the heat source ($Q=0$), as

$$\begin{aligned} &K_m^* e^{n_K(2z-h)/h}\left(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial z^2}+\frac{2n_K}{h}\frac{\partial\varphi}{\partial z}\right)\\ &=\frac{\partial^2}{\partial t^2}\left\{\rho_m C_m^e e^{n_{\rho C^e}(2z-h)/h}\left[1-a\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\right]\varphi - z\gamma_m e^{n_\gamma(2z-h)/h}T_0\frac{\partial^2 w}{\partial x^2}\right\}, \end{aligned} \tag{7}$$

where K_m^*, ρ_m, γ_m and C_m^e are, respectively, the material constant, the material density, thermal modulus and the specific heat per unit mass at constant strain of the metal material. Note that

$$\gamma_m = E_m\alpha_m/(1-2\nu_m), \quad \rho_m C_m^e = K_m/\chi_m, \tag{8}$$

where α_m, E_m, ν_m and χ_m are the thermal expansion coefficient, Young's modulus, Poisson's ratio and the thermal diffusivity of the metal material, respectively.

The present nanobeam is thermally insulated, i.e., there is no heat flow across its upper and lower surfaces, so that $\partial\varphi/\partial z$ should be vanished at these surfaces ($z=\pm h/2$). So, let us assume that the conductive temperature varies sinusoidally along the thickness direction of the nanobeam. That is

$$\varphi(x,z,t) = \varphi_1(x,t)\sin\left(\frac{\pi z}{h}\right). \tag{9}$$

Substituting Equation (9) into Equation (7) and integrating the resulting equation with respect to z through the beam thickness from $-h/2$ to $h/2$, yields

$$\frac{\partial^2\varphi_1}{\partial x^2} = \frac{\partial^2}{\partial t^2}\left[\bar{\mu}_{\rho C^e}\eta\left(1-\frac{a\pi^2}{h^2}-a\frac{\partial^2}{\partial x^2}\right)\varphi_1 - \frac{\bar{\mu}_\gamma T_0 h\gamma_m}{K_m^*}\frac{\partial^2 w}{\partial x^2}\right], \tag{10}$$

where $\eta=\rho_m C_m^e/K_m^*$, $\bar{\mu}_{\rho C^e}=\mu_{\rho C^e}/\mu_K$ and $\bar{\mu}_\gamma=\mu_\gamma/\mu_K$ in which

$$\mu_K = \frac{2n_K(1+e^{-2n_K})}{\pi^2+4n_K^2},\ \mu_{\rho C^e} = \frac{2n_{\rho C^e}(1+e^{-2n_{\rho C^e}})}{\pi^2+4n_{\rho C^e}^2},\ \mu_\gamma = \frac{n_\gamma(1+e^{-2n_\gamma})-1+e^{-2n_\gamma}}{4n_\gamma^2}. \tag{11}$$

It is to be noted that the nonlocal theory assumes that stress at a point depends not only on the strain at that point but also on strains at all other points of the body. So, the one-dimensional constitutive equation gives the uniaxial tensile stress only, according to the differential form of the nonlocal constitutive relation proposed by Eringen (1972, 1983); Eringen & Edelen, 1972), as

$$\sigma_x - \xi \frac{\partial^2 \sigma_x}{\partial x^2} = -E_m \left[e^{n_E(2z-h)/h} z \frac{\partial^2 w}{\partial x^2} + \alpha_m e^{n_{E\alpha}(2z-h)/h} \left(1 - \frac{a\pi^2}{h^2} - a \frac{\partial^2}{\partial x^2} \right) \varphi \right], \qquad (12)$$

where $n_{E\alpha} = \ln \sqrt{E_m \alpha_m / E_c \alpha_c}$ in which α_c and E_c are the thermal expansion coefficient and Young's modulus of the ceramic material, respectively. Note that $\xi = (e_0 L)^2$ is the nonlocal parameter, in which e_0 is a constant appropriate to each material and L is the internal characteristic length. In general, a conservative estimate of the nonlocal parameter is $e_0 L < 2.0$ nm for a single-wall carbon nanotube (Wang & Wang, 2007).

The flexure moment of the cross-section is given, with the aid of Equation (9), by

$$M - \xi \frac{\partial^2 M}{\partial x^2} = -bh^2 E_m \left[h\mu_E \frac{\partial^2 w}{\partial x^2} + \alpha_m \mu_{E\alpha} \left(1 - \frac{a\pi^2}{h^2} - a \frac{\partial^2}{\partial x^2} \right) \varphi_1 \right], \qquad (13)$$

where

$$\mu_E = \frac{(n_E^2 + 2)(1 - e^{-2n_E}) - 2n_E(1 + e^{-2n_E})}{8n_E^3},$$
$$\mu_{E\alpha} = \frac{n_{E\alpha}(4n_{E\alpha}^2 + \pi^2)(1 - e^{-2n_{E\alpha}}) + (\pi^2 - 4n_{E\alpha}^2)(1 + e^{-2n_{E\alpha}})}{(\pi^2 + 4n_{E\alpha}^2)^2}. \qquad (14)$$

The differential equation of thermally induced lateral vibration of the nanobeam may be expressed in the form:

$$\frac{\partial^2 M}{\partial x^2} = \frac{1 - e^{-2n_\rho}}{2n_\rho} \rho_m A \frac{\partial^2 w}{\partial t^2}, \qquad (15)$$

where $A = bh$ is the cross-section area. Substituting Equation (13) into Equation (15), one can get the motion equation of the beam as

$$\frac{\partial^4 w}{\partial x^4} + \frac{1 - e^{-2n_\rho}}{2n_\rho \mu_E \varepsilon^2 h^2} \left(\frac{\partial^2 w}{\partial t^2} - \xi \frac{\partial^4 w}{\partial x^2 \partial t^2} \right) + \frac{\alpha_m \bar{\mu}_{E\alpha}}{h} \left(1 - \frac{a\pi^2}{h^2} - a \frac{\partial^2}{\partial x^2} \right) \frac{\partial^2 \varphi_1}{\partial x^2} = 0, \qquad (16)$$

where $\bar{\mu}_{E\alpha} = \mu_{E\alpha}/\mu_E$ and $\varepsilon = \sqrt{E_m/\rho_m}$.

The initial and boundary conditions should be considered to solve the present problem. The initial conditions of the problem are taken as

$$w(x,t)\Big|_{t=0} = \frac{\partial w(x,t)}{\partial t}\Big|_{t=0} = 0, \quad \varphi_1(x,t)\Big|_{t=0} = \frac{\partial \varphi_1(x,t)}{\partial t}\Big|_{t=0} = 0. \qquad (17)$$

Now, the problem is to solve the Equations (10) and (16) subject to the boundary conditions:

(1) The two ends of the nanobeam satisfy the conditions:

$$w(x,t)\Big|_{x=0,L} = 0, \quad \frac{\partial^2 w(x,t)}{\partial x^2}\Big|_{x=0,L} = 0. \qquad (18)$$

(2) Let us also consider the nanobeam is thermally loaded by harmonically varying heat incidents into the surface of the nanobeam $x = 0$. In this case, we take the conductive temperature in the form

$$\varphi_1(x,t)|_{x=0} = f(t) = H(t)\, \Phi_0 \cos(\omega t), \qquad (19)$$

where $H(t)$ is called Heaviside's unite step function, Φ_0 is constant and ω is the angular frequency of thermal vibration ($\omega = 0$ for a thermal shock problem). In addition, the conductive temperature at the end boundary should satisfy the following relation:

$$\frac{\partial \varphi_1}{\partial x} = 0 \quad \text{on} \quad x = L. \tag{20}$$

Let us introduce the following dimensionless quantities:

$$\begin{gathered}(x', L', u', w', z', h') = \eta\varepsilon(x, L, u, w, z, h), \quad t' = \eta\varepsilon^2 t, \quad \omega' = \tfrac{\omega}{\eta\varepsilon^2}, \\ (a', \xi') = \eta^2\varepsilon^2(a, \xi), \quad \varphi_1' = \tfrac{\varphi_1}{T_0}, \quad \sigma_x' = \tfrac{\sigma_x}{E_m}.\end{gathered} \tag{21}$$

Now omitting primes, the governing equations, Equations (2), (10), (12) and (16) can be re-written in the dimensionless forms as

$$\frac{\partial^4 w}{\partial x^4} + A_1\left(\frac{\partial^2 w}{\partial t^2} - \xi\frac{\partial^4 w}{\partial x^2 \partial t^2}\right) + A_2\left(c - a\frac{\partial^2}{\partial x^2}\right)\frac{\partial^2 \varphi_1}{\partial x^2} = 0, \tag{22}$$

$$\frac{\partial^2 \varphi_1}{\partial x^2} = A_3\left(c - a\frac{\partial^2}{\partial x^2}\right)\frac{\partial^2 \varphi_1}{\partial t^2} - A_4\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w}{\partial x^2}\right), \tag{23}$$

$$\theta = \sin\left(\frac{\pi z}{h}\right)\left(c - a\frac{\partial^2}{\partial x^2}\right)\varphi_1, \tag{24}$$

where

$$c = 1 - a\frac{\pi^2}{h^2}, \quad A_1 = \frac{1 - e^{-2n_\rho}}{2n_\rho \mu_E h^2}, \quad A_2 = \frac{\alpha_m \bar{\mu}_{E\alpha} T_0}{h}, \quad A_3 = \bar{\mu}_{\rho C^e}, \quad A_4 = \frac{\bar{\mu}_\gamma h \gamma_m}{\eta K_m}. \tag{25}$$

3. Method of solution

Applying the Laplace transform to Equations (22) and (23), one gets the field equations as

$$\left[\frac{\mathrm{d}^4}{\mathrm{d}x^4} + A_1 s^2\left(1 - \xi\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)\right]\bar{w} = A_2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(a\frac{\mathrm{d}^2}{\mathrm{d}x^2} - c\right)\bar{\varphi}_1, \tag{26}$$

$$s^2 A_4\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}x^2} = -\left((1 + s^2 A_3 a)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - s^2 c A_3\right)\bar{\varphi}_1, \tag{27}$$

where an over bar symbol denotes its Laplace transform, s denotes the Laplace transform parameter. Elimination $\bar{\varphi}_1$ or $\bar{w}$ from Equations (26) and (27) gives the following differential equation for either $\bar{w}$ or $\bar{\varphi}_1$:

$$\left[\frac{\mathrm{d}^6}{\mathrm{d}x^6} - A\frac{\mathrm{d}^4}{\mathrm{d}x^4} + B\frac{\mathrm{d}^2}{\mathrm{d}x^2} - C\right]\{\bar{w}, \bar{\varphi}_1\} = 0, \tag{28}$$

where the coefficients A, B and C are given by

$$A = \frac{cs^2(A_3 + A_2A_4) + A_1s^2\xi(1 + as^2A_3)}{1 + as^2(A_3 + A_2A_4)},$$
$$B = \frac{A_1s^2(1 + as^2A_3 + A_3cs^2\xi)}{1 + as^2(A_3 + A_2A_4)}, \tag{29}$$
$$C = \frac{cA_1A_3s^4}{1 + as^2(A_3 + A_2A_4)}.$$

Introducing m_i $(i = 1, 2, 3)$ into Equation (28), one gets

$$(D^2 - m_1^2)(D^2 - m_2^2)(D^2 - m_3^2)\{\bar{w}, \bar{\varphi}_1\} = 0, \tag{30}$$

where $D = \mathrm{d}/\mathrm{d}x$ and m_1^2, m_2^2 and m_3^2 are the roots of the characteristic equation

$$m^6 - Am^4 + Bm^2 - C = 0. \tag{31}$$

These roots are given by

$$m_1^2 = \tfrac{1}{3}[2p_0\sin(q_0) + A], \quad m_2^2 = -\tfrac{1}{3}p_0[\sqrt{3}\cos(q_0) + \sin(q_0)] + \tfrac{1}{3}A,$$
$$m_3^2 = \tfrac{1}{3}p_0[\sqrt{3}\cos(q_0) - \sin(q_0)] + \tfrac{1}{3}A, \tag{32}$$

where

$$p_0 = \sqrt{A^2 - 3B}, \quad q_0 = \frac{1}{3}\sin^{-1}\left(-\frac{2A^3 - 9AB + 27C}{2p_0^3}\right). \tag{33}$$

The solution of the governing equations, Equation (30), in the Laplace transformation domain can be represented as

$$\{\bar{w}, \bar{\varphi}_1\} = \sum_{i=1}^{3}(\{C_i, F_i\}e^{-m_ix} + \{C_{i+3}, F_{i+3}\}e^{m_ix}), \tag{34}$$

where C_i and F_i are parameters depending on s. The compatibility between these two equations and Equations (26) and (27) gives

$$F_i = \beta_iC_i, \quad F_{i+3} = \beta_iC_{i+3}, \quad \beta_i = \frac{m_i^4 + A_1s^2}{A_2(am_i^4 - cm_i^2)}. \tag{35}$$

So,

$$\{\bar{w}, \bar{\varphi}_1\} = \sum_{i=1}^{3}\{1, \beta_i\}(C_ie^{-m_ix} + C_{i+3}e^{m_ix}). \tag{36}$$

Then, the thermodynamical temperature θ in the Laplace domain with the aid of the above equation becomes

$$\bar{\theta} = \sin\left(\frac{\pi z}{h}\right)\sum_{i=1}^{3}\beta_i(c - am_i^2)(C_ie^{-m_ix} + C_{i+3}e^{m_ix}). \tag{37}$$

Also, the axial displacement and the strain after using Equation (36) take the forms

$$\bar{u} = -z\frac{\mathrm{d}\bar{w}}{\mathrm{d}x} = z\sum_{i=1}^{3} m_i(C_i e^{-m_i x} - C_{i+3}e^{m_i x}),$$
$$\bar{e} = \frac{\mathrm{d}\bar{u}}{\mathrm{d}x} = -z\sum_{i=1}^{3} m_i^2(C_i e^{-m_i x} + C_{i+3}e^{m_i x}). \tag{38}$$

Using Laplace transformation to Equations (18)–(20), the boundary conditions take the forms

$$\bar{w}(x,s)\Big|_{x=0,L} = 0, \quad \frac{\mathrm{d}^2\bar{w}(x,s)}{\mathrm{d}x^2}\Big|_{x=0,L} = 0,$$
$$\bar{\varphi}_1(x,s)\Big|_{x=0} = \frac{s\,\varphi_0}{\omega^2+s^2} = \bar{G}(s), \quad \frac{\mathrm{d}\bar{\varphi}_1(x,s)}{\mathrm{d}x}\Big|_{x=L} = 0. \tag{39}$$

Substituting Equation (36) into the above boundary conditions, one obtains six linear equations in the matrix form as

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ e^{-m_1L} & e^{-m_2L} & e^{-m_3L} & e^{m_1L} & e^{m_2L} & e^{m_3L} \\ m_1^2 & m_2^2 & m_3^2 & m_1^2 & m_2^2 & m_3^2 \\ m_1^2e^{-m_1L} & m_2^2e^{-m_2L} & m_3^2e^{-m_3L} & m_1^2e^{m_1L} & m_2^2e^{m_2L} & m_3^2e^{m_3L} \\ \beta_1 & \beta_2 & \beta_3 & \beta_1 & \beta_2 & \beta_3 \\ -m_1\beta_1e^{-m_1L} & -m_2\beta_2e^{-m_2L} & -m_3\beta_3e^{-m_3L} & m_1\beta_1e^{m_1L} & m_2\beta_2e^{m_2L} & m_3\beta_3e^{m_3L} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \bar{G}(s) \\ 0 \end{Bmatrix}. \tag{40}$$

The solution of the above system of linear equations gives the unknown parameters C_i and C_{i+3}. This completes the solution of the problem in the Laplace transform domain.

4. Inversion of Laplace transforms

There are many problems whose solution may be found in terms of Laplace transform which is then, however, too complicated for inversion using the techniques of complex analysis. Numerous methods have been devised for the numerical evaluation of the Laplace inversion integral. In order to determine the conductive and thermal temperature, displacement and stress distributions in the time domain, the Riemann-sum approximation method is used to obtain the numerical results. In this method, any function in the Laplace domain can be inverted to the time domain as

$$f(t) = \frac{e^{\zeta t}}{t}\left[\frac{1}{2}\ \mathrm{Re}[\bar{F}(\zeta)] + \mathrm{Re}\sum_{n=0}^{N}\left(\bar{F}\left(\zeta + \frac{in\pi}{t}\right)(-1)^n\right)\right], \tag{41}$$

where Re is the real part and i is the imaginary number unit. For faster convergence, numerical experiments have shown that the value that satisfies the above relation is $\zeta \approx 4.7/t$ (Tzou, 1996).

5. Numerical results

In terms of the Riemann-sum approximation defined in Equation (41), numerical Laplace inversion is performed to obtain the nondimensional lateral vibration, temperature, displacement, stress, moment and strain energy in the nanobeam. In the present work, the thermoelastic coupling effect is analysed by considering a beam made of metal-ceramic FGM.

The *aluminium* as lower metal surface and *alumina* as upper ceramic surface are used for the present nanobeam. The material properties are assumed to be:

$$
\begin{gathered}
E_m = 70\,\text{GPa}, \quad E_c = 116\,\text{GPa}, \quad v_m = 0.35, \quad v_c = 0.33, \quad \rho_m = 2700\,\text{Kg}/m^3, \\
\rho_c = 3000\,\text{Kg}/m^3, \quad \alpha_m = 23.1\times10^{-6}\text{K}^{-1}, \quad \alpha_c = 8.7\times10^{-6}\text{K}^{-1}, \\
\chi_m = 84.18\times10^{-6}\text{m}^2/\text{s}, \quad \chi_c = 1.06\times10^{-6}\text{m}^2/\text{s}, \\
K_m^* = 237\,\text{W}/(\text{mK}), \quad K_c^* = 1.78\,\text{W}/(\text{mK}).
\end{gathered} \tag{42}
$$

The computations are carried out for $t = .12$ and $\Phi_0 = 1.0$. The conductive temperature, the dynamical temperature, the stress and the strain distributions are represented graphically with respect to wide range of $x\,(0 \le x \le 1)$. The reference temperature of the nanobeam is $T_0 = 293$ K. The aspect ratio of the beam is fixed as $L/h = 10$. The dimensionless variables defined in Equation (21) are plotted for a wide range of beam length when $L = 1$ and $h = 0.1$. The thickness position is assumed, except otherwise stated, to be $z = h/6$. Some plots considered the present quantities through the length of the beam and others took into account both the length and thickness directions.

Equation (41) was used to invert the Laplace transforms in Equations (36)–(38) to graphically present the transverse deflection, the conductive temperature, the dynamical

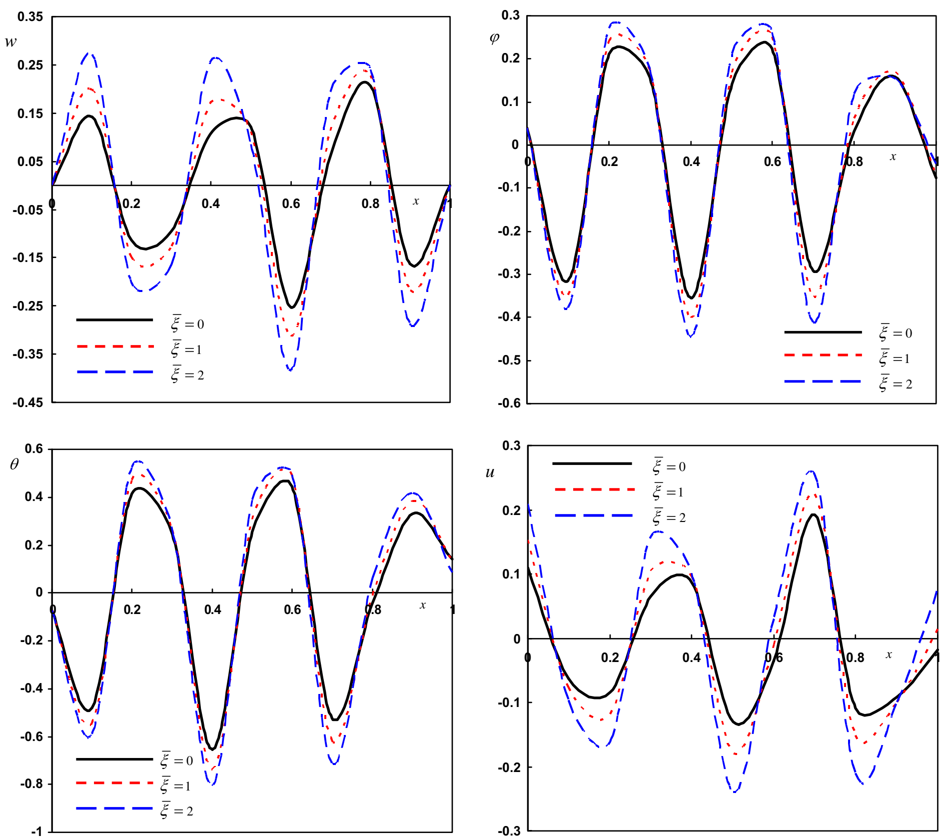

Figure 2. (a) The transverse deflection, (b) the conductive temperature, (c) the dynamical temperature and (d) the displacement distributions of the FG nanobeam for different values of the nonlocal parameter $\bar{\xi}$ ($z = h/6$, $a = 0.01$, $t = 0.12$, $\omega = 5$).

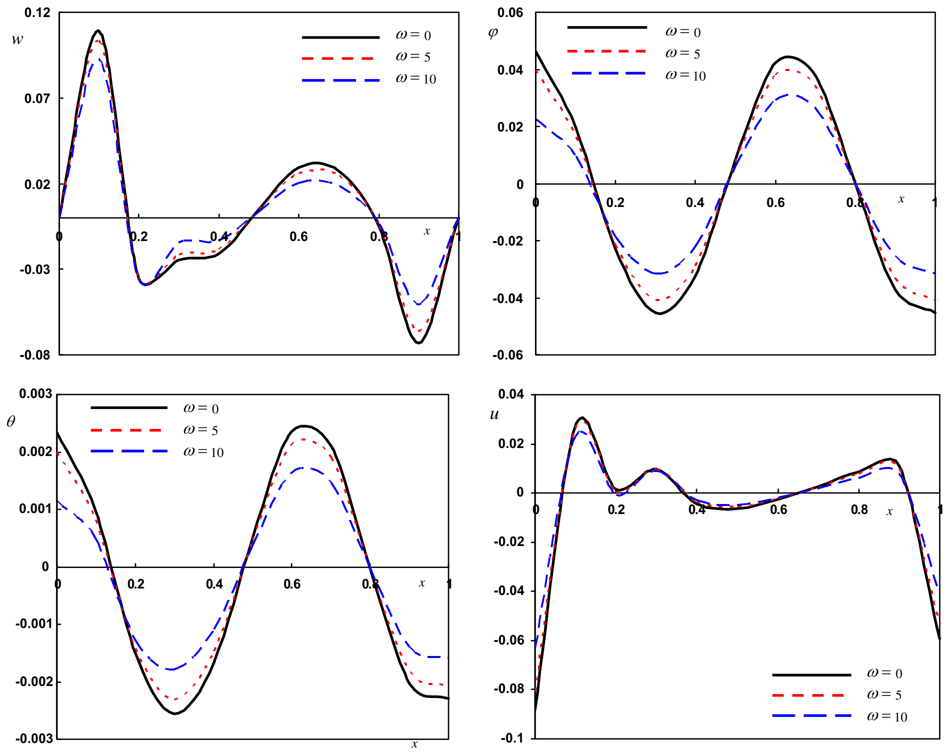

Figure 3. (a) The transverse deflection, (b) the conductive temperature, (c) the dynamical temperature and (d) the displacement distributions of the FG nanobeam for different values of temperature discrepancy a ($z = h/6$, $t = 0.12$, $\omega = 5$, $\bar{\xi} = 2$).

temperature, and the displacement distributions with respect to x and z directions. For all numerical calculations, *Mathematica* programming language has been used. Figures 2–5 represent the curves predicted for these quantities. Numerical calculations are carried out for three cases as follows:

Case I: The effect of nonlocal parameter $\xi = 10^{-7}\bar{\xi}$, ($\bar{\xi} = 0, 1, 2$) on the dimensionless transverse deflection, the conductive temperature, the thermodynamic temperature and the displacement (see Figure 2). Note that the value of $\bar{\xi} = 0$ indicates the local theory while the values $\bar{\xi} = 1$ and 2 indicate the nonlocal theory. In this case, one assumes that the angular frequency of the thermal vibration is $\omega = 5$ and the dimensionless temperature discrepancy $a = 0.01$.

Case II: Investigating how the dimensionless conductive temperature, thermodynamic temperature and displacement vary with different values of the dimensionless temperature discrepancy a (see Figure 3). Note that, the value of $a = 0$ indicates the old situation (GN model or the one-temperature theory (1TT)) while the values $a = 0.01$ and 0.02 indicate the nonlocal two-temperature theory (2TT). In this case, one assumes that the angular frequency of the thermal vibration is $\omega = 5$ and the nonlocal parameter is $\bar{\xi} = 2$.

Case III: Illustrating in Figure 4 how the field quantities vary with the different values of the angular frequency of thermal vibration $\omega = 0, 5, 10$ ($\omega = 0$ for a thermal shock problem) with constant value of the temperature discrepancy parameter ($a = 0.01$) and the nonlocal parameter ($\bar{\xi} = 2$).

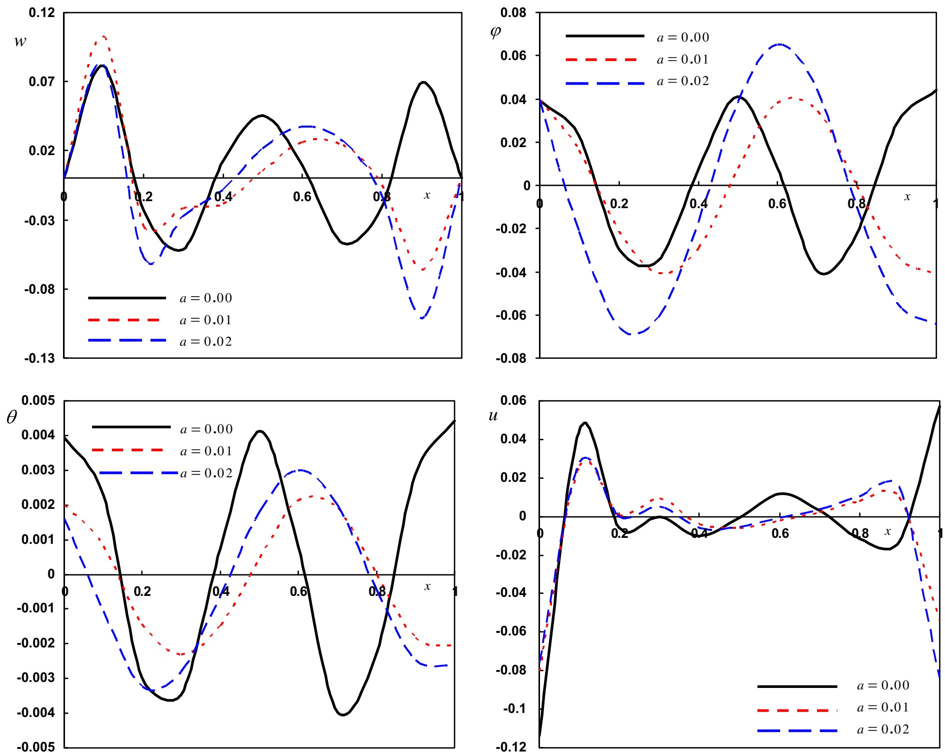

Figure 4. The effect of angular frequency of thermal vibration on (a) the transverse deflection, (b) the conductive temperature, (c) the dynamical temperature and (d) the displacement distributions of the FG nanobeam ω $(z = h/6,\ t = 0.12,\ a = 0.01,\ \bar{\xi} = 2)$.

Case IV: Discussing the behaviour of the field quantities through-the-thickness of the nanobeam for fixed values of the angular frequency, the nonlocal parameter and the temperature discrepancy parameter is $(\omega = 5,\ \bar{\xi} = 2,\ a = 0.01)$ (see Figure 5).

In the first case, the values of the nonlocal parameter $\bar{\xi}$ with the constant parameters $a = 0.01$ and $\omega = 5$ are considered. For a local theory, one puts $\bar{\xi} = 0$ and for a nonlocal theory, $\bar{\xi}$ may be 1 or 2. It is found from Figure 2 that the nonlocal parameter $\bar{\xi}$ has significant effects on all field quantities. The nonlocal parameter enlarges the waves of all field quantities. It is observed that as the value of the nonlocal parameter $\bar{\xi}$ increases the peak of thermal waves of all field quantities. It is seen from figures that as the value of $\bar{\xi}$ increases, the magnitude of the temperature and other fields decreases for fixed x.

In the second case, three different values of the temperature discrepancy parameter a are considered. During the x-axis, all field quantities reflect their behaviours when the value of a increases. This shows the difference between the nonlocal one-temperature generalised thermoelasticity of Green and Naghdi $(a = 0)$ and the nonlocal two-temperature generalised thermoelasticity $(a = 0.01,\ 0.02)$ models. The figures show that this parameter has significant effect on all studied field. According to the figures, the two-temperature parameter plays vital role on the speed of the wave propagation of all the studied fields. The field quantities waves may be reach

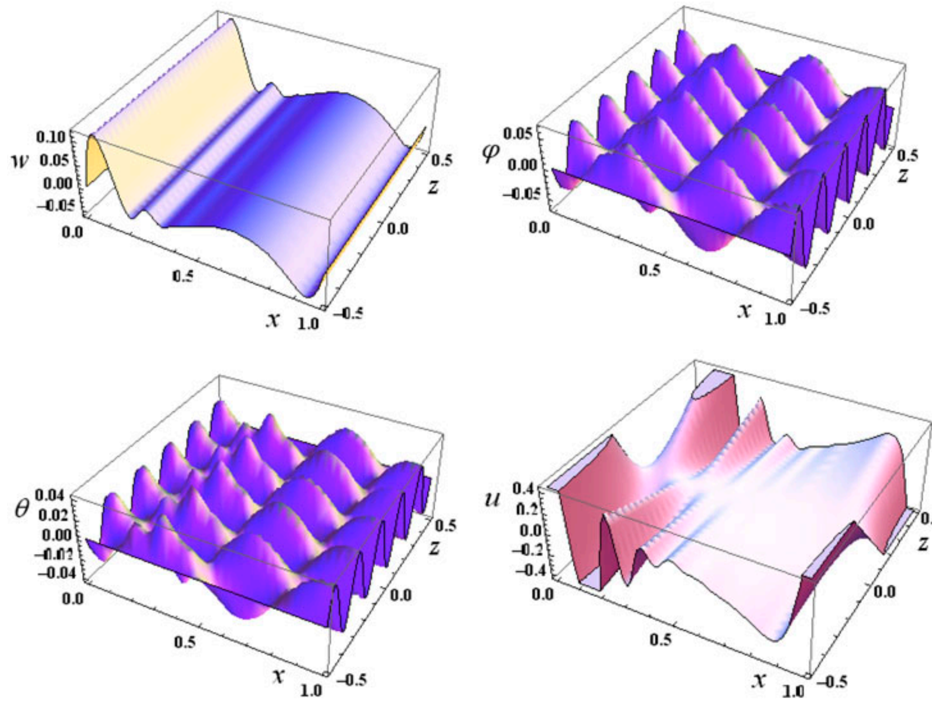

Figure 5. (a) The transverse deflection, (b) the conductive temperature, (c) the dynamical temperature and (d) the displacement distributions versus the axial and thickness directions ($a = 0.01$, $t = 0.12$, $\omega = 5$, $\bar{\xi} = 2$).

the steady state or/and reflect their behaviours depending on the value of the temperature discrepancy a. So, according to the results of this work, it is important to distinguish between the dynamical temperature and the conductive temperature.

In the third case, we considered values of the angular frequency of thermal vibration ω with the constant parameter $a = 0.01$ and $\bar{\xi} = 2$. For a thermal shock problem, we put $\omega = 0$ and for a harmonically heat problem, ω may be 5 or 10. From Figure 4, it is found that, the angular frequency of thermal vibration ω has significant effects on all field quantities. The amplitudes of the field waves are increasing as ω increases. The angular frequency of thermal vibration ω makes the difference between the results in the context of the nonlocal theory two-temperature generalised thermoelasticity and GN theory. It is observed from these figures that the angular frequency of thermal vibration ω increases as the magnitude of the field quantities decrease.

In the last case, the field quantities of the nanobeam at constant values of $\omega = 5$ and $\bar{\xi} = 2$, and in a wide range of thickness $-1/2 \le z/h \le 1/2$ are presented in three-dimensions in Figure 5. In those figures, we can see the effects of the changing of the thickness on all the studied fields. When the thickness increases, all values of the studied fields increase and this is very obvious at the peak points of the curves. It is concluded that the different gradient parameters produce different distributions of the thermal stresses.

6. Conclusion

In this paper, a new model of two-temperature generalised thermoelasticity without energy dissipation and a nonlocal Euler-Bernoulli beam theory for a FG nanobeam is constructed. The vibration characteristics of the deflection, conductive temperature, thermodynamics temperature, displacement, stress and strain energy of nanobeams due to harmonically varying heat are investigated in the context of two-temperature theory of thermoelasticity without energy dissipation. The effects of the nonlocal parameter, the angular frequency of thermal vibration ω and the two-temperature parameter a of thermal vibration on the field variables are investigated. Numerical technique based on the Laplace transformation has been used. The effects of different parameters on all the studied field quantities have been shown and presented graphically.

According to the results shown in all figures, it is found that the nonlocal parameter as well as the two-temperature parameter plays a vital role on the speed of the wave propagation of the heat conduction and the dynamical heat. The presence of the nonlocal parameter $\bar{\xi}$ has significant effect on the solutions of all studied fields.

The significant differences in the physical quantities are observed for all the one-temperature model (GN model) and two-temperature models. Two-temperature theory is more realistic than the one-temperature theory in the case of generalised thermoelasticity. The two-temperature generalised theory of thermoelasticity describes the behaviour of the particles of an elastic body which is more realistic than the one-temperature theory of generalised thermoelasticity. On the other hand, the thermoelastic stresses, displacement and temperature have a strong dependency on the angular frequency parameter.

References

Al-Huniti, N. S., Al-Nimr, M. A., & Naij, M. (2001). Dynamic response of a rod due to a moving heat source under the hyperbolic heat conduction model. *Journal of Sound and Vibration, 242*, 629–640.

Boley, M. (1956). Thermoelastic and irreversible thermo dynamics. *Journal of Applied Physics, 27*, 240–253.

Boley, B. A. (1972). Approximate analyses of thermally induced vibrations of beams and plates. *Journal of Applied Mechanics, 39*, 212–216.

Chen, P. J., & Gurtin, M. E. (1968). On a theory of heat conduction involving two temperatures. *Zeitschrift für angewandte Mathematik und Physik ZAMP, 19*, 614–627.

Chen, P. J., Gurtin, M. E., & Willams, W. O. (1969). On the thermodynamics of non-simple elastic materials with two temperatures. *Zeitschrift für angewandte Mathematik und Physik ZAMP, 20*, 107–112.

Ching, H. K., & Yen, S. C. (2006). Transient thermoelastic deformations of 2-D functionally graded beams under nonuniformly convective heat supply. *Composite Structures, 73*, 381–393.

Eringen, A. C. (1972). Nonlocal polar elastic continua. *International Journal of Engineering Science, 10*, 1–16.

Eringen, A. C. (1983). On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves. *Journal of Applied Physics, 54*, 4703–4710.

Eringen, A. C., & Edelen, D. G. B. (1972). On nonlocal elasticity. *International Journal of Engineering Science, 10*, 233–248.

Green, A. E., & Lindsay, K. A. (1972). Thermoelasticity. *Journal of Elasticity, 2*, 1–7.

Green, A. E., & Naghdi, P. M. (1993). Thermoelasticity without energy dissipation. *Journal of Elasticity, 31*, 189–209.

Kidawa-Kukla, J. (2003). Application of the green functions to the problem of the thermally induced vibration of a beam. *Journal of Sound and Vibration, 262*, 865–876.

Lord, H. W., & Shulman, Y. (1967). A generalized dynamical theory of thermoelasticity. *Journal of the Mechanics and Physics of Solids, 15*, 299–309.

Manolis, G. D., & Beskos, D. E. (1980). Thermally induced vibrations of beam structures. *Computer Methods in Applied Mechanics and Engineering, 21*, 337–355.

Malekzadeh, P. & Shojaee, A. (in press). Dynamic response of functionally graded beams under moving heat source. *Journal of Vibration and Control*. doi:10.1177/1077546312464990

Mareishi, S., Mohammadi, M., & Rafiee, M. (2013). An analytical study on thermally induced vibration analysis of FG beams using different HSDTs. *Applied Mechanics and Materials, 249–250*, 784–791.

Quintanilla, R. (2004). On existence, structural stability, convergence and spatial behavior in thermoelastic with two temperature. *Acta Mechanica, 168*, 161–173.

Quintanilla, R. (2004). Exponential stability and uniqueness in thermoelasticity with two temperatures. *Series A: Mathematics Analysis, 11*, 57–68.

Tzou, D. (1996). *Macro-to-micro heat transfer*. Washington, DC: Taylor & Francis.

Wang, Q., & Wang, C. M. (2007). The constitutive relation and small scale parameter of nonlocal continuum mechanics for modelling carbon nanotubes. *Nanotechnology, 18*, 075702.

Warren, W. E., & Chen, P. J. (1973). Wave propagation in the two temperature theory of thermoelasticity. *Acta Mechanica, 16*, 21–23.

Zenkour, A. M. (2006). Steady-state thermoelastic analysis of a functionally graded rotating annular disk. *International Journal of Structural Stability and Dynamics, 6*, 1–16.

Zenkour, A. M. (2014). On the magneto-thermo-elastic responses of FG annular sandwich disks. *International Journal of Engineering Science, 75*, 54–66.

Zenkour, A. M., & Abouelregal, A. E. (2014). State-space approach for an infinite medium with a spherical cavity based upon two-temperature generalized thermoelasticity theory and fractional heat conduction. *Zeitschrift für angewandte Mathematik und Physik ZAMP*, *65*, 149–164.