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Mathematical and numerical results on the parametric sensitivity of a ROM-POD of the Burgers equation

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We are interested in the mathematical study of the sensitivity of a reduced order model (ROM) of a particular single-parameterised quasi-linear equation, via the parametric evolution. More precisely, the ROM of interest is obtained in two different ways: First, we reduce the complete parametric equation using a proper orthogonal decomposition (POD) basis computed at a given reference value of the parameter, and second the parametric ROM is obtained by an expanded POD basis associated this time to a reference solution and its parametric derivative. The second case of our study was considered in a nearly similar way in Ito and Ravindran (1998), but in the context of the reduced basis (RB) method of the Navier–Stokes equations reduction. Indeed, the authors, Ito and Ravindran (1998) proposed to use an expanded set of basis functions, including solution flows for different values of the Reynolds number and their associated first-order derivatives with respect to this parameter. Beside this work, our second strategy for the parametric ROM-POD construction is to consider a temporal snapshots set including a reference solution and its first-order derivative with respect to the corresponding parameter reference value. We give in both proposed cases of the POD basis construction, an *a priori* estimate of the parametric squared L^2-error between the ROM's solution and the one associated to the full semi-discrete problem. We will show that this estimate will be depending on the distance between two distinct parameters and the evolution of the ROM's dimension. Moreover, we show that an *a priori* upper bound of the squared L^2-ROM-POD error is much better in the case of an expanded POD basis functions. In particular, we apply our theoretical study to the one-dimensional Burgers equation. Numerical tests are done for the one-dimensional Burgers equation, only in the case of a POD basis associated with a reference solution at a fixed value of the viscosity.

Keywords: ROM; POD; sensitivity; parametric evolution; error estimate; Burgers equation

1. Introduction and main results

1.1. Statement of the problem

The resolution cost of the fluid mechanics equations for a given Reynolds number is too high from the two points of view: PC storage capacity and time for computations. Also, the boundary condition problems with parametric evolution often require large computation time: Inverse problems form optimisation problems. There are also the optimal control problems by variation of the fluid flow characteristic parameters

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(Allery, Béghein, & Hamdouni, 2005, 2008; Kunisch & Volkwein, 1999; Ly & Tran, 1998). To solve this in practice, one reduces the degrees of freedom of the Navier–Stokes system. One reduced order model (ROM) is then used to predict the behaviour of fluid flows at different parametric values.

Several techniques of models reduction exist to build a good candidate within the parametric ROMs.

We consider first the reduced basis (RB) method. It is based on showing a given parametric solution as a finite linear combination of solutions associated, respectively, with particular parameters values. There is a detailed literature of the RB method in Buffa, Maday, Patera, Prud'homme, and Turinici (2012), Chen, Hesthaven, Maday, Rodriguez, and Zhu (2012), Grepl, Maday, Nguyen, and Patera (2007), Grepl and Patera (2005), Machiels, Maday, and Patera (2001), Maday, Patera, and Turinici (2002), Nguyen, Rozza, and Patera (2009), and Veroy, Prudhomme, and Patera (2003).

A very adaptive technique for building a parametric ROM is the proper generalised decomposition method. It is based on building an approximation of the initial PDE as a finite combination of functions of separate variables, including not only the space and time variables, but also all eventual parameters that could be associated with the initial equations. These functions and there coefficients in the later expression are obtained by an iterated algorithm which minimises the error with respect to the initial problem. This method was introduced by Ladeveze in the LATIN method (Ladeveze, 1999; Ladeveze & Nouy, 2003), where he started by a space–time separation. Then, it was generalised by Chinesta et al. for multidimensional problems (Ammar, Chinesta, Diez, & Huerta, 2010; Ammar, Mokdad, Chinesta, & Keunings, 2006; Ammar, Normandin, & Chinesta, 2010; Chinesta, Ammar, & Cueto, 2010; Chinesta et al., 2013; Gonzalez et al., 2012).

In this paper, we are interested in the models reduction by projection, the proper orthogonal decomposition (POD) technique (Lumley, 1967).

Given X a Hilbert space, and $(u(t))_t$ a family in X living on a time interval $(0, T)$ and integrable in the sense of $L^2(0, T; X)$, it is about to construct a projection subspace spanned by a basis Φ and which minimises the error obtained *a posteriori* by the orthogonal projection Π_N^Ψ of u on a subspace of X of finite dimension N spanned by an orthonormal Hilbert basis Ψ:

$$\left\|u - \Pi_N^\Phi u\right\|^2_{L^2(0,T;X)} \leq \left\|u - \Pi_N^\Psi u\right\|^2_{L^2(0,T;X)}, \qquad \forall N,$$

$$\text{and } \forall\Psi \text{ an orthonormal Hilbert basis.}$$

This minimisation problem is equivalent to the eigenvalues; one of the compact, self-adjoint and positive operator $\mathcal{R}$ defined as follows:

$$\mathcal{R}\Phi = \mu\Phi.$$

Where,

$$\mathcal{R}: \quad \begin{array}{c} X \to X \\ \varphi \mapsto \mathcal{R}\varphi = \frac{1}{T}\int_0^T (u(t), \varphi)_X u(t)dt. \end{array}$$

$(.,.)_X$ denotes the scalar product of the Hilbert space X.

One can verify that the optimal error by POD is equal to the remainder of the POD-eigenvalues series of $\mathcal{R}$:

$$\frac{1}{T}\left\|u-\Pi_N^{\Phi}u\right\|^2_{L^2(0,T;X)}=\sum_{n=N+1}^{+\infty}\mu_n.$$

The problem that appears naturally is to find a way to control the parametric evolution in a ROM by POD of high-dimensional parabolic parameterised partial differential equations.

In order to determine a strong ROM-POD via parametric evolution, several algorithms were based on a geometric interpolation of POD basis functions for different values of the parameters. We cite the work of Amsallem, Cortial, Carlberg, and Farhat (2009), Amsallem and Farhat (2008), and Amsallem, Cortial, and Farhat (2009).

Nevertheless, there are other methods for computing parametric ROM-PODs and which are essentially based on constructing an expanded basis by including the derivatives of the POD basis functions with respect to the corresponding system parameters. Hay et al. have applied the last method (Hay, Akhtar, & Borggaard, 2012; Hay, Borggaard, Akhtar, & Pelletier, 2010; Hay, Borggaard, & Pelletier, 2009), in order to construct a reduced model ROM-POD which is strong and efficient via the variation of the viscosity of a two-dimensional flow around a square cylinder. Moreover, the technique of including the parametric sensitivity was also used in the context of the RB method, for having more efficient parametric ROMs. We cite the work of Ito and Ravindran (1998) where they proposed to use an expanded set of basis functions, including solution flows for different values of the Reynolds number and their associated first-order derivatives with respect to this parameter. In this paper, we try to investigate the efficiency of such technique, theoretically and from the point of view of the POD technique, i.e. an expanded POD basis associated with a reference solution and its first-order parametric derivative.

Besides, Terragni and Vega (2012) presented an argument to use the POD approximation at a given parametric value to construct the bifurcation diagram of dissipative systems, which is not the aim of the present paper.

We are interested then in mathematical contributions concerning the study of the sensitivity of a ROM of high-dimensional parabolic parameterised partial differential equations, by a reference POD basis and an expanded POD one, associated with a reference solution at a given value of the parameter and to its first-order derivative with respect to this later. More precisely, the organisation of the paper remainder is:

1.2. Organisation of the paper

In the remainder of this section, we give the mathematical formulation of the problem of interest and write formally the main results. In Section 2, we prove our results. In Section 3, we show that the same proof applies in particular to the one-dimensional model equation of Burgers. The parameter of this equation will be the viscosity $\lambda \in \mathbb{R}^{+*}$. In Section 4, we present numerical tests relative to the Burgers equation. In Section 5, we conclude by giving some prospects of this work.

1.3. Mathematical formulation of the problem

Let us consider a problem describing the evolution of a parametric solution u_λ ($\lambda \in \mathbb{R}^{+*}$) in a Hilbert space X. V denotes a subspace of X and V_h a subspace of V of dimension M:

$$\begin{cases} \frac{\partial u_\lambda}{\partial t} = A_\lambda(u_\lambda(t)) \\ u_\lambda(0) = u_0 \end{cases} \tag{1}$$

From now on, we are interested in the ordinary differential equation of dimension M obtained by a Galerkin projection of the initial complete problem (1) on V_h: $(.,.)_X$ denotes the scalar product of the space X.

$$\begin{cases} \left(\frac{\partial u_\lambda^h}{\partial t}, v^h\right)_X = (A_\lambda^h(u_\lambda^h(t)), v^h)_X \quad \forall v^h \in V_h \\ (u_\lambda^h(0), v^h)_X = (u_0, v^h)_X \quad \forall v^h \in V_h \end{cases} \tag{2}$$

A_λ^h is the projection of the operator A_λ on V_h.

A solution $u_{\lambda_0}^h$ of this equation associated to a parameter λ_0 is computed once and for all. A POD basis $\Phi^{\lambda_0} = (\Phi_n^{\lambda_0})_{n=1,\ldots,M}$ of dimension M in X, associated to $u_{\lambda_0}^h$ on a time interval $(0, T)$, will lead to construct a ROM-POD describing the evolution of an approximation $\hat{u}_{\lambda,\lambda_0}$ of u_λ^h, in a subspace of dimension N very small compared to M. We denote $(\mu_n^{\lambda_0})_{n=1,\ldots,M}$ the POD eigenvalues sequence associated to the POD basis:

$$\forall t \in (0, T) \text{ and } \forall x \in X :$$

$$\hat{u}_{\lambda,\lambda_0}(t,x) = \sum_{n=1}^{N} a_n^{\lambda,\lambda_0}(t)\Phi_n^{\lambda_0}(x)$$

where, $\forall n = 1,\ldots,N, a_n^{\lambda,\lambda_0}(t)$ is the solution of the following dynamical system thanks to the orthonormal construction of a POD basis:

$$\begin{cases} \frac{da_n^{\lambda,\lambda_0}}{dt} = \left(\hat{A}_{\lambda,\lambda_0}(\hat{u}_{\lambda,\lambda_0}(t)), \Phi_n^{\lambda_0}\right)_X \\ a_n^{\lambda,\lambda_0}(0) = (u_0, \Phi_n^{\lambda_0})_X \end{cases} \tag{3}$$

$\hat{A}_{\lambda,\lambda_0}$ is the projection of the operator A_λ^h on the POD subspace spanned by the orthonormal set $(\Phi_1^{\lambda_0}, \ldots, \Phi_{\lambda_0}^N)$.

The question which arises naturally is the following: To what extent $\hat{u}_{\lambda,\lambda_0}$ stays adapted to u_λ^h?

Then, the main problem is to give a mathematical criteria in order to determine a confidence region for a reference POD basis. It appears also the control problem of the dimension of the reduced model, so we can improve its performance via *the parametric evolution*, without loss of the reduction concept. It is worth noting that, Terragni, Valero and Vega (2011) developed a numerical adaptive method to accelerate time-dependent numerical solvers of PDEs, by the use of error estimates to predict the required number of POD modes in order to preserve stability of a ROM by POD in connection with high-order modes truncation. Also, this is not the problem treated in this paper.

1.4. Main results

The results we are proposing, for a particular class of evolution problems where the operator A_λ^h is α – Holder with respect to $\lambda(\alpha \in (0, 1])$, are *a priori* upper bounds of the squared error in the space $L^2(0, T; X)$ induced after a Galerkin approach for u_λ^h by a reference POD basis and an expanded one:

Reference POD basis

Theorem 1. *There exist two decreasing sequences* $(f_1^{\lambda_0}(N))_{N=1,\ldots,M}$ and $\left(f_2^{\lambda_0}(N)\right)_{N=1,\ldots,M}$, such that:

$$\left\|u_\lambda^h - \hat{u}_{\lambda,\lambda_0}\right\|^2_{L^2(0,T;X)} \leq f_1^{\lambda_0}(N) + f_2^{\lambda_0}(N)\frac{\mathcal{B}_{\lambda_0}|\lambda-\lambda_0|^{2\alpha}}{\lambda_0}. \tag{4}$$

The estimate (4) shows that the parametric squared ROM-POD error $\left\|u_\lambda^h - \hat{u}_{\lambda,\lambda_0}\right\|^2_{L^2(0,T;X)}$, is controlled by a first term $f^1_{\lambda_0}(N)$ which will estimate the squared POD-Galerkin error relative to the parameter λ_0, and a second term which is a function of the alpha-th power of the distance between λ and λ_0, multiplied by an $f^2_{\lambda_0}(N)$ which tends to zero when N becomes close to M.

This result establishes an *a priori* estimate of the decrease rate of the squared ROM-POD error, especially when the two parameters λ and λ_0 are distant. This rate depends on the one of the sequence $f^2_{\lambda_0}(N)$. Moreover, for a fixed POD modes number N, this result shows the dependency of the ROM-POD error with respect to the distance between the parameters.

Precision on result 1

Heuristic result 1. *Under regularity conditions on the solutions difference $u_\lambda^h - u_{\lambda_0}^h$, an a priori estimate of the squared ROM-POD error is given by:*

$$\left\|u_\lambda^h - \hat{u}_{\lambda,\lambda_0}\right\|^2_{L^2(0,T;X)} \leq f_1^{\lambda_0}(N) + c(N)\frac{\mathcal{B}_{\lambda_0}|\lambda-\lambda_0|^{2\alpha}}{\lambda_0} \tag{5}$$

where, $c(N)$ is of the order of $\frac{1}{N^m}$.

Improvement of the ROM-POD confidence interval: Expanded POD basis

Theorem 2. *Under the following restrictive conditions:*

- $\frac{\partial u_\lambda^h}{\partial \lambda}(\lambda_0) \in L^2(0,T;X)$.
- Φ^{λ_0} a POD basis associated to snapshots of $u^h_{\lambda_0}(t)$ and $\frac{\partial u_\lambda^h}{\partial \lambda}(\lambda_0)(t)$ on $(0, T)$.

We can show that:

$$\left\|u_\lambda^h - \hat{u}_{\lambda,\lambda_0}\right\|^2_{L^2(0,T;X)} \leq f_1^{\lambda_0}(N) + \sum_{n=N+1}^{M} \mu_n^{\lambda_0}|\lambda-\lambda_0|^2 + C|\lambda-\lambda_0|^4 \tag{6}$$

For a fixed POD modes number N, the estimate (6) improves the validity domain of the ROM beside our previous result (estimate (4)) thanks to the term depending on $(\lambda-\lambda_0)^4$, and to the remainder of the expanded-POD eigenvalues sum multiplying $(\lambda-\lambda_0)^2$ and which decreases so fast.

2. Proof of the main results for a quasi-linear parabolic equation

We place our problem in the case of a semi-discrete quasi-linear parabolic equation with a dissipative term:

We denote $X = L^2(\Omega)$, $V = H^1(\Omega)$. Where, Ω is a bounded open set, connected and lipschitz of $\mathbb{R}^d$, and V_h is a subspace of V of dimension M. f is given in $L^2_{loc}([0,+\infty[,X)$. This equation is given by its weak formulation as follows:

$$\begin{cases} \left(\frac{\partial u_\lambda^h}{\partial t}, v^h\right)_X + b(u_\lambda^h(t), u_\lambda^h(t), v^h) + a(u_\lambda^h(t), v^h, \lambda) = F_t(v^h) & \forall v^h \in V_h \\ (u_\lambda^h(0), v^h)_X = (u_0, v^h)_X \quad \forall v^h \in V_h \end{cases}. \tag{7}$$

where, b, a and F_t are, respectively, a trilinear form, a bilinear one and a linear one verifying the following:

- $\forall v_1, v_2 \in V, a(v_1, v_2; \lambda)$ is a bilinear symmetric and positive form, which is continuous on $V \times V$ and coercive on $H_0^1(\Omega) \times H_0^1(\Omega)$. Moreover, we suppose that a is α – Holder with respect to λ, with Lipschitz constant equal to 1.
- $\forall v_1, v_2, v_3 \in V, b(v_1, v_2, v_3) = (\boldsymbol{\mathcal{A}}(v_1).\boldsymbol{\mathcal{B}}(\nabla v_2), v_3)_X$. Where, $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{B}}$ are, respectively, linear forms in v_1 and ∇v_2.
- $\forall v \in V, F_t(v) = (f(t), v)_X$.
- (.,.) denotes the scalar product of X.

We precise some additional notations that could be useful to our proof:

- C_p^λ denotes the constant relative to the coercivity of the bilinear form a.
- C_a, C_b are, respectively, the two norms of the forms a and b on (V_h, $\|.\|_X$) and $K = \left\| u_{\lambda_0}^h \right\|_{L^\infty(0,T;X)}$.

For simplicity reasons, we suppose from now on that $C_p^\lambda = \lambda C_p$ and $\alpha = 1$, without any loss of generality.

In what follows, we give four required key points for completing the proof of our formal results. These key points will be divided through four subsections:

2.1. *Key point 1: the study of the squared POD-Galerkin error relative to a fixed parameter λ_0*

We denote $\hat{u}_{\lambda_0} = \hat{u}_{\lambda,\lambda_0}$ for $\lambda = \lambda_0$.

We prove a result on the decrease rate of the squared POD-Galerkin error relative to the parameter λ_0: $\left\| u_{\lambda_0}^h - \hat{u}_{\lambda_0} \right\|_{L^2(0,T;X)}^2$.

Thanks to the optimal property of the POD reference basis functions, energetic wise with respect to $u_{\lambda_0}^h$, i.e. $\left\| u_{\lambda_0}^h - \Pi_N^{\Phi^{\lambda_0}} u_{\lambda_0}^h \right\|_{L^2(0,T;X)}^2 = T \sum_{n=N+1}^M \mu_n^{\lambda_0}$, we will just study an *a priori* control of $\left\| \Pi_N^{\Phi^{\lambda_0}} u_{\lambda_0}^h - \hat{u}_{\lambda_0} \right\|_{L^2(0,T;X)}^2$.

Proposition 1. *$\forall t \in (0, T)$ and $\forall \varepsilon > 0$, one has:*

$$\left\| (\Pi_N^{\Phi^{\lambda_0}} u_{\lambda_0}^h - \hat{u}_{\lambda_0})(t) \right\|_X^2 \leq \exp\left[(\varepsilon - 2\lambda_0 C_p + 6C_b K)t\right] \left(2C_b K + \frac{\lambda_0^2 C_a^2}{\varepsilon} \right) T \sum_{n=N+1}^{M} \mu_n^{\lambda_0}. \quad (8)$$

Proof. We replace v^h by $\Phi_n^{\lambda_0}$ in Equation (7) (written for the parameter λ_0) and we use the commutativity between the projector operator $\Pi_N^{\Phi^{\lambda_0}}$ and the time derivative $\frac{d}{dt}$, then we can deduce the following equality:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \left\| \Pi_N^{\Phi^{\lambda_0}} u_{\lambda_0}^h - \hat{u}_{\lambda_0} \right\|_X^2 = \quad & -b((u_{\lambda_0}^h - \hat{u}_{\lambda_0})(t), u_{\lambda_0}^h(t), (\Pi_N^{\Phi^{\lambda_0}} u_{\lambda_0}^h - \hat{u}_{\lambda_0})(t)) \\ & -b\left(\hat{u}_{\lambda_0}(t), (u_{\lambda_0}^h - \hat{u}_{\lambda_0})(t), (\Pi_N^{\Phi^{\lambda_0}} u_{\lambda_0}^h - \hat{u}_{\lambda_0})(t)\right) \\ & -a((u_{\lambda_0}^h - \hat{u}_{\lambda_0})(t), \left(\Pi_N^{\Phi^{\lambda_0}} u_{\lambda_0}^h - \hat{u}_{\lambda_0}\right)(t), \lambda_0). \end{aligned}$$

Thanks to the continuity of the form b on $(V_h, ||\cdot||_X)$, we write:

$$
\begin{aligned}
&- b((u^h_{\lambda_0} - \hat{u}_{\lambda_0})(t), u^h_{\lambda_0}(t), \left(\Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0} - \hat{u}_{\lambda_0}\right)(t)) \\
&\leq C_b \left\| (u^h_{\lambda_0} - \hat{u}_{\lambda_0})(t) \|_X \| u^h_{\lambda_0}(t) \|_X \| (\Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0} - \hat{u}_{\lambda_0})(t) \right\|_X \\
&\leq C_b K \left\| (u^h_{\lambda_0} - \Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0})(t) \|_X \| (\Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0} - \hat{u}_{\lambda_0})(t) \right\|_X + C_b K \left\| (\Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0} - \hat{u}_{\lambda_0})(t) \right\|_X^2
\end{aligned}
$$

We apply a Young inequality to the product

$\left\| (u^h_{\lambda_0} - \Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0})(t) \|_X \| (\Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0} - \hat{u}_{\lambda_0})(t) \right\|_X$, then we get the following inequality:

$$
\begin{aligned}
&- b\Big((u^h_{\lambda_0} - \hat{u}_{\lambda_0})(t), u^h_{\lambda_0}(t), (\Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0} - \hat{u}_{\lambda_0})(t) \Big) \leq C_b K \| (\Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0} - \hat{u}_{\lambda_0})(t) \|_X^2 \\
&+ \frac{1}{2} C_b K \| (u^h_{\lambda_0} - \Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0})(t) \|_X^2 \\
&+ \frac{1}{2} C_b K \| (\Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0} - \hat{u}_{\lambda_0})(t) \|_X^2,
\end{aligned}
$$

Similarly, we have the following:

$$
\begin{aligned}
-b(\hat{u}_{\lambda_0}(t), (u^h_{\lambda_0} - \hat{u}_{\lambda_0})(t), (\Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0} - \hat{u}_{\lambda_0})(t)) &\leq C_b K \| (\Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0} - \hat{u}_{\lambda_0})(t) \|_X^2 \\
+ \frac{1}{2} C_b K \| (u^h_{\lambda_0} - \Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0})(t) \|_X^2 \quad &+ \frac{1}{2} C_b K \| (\Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0} - \hat{u}_{\lambda_0})(t) \|_X^2,
\end{aligned}
$$

Thanks to the coercivity of the form a on $H^1_0(\Omega) \times H^1_0(\Omega)$ and its continuity on $(V_h, \|\cdot\|_X)$, we write:

$$
\begin{aligned}
-a((u^h_{\lambda_0} - \hat{u}_{\lambda_0})(t), (\Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0} - \hat{u}_{\lambda_0})(t), \lambda_0) &\leq -\lambda_0 C_p \left\| (\Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0} - \hat{u}_{\lambda_0})(t) \right\|_X^2 \\
+ C_a \left\| (u^h_{\lambda_0} - \Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0})(t) \|_X \| (\Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0} - \hat{u}_{\lambda_0})(t) \right\|_X &.
\end{aligned}
$$

We apply a Young inequality to the product

$C_a \left\| (u^h_{\lambda_0} - \Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0})(t) \|_X \| (\Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0} - \hat{u}_{\lambda_0})(t) \right\|_X$, then $\forall \varepsilon > 0$ we have the following inequality:

$$
\begin{aligned}
&- a((u^h_{\lambda_0} - \hat{u}_{\lambda_0})(t), (\Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0} - \hat{u}_{\lambda_0})(t), \lambda_0) \leq \frac{C_a^2}{2\varepsilon} \left\| (u^h_{\lambda_0} - \Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0})(t) \right\|_X^2 \\
&+ \frac{\varepsilon}{2} \left\| (\Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0} - \hat{u}_{\lambda_0})(t) \right\|_X^2 \\
&- \lambda_0 C_p \left\| (\Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0} - \hat{u}_{\lambda_0})(t) \right\|_X^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{d}{dt} \left\| \Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0} - \hat{u}_{\lambda_0} \right\|_X^2 &\leq (6C_b K + \varepsilon - 2\lambda_0 C_p) \left\| (\Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0} - \hat{u}_{\lambda_0})(t) \right\|_X^2 \\
+ \left(2C_b K + \frac{C_a^2}{\varepsilon} \right) &\left\| (u^h_{\lambda_0} - \Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0})(t) \right\|_X^2
\end{aligned}
$$

We integrate over an interval $(0, t) \subset (0, T)$ and we apply the Gronwall lemma, then we get:

$$
\left\| (\Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0} - \hat{u}_{\lambda_0})(t) \right\|_X^2 \leq \exp[(6C_b K + \varepsilon - 2\lambda_0 C_p)t] \left(2C_b K + \frac{C_a^2}{\varepsilon} \right) T \sum_{n=N+1}^{M} \mu_n^{\lambda_0}.
$$

This ends the proof.

By choosing ε small in the preceding estimate (8), we can see the competition between the quasi-linear term and the one corresponding to the coercivity of the bilinear form a. This could be rectified by increasing the POD modes number N. Then, a solution to this problem could be, by doing an *a priori* estimate of the two constants C_b and C_p. This could lead to choose an artificial viscosity verifying a certain condition, so that the term in the exponential becomes negative (8).

Then, under this hypothesis, we prove the following proposition:

Proposition 2. *If we choose an artificial viscosity λ_{sm} that verifies the condition:*

$$\lambda_{sm} > \frac{6C_bK + \varepsilon}{2C_p} - \lambda_0, \tag{9}$$

then the squared POD-Galerkin error for the parameter λ_0. will be controlled as follows:

$$\|u^h_{\lambda_0} - \hat{u}_{\lambda_0}\|^2_{L^2(0,T;X)} \leq \underbrace{2T\left(1 + 2C_bK + \frac{C_a^2}{\varepsilon}\right)\sum_{n=N+1}^{M} \mu_n^{\lambda_0}}_{f_1^{\lambda_0}(N)}. \tag{10}$$

Proof.

$$\|u^h_{\lambda_0} - \hat{u}_{\lambda_0}\|^2_{L^2(0,T;X)} \leq 2\left\|u^h_{\lambda_0} - \Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0}\right\|^2_{L^2(0,T;X)} + 2\left\|\Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0} - \hat{u}_{\lambda_0}\right\|^2_{L^2(0,T;X)}.$$

- $\|u^h_{\lambda_0} - \Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0}\|^2_{L^2(0,T;X)} = T\sum_{n=N+1}^{M} \mu_n^{\lambda_0}$.
- Thanks to (9) and to the previous proposition, $\|\Pi_N^{\Phi^{\lambda_0}} u^h_{\lambda_0} - \hat{u}_{\lambda_0}\|^2_{L^2(0,T;X)} \leq (2C_bK + \frac{C_a^2}{\varepsilon})T\sum_{n=N+1}^{M} \mu_n^{\lambda_0}$.

This ends the proof.

2.2. Key point 2: control of $\|u^h_\lambda - u^h_{\lambda_0}\|^2_{L^2(0,T;X)}$

We prove a result on the parametric sensitivity of the solutions u^h_λ of the Equation (7):

Proposition 3. *We denote β a positive real number close to zero. Then, the mapping defined by:*

$$\begin{aligned}]\tfrac{\beta}{2C_p}, +\infty[\quad &\to L^2(0,T;X) \\ \lambda \quad &\mapsto u^h_\lambda \end{aligned}$$

is locally lipschitz. Moreover, at a given neighbourhood of λ_0, we have the following inequality:

$$\|u^h_\lambda - u^h_{\lambda_0}\|^2_{L^2(0,T;X)} \leq \mathcal{B}_{\lambda_0} \frac{|\lambda - \lambda_0|^2}{\lambda_0}.$$

$$\mathcal{B}_{\lambda_0} := \frac{TC_a^2K}{4C_b}\frac{\lambda_0}{\gamma}[\exp(4C_bKT) - 1].$$

Where, γ is a positive real number verifying: $\gamma = \beta$.

PROOF. We denote $w^h(t) = (u^h_\lambda - u^h_{\lambda_0})(t)$, which verifies the following weak formulation:

$$\left(\frac{d}{dt}w^h, v\right)_X + b(w^h(t), u^h_\lambda(t), v) + b(u^h_{\lambda_0}, w^h(t), v)$$
$$+ a(w^h(t), v, \lambda) + a(u^h_{\lambda_0}(t), v, \lambda) - a(u^h_{\lambda_0}(t), v, \lambda_0) = 0.$$

We replace v by $w^h(t)$ in the preceding equation, then we get:

$$\frac{1}{2}\frac{d}{dt}\|w^h\|^2_X = -b(w^h(t), u^h_\lambda(t), w^h(t)) - b(u^h_{\lambda_0}(t), w^h(t), w^h(t))$$
$$- a(w^h(t), w^h(t), \lambda) - a(u^h_{\lambda_0}(t), w^h(t), \lambda) + a(u^h_{\lambda_0}(t), w^h(t), \lambda_0).$$

Thanks to the continuity of the form b on $(V_h, \|.\|_X)$, we obtain the following:

$$-b(w^h(t), u^h_\lambda(t), w^h(t)) \leq C_b K \|w^h(t)\|^2_X.$$

$$-b(u^h_{\lambda_0}(t), w^h(t), w^h(t)) \leq C_b K \|w^h(t)\|^2_X.$$

And, thanks to the coercivity of the form a on $H^1_0(\Omega) \times H^1_0(\Omega)$, its Lipschitz character with respect to λ and its continuity on $(V_h, \|.\|_X)$, we deduce the following:

$$-a(w^h(t), w^h(t), \lambda) - a(u^h_{\lambda_0}(t), w^h(t), \lambda) + a(u^h_{\lambda_0}(t), w^h(t), \lambda_0) \leq -\lambda C_p \|w^h(t)\|^2_X$$
$$+ |\lambda - \lambda_0| C_a K \|w^h(t)\|_X.$$

Now, after applying a Young inequality to the product $|\lambda - \lambda_0| C_a K \|w^h(t)\|_X$, we write:

$$-a(w^h(t), w^h(t), \lambda) - a(u^h_{\lambda_0}(t), w^h(t), \lambda) + a(u^h_{\lambda_0}(t), w^h(t), \lambda_0) \leq -\lambda C_p \|w^h(t)\|^2_X$$
$$+ \frac{\gamma}{2}\|w^h(t)\|^2_X \quad + |\lambda - \lambda_0|^2 \frac{C_a^2 K^2}{2\gamma}.$$

Based on the conditions satisfied by γ, we can prove that $-2\lambda C_p + \gamma < 0$. Therefore,

$$\frac{d}{dt}\|w^h\|^2_X \leq 4 C_b K \|w^h(t)\|^2_X + \frac{C_a^2 K^2}{\gamma}|\lambda - \lambda_0|^2.$$

We integrate over an interval $(0, t) \subset (0, T)$ and we apply the Gronwall lemma, then we get:

$$\|w^h(t)\|^2_X \leq T C_a^2 K^2 \frac{\lambda_0}{\gamma} \exp(4 C_b K t) \frac{|\lambda - \lambda_0|^2}{\lambda_0}.$$

2.3. *Key point 3: choice of the sequence $(f_2^{\lambda_0}(N))_{N=1,\ldots,M}$ and an* **a priori** *estimate of its terms*

We prove the following proposition that will be a key point in order to choose the sequence $(f_2^{\lambda_0}(N))_{N=1,\ldots,M}$ and give an *a priori* estimate of its terms.

For simplicity reasons, we suppose that $d = 1$ and $\Omega = (0, 1)$, without any loss of generality:

Proposition 4. *Let $(\Phi_n)_{n=1,\ldots,M}$ be an orthonormal basis of $(V_h, \|.\|_X)$. We define f_n such that:*

$$f_n(x) = \int_0^x \Phi_n(y)dy.$$

Then, $\sum_{n=1}^{M} \|f_n\|_X^2 < \frac{1}{2}$.

PROOF. $f_n(x) = (\Phi_n, 1_{[0,x]})_X$, then $\|f_n\|_X^2 = \int_0^1 |(\Phi_n, 1_{[0,x]})_X|^2 dx$.

Therefore, $\sum_{n=1}^{M} \|f_n\|_X^2 < \int_0^1 \|1_{[0,x]}\|_X^2 dx$.

Which concludes to the result.

Thus, $f_2^{\lambda_0}(N) = \sum_{n=N+1}^{M} \|f_n^{\lambda_0}\|_X^2$. Where $f_n^{\lambda_0}(x) = \int_0^x \Phi_n^{\lambda_0}(y)dy$.

And, $\forall N = 0, \ldots, M-1, f_2^{\lambda_0}(N) < \frac{1}{2}$.

2.4. Key point 4: an upper bound for the parametric squared ROM-POD error $\|u_\lambda^h - \hat{u}_{\lambda,\lambda_0}\|^2_{L^2(0,T;X)}$

$$\begin{aligned}\|u_\lambda^h - \hat{u}_{\lambda,\lambda_0}\|^2_{L^2(0,T;X)} &\leq 2\|u_{\lambda_0}^h - \hat{u}_{\lambda_0}\|^2_{L^2(0,T;X)} \\ &+ 2\|(u_\lambda^h - u_{\lambda_0}^h) - \Pi_N^{\Phi^{\lambda_0}}(u_\lambda^h - u_{\lambda_0}^h)\|^2_{L^2(0,T;X)} \\ &+ 2\|\Pi_N^{\Phi^{\lambda_0}}(u_\lambda^h - u_{\lambda_0}^h) - (\hat{u}_{\lambda,\lambda_0} - \hat{u}_{\lambda_0})\|^2_{L^2(0,T;X)}\end{aligned}$$

- The squared POD-Galerkin error $\|u_{\lambda_0}^h - \hat{u}_{\lambda_0}\|^2_{L^2(0,T;X)}$ is estimated by $f_1^{\lambda_0}(N)$, based on proposition 2.
- $\|(u_\lambda^h - u_{\lambda_0}^h) - \Pi_N^{\Phi^{\lambda_0}}(u_\lambda^h - u_{\lambda_0}^h)\|^2_{L^2(0,T;X)} = \sum_{n=N+1}^{M} \|(u_\lambda^h - u_{\lambda_0}^h, \Phi_n^{\lambda_0})_X\|^2_{L^2(0,T)}$.
- The Galerkin error $\|\Pi_N^{\Phi^{\lambda_0}}(u_\lambda^h - u_{\lambda_0}^h) - (\hat{u}_{\lambda,\lambda_0} - \hat{u}_{\lambda_0})\|^2_{L^2(0,T;X)}$ is controlled by $\sum_{n=N+1}^{M} \|(u_\lambda^h - u_{\lambda_0}^h, \Phi_n^{\lambda_0})_X\|^2_{L^2(0,T)}$; This is shown easily by reducing the semi-discrete model describing the evolution of $(u_\lambda^h - u_{\lambda_0}^h)(t)$.

Therefore, the parametric squared POD-Galerkin error is essentially controlled by the remainder $\sum_{n=N+1}^{M} \|(u_\lambda^h - u_{\lambda_0}^h, \Phi_n^{\lambda_0})_X\|^2_{L^2(0,T)}$.

2.4.1. *Completion of the proof of result 1*

Based on proposition 4, a way to study the decrease rate of this remainder will be, by considering the remainder of the primitives sum of the reference POD modes $\Phi_n^{\lambda_0}$: This is shown by applying successively the Green formula and the Cauchy-Shwarz inequality to each one of the orthogonal projection coefficients $((u_\lambda^h - u_{\lambda_0}^h)(t), \Phi_n^{\lambda_0})_X$. Therefore,

$\sum_{n=N+1}^{M} \|(u_\lambda^h - u_{\lambda_0}^h, \Phi_n^{\lambda_0})_X\|^2_{L^2(0,T)}$ is controlled by $\frac{\mathcal{B}_{\lambda_0}|\lambda-\lambda_0|^2}{\lambda_0} f_2^{\lambda_0}(N)$.
This ends the proof of result 1.

2.4.2. *Completion of the proof of the heuristic result*

Under the following restrictive condition:

$$u_\lambda^h - u_{\lambda_0}^h \in L^2(0,T;H^m(\Omega)),$$

a precision on the decrease rate of the squared ROM-POD error is given by the *a priori* estimate (5). This is shown as follows:

By applying successively the Green formula and the Cauchy-Schwarz inequality and by repeating this step m-times to each $((u_\lambda^h - u_{\lambda_0}^h)(t), \Phi_n^{\lambda_0})_X$, we prove that

$\sum_{n=N+1}^{M} \|(u_\lambda^h - u_{\lambda_0}^h, \Phi_n^{\lambda_0})_X\|_{L^2(0,T)}^2$ is finally controlled by the remainder of the m-iterated primitives sum of the reference POD modes.

Which concludes to the result.

2.4.3. Completion of the proof of result 2

Thanks to proposition (3), the first restrictive condition of theorem 2 is verified for a quasi-linear equation of the form (7). Then, we write the Taylor expansion of u_λ^h to the order 1:

$u_\lambda^h = u_{\lambda_0}^h + \frac{\partial u_\lambda^h}{\partial \lambda}(\lambda_0)(\lambda - \lambda_0) + R_1(\lambda)$. Where, $\|R_1(\lambda)\|_{L^2(0,T;X)}$ is a function of $|\lambda - \lambda_0|^2$.

Now, we impose the second restrictive condition of theorem 2. In this case, the remainder $\sum_{n=N+1}^{M} \|(u_\lambda^h - u_{\lambda_0}^h, \Phi_n^{\lambda_0})_X\|_{L^2(0,T)}^2$ is better controlled by the POD modes:

$$\sum_{n=N+1}^{M} \|(u_\lambda^h - u_{\lambda_0}^h, \Phi_n^{\lambda_0})_X\|_{L^2(0,T;\mathbb{R})}^2 \le |\lambda - \lambda_0|^2 \sum_{n=N+1}^{M} \left\| \left(\frac{\partial u_\lambda^h}{\partial \lambda}(\lambda_0), \Phi_n^{\lambda_0} \right)_X \right\|_{L^2(0,T)}^2 + \|R_1(\lambda)\|_{L^2(0,T;X)}^2.$$

Now, $\sum_{n=N+1}^{M} \| \left(\frac{\partial u_\lambda^h}{\partial \lambda}(\lambda_0), \Phi_n^{\lambda_0} \right)_X \|_{L^2(0,T)}^2$ is controlled by the remainder of the POD-eigenvalues sum, as the snapshots which are used to obtain the POD basis have included the parametric sensitivity of the solution at the reference parameter value λ_0.

Therefore, we deduce the *a priori* estimate (6).

This ends the proof.

3. Application of results 1 and 2 to the model equation of Burgers

We consider the model equation of Burgers given by its weak formulation:

$$\begin{cases} \left(\frac{\partial u_\lambda}{\partial t}, v\right)_X + b(u_\lambda(t), u_\lambda(t), v) = a(u_\lambda(t), v, \lambda) + F_t(v) \quad \forall v \in V \\ (u_\lambda(0), v)_X = (u_0, v)_X \ \forall v \in V \end{cases}$$

- $\forall v_1, v_2 \in V, a(v_1, v_2, \lambda) = -\lambda(\partial_x v_1, \partial_x v_2)_X$.
- $\forall v_1, v_2, v_3 \in V, b(v_1, v_2, v_3) = (v_1 \partial_x v_2, v_3)_X$.
- $\forall v \in V, F_t(v) = (f(t), v)_X$.

It is a classical parabolic equation, and we suppose existence and uniqueness of the solution under sufficient conditions of regularity.

It is obvious that for this particular case of the Burgers equation, the procedure we did in the previous section could be repeated exactly in the same way. Therefore, the *a priori* estimate (4) of result 1, the heuristic estimate (5) and the *a priori* estimate (6) of result 2, are all valid.

4. Numerical tests

4.1. Numerical solution for $\lambda = \lambda_0$.

For the numerical computations, we will consider, for instance, the following initial and boundary conditions:

$$\begin{cases} u_\lambda(x, 0) = \sin(\pi x), 0 < x < 1 \\ u_\lambda(0, t) = u_\lambda(1, t) = 0, t \in [0, 1] \end{cases}$$

We suppose $f = 0$.

We discretise the time and spatial domains into $M = 200$ nodes.

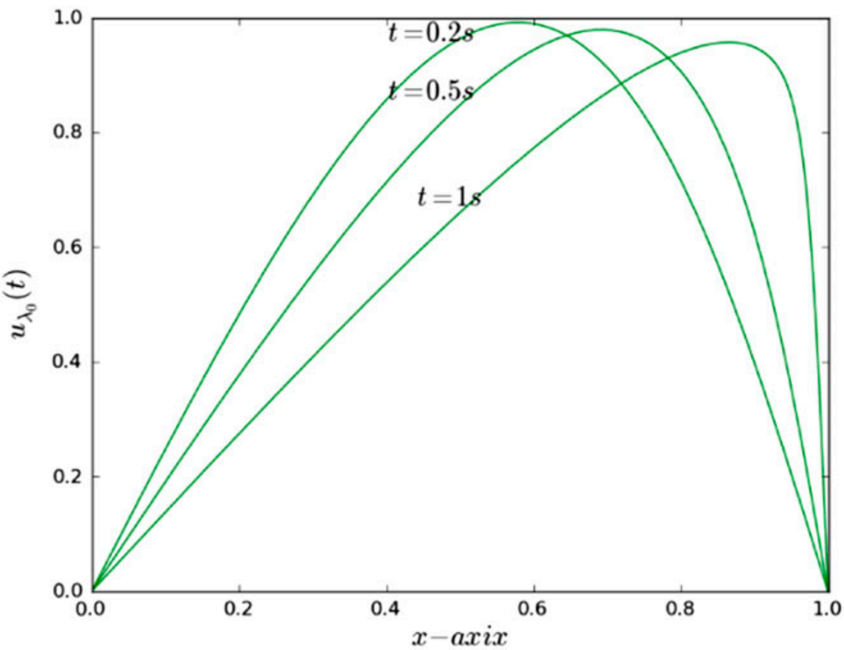

Figure 1. The numerical solution of the Burgers equation associated to $\lambda_0 = 10^{-2}$.

We discretise the Burgers equation using an implicit Euler scheme with respect to time; As for the non-linear term $u_\lambda(t)\partial_x u_\lambda(t)$, we use a semi-implicit scheme.

For instance, we consider $\lambda_0 = 10^{-2}$. The numerical solution of the Burgers equation, associated to this viscosity, is presented in Figure 1. More precisely, we show in this figure the profile of $u^h_{\lambda_0}(t_i)$ for different instants t_i.

4.2. POD basis for $u^h_{\lambda_0}$

To discretise the POD eigenvalue problem, we use the snapshots method (Sirovitch, 1987).

Then, we denote:

$$R_{\lambda_0} = \left(\frac{1}{M} (u^h_{\lambda_0}(t_i), u^h_{\lambda_0}(t_j))_{\mathbb{R}^M} \right)_{1 \le i,j \le M}.$$

We denote by $v_n = (v_{i,n})_{1 \le i \le M}$ for $n = 1, \ldots, M$, a set of orthonormal eigenvectors of the matrix R_{λ_0}. Then, the POD-eigenvectors associated to u_{λ_0}, are given by:

$$\Phi_n^{\lambda_0} = \frac{1}{\sqrt{M}} \sum_{i=1}^{M} v_{i,n} u(t_i).$$

The 10 first values of the error in $L^2(0, T; X)$, because of the orthogonal projection of $u^h_{\lambda_0}$ on the POD subspaces of dimensions $N = 1, \ldots, 10$, are presented in the Table 1. We remark that the five first POD-modes contain almost all the kinetic energy $\|u^h_{\lambda_0}\|^2_{L^2(0,T;X)} \cong \sum_{n=1}^{M} \mu_n^{\lambda_0}$.

Table 1. Errors in $L^2(0, T; X)$ of the orthogonal projection of $u^h_{\lambda_0}$ on the POD subspaces of dimensions $N = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$.

POD-modes	POD-error
1	3.58784914*e*-01
2	1.55734895*e*-02
3	6.39895831*e*-04
4	3.81187212*e*-05
5	3.68559574*e*-06
6	2.56290577*e*-08
7	2.08562743*e*-09
8	1.93515109*e*-10
9	1.79758134*e*-11
10	2.09821738*e*-13

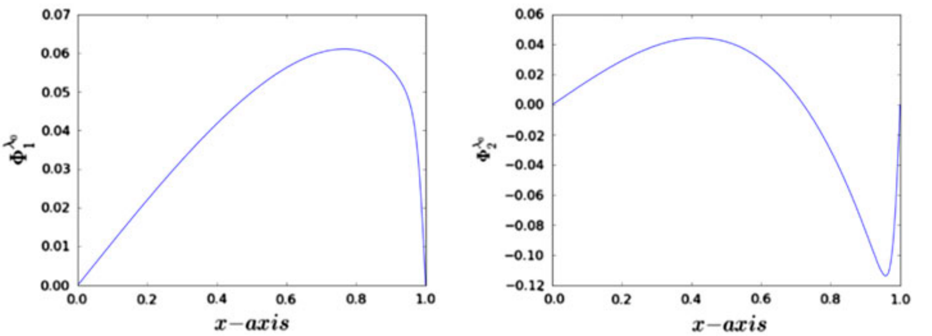

Figure 2. The POD-modes $\Phi_1^{\lambda_0}$ and $\Phi_2^{\lambda_0}$.

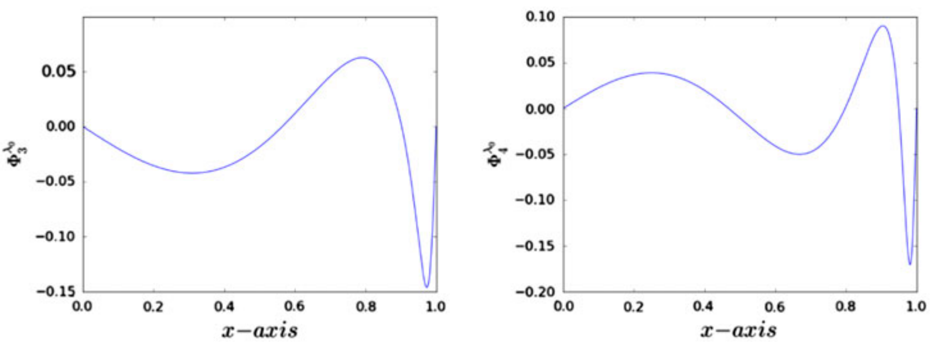

Figure 3. The POD-modes $\Phi_3^{\lambda_0}$ and $\Phi_4^{\lambda_0}$.

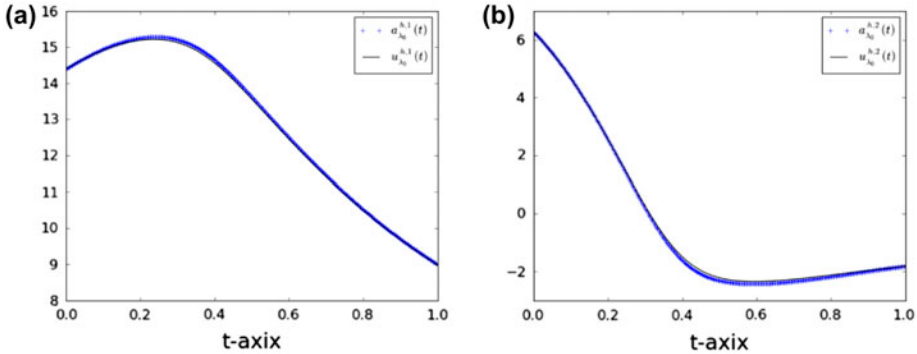

Figure 4. (a) $a^{h,1}_{\lambda_0}(t)$ (dotted blue line) and $u^{h,1}_{\lambda_0}(t)$ (black line). (b) $a^{h,2}_{\lambda_0}(t)$ (dotted blue line) and $u^{h,2}_{\lambda_0}(t)$ (black line).

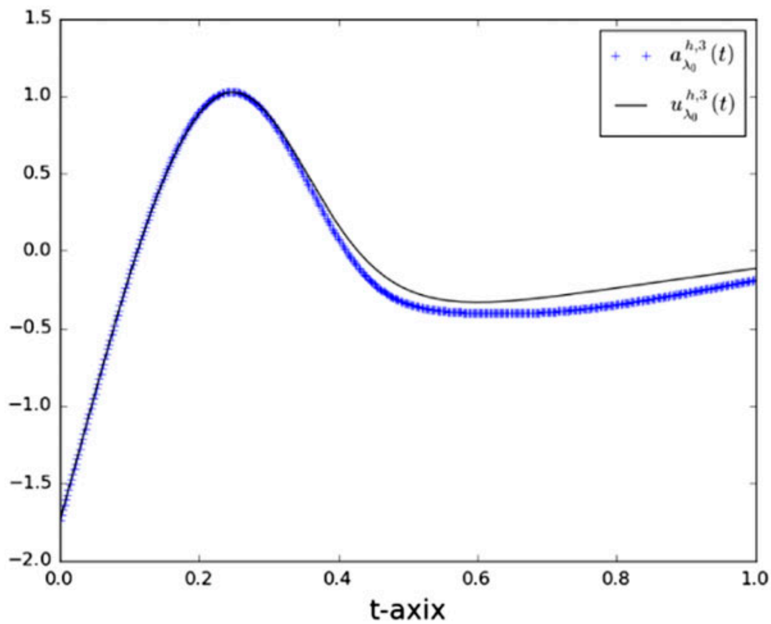

Figure 5. $a^{h,3}_{\lambda_0}(t)$ (dotted blue line) and $u^{h,3}_{\lambda_0}(t)$ (black line).

The four first POD-eigenvectors are presented in Figures 2 and 3.

4.3. ROM-POD for $\lambda = \lambda_0$

The three first coefficients of the ROM-POD, $a^{h,n}_{\lambda_0}(t)$ for $n = 1, 2, 3$, are represented in comparison with those of the direct POD-approximation of the full solution $u^h_{\lambda_0}$, $u^{h,n}_{\lambda_0}(t)$ for $n = 1, 2, 3$, in the Figures 4 and 5.

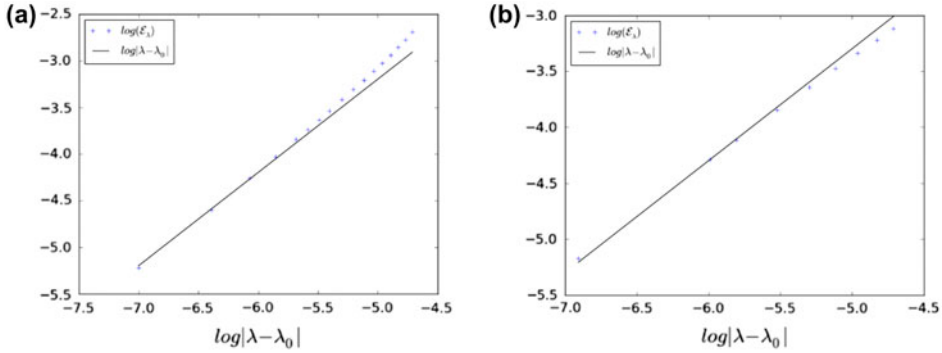

Figure 6. (a) Evolution of $\ln(\mathcal{E}_\lambda)$ for $\lambda \leq \lambda_0$ (dotted blue line) and the theoretical error (black line). (b) The same as (a) for $\lambda \geq \lambda_0$.

4.4. Sensitivity tests via the parametric evolution and the POD modes number N

4.4.1. Parametric evolution

We take $N = 5$. And, we consider $\lambda \in [10^{-3}, 1.9 \times 10^{-2}]$. We denote:

$$\mathcal{E}_\lambda = \|u_\lambda^h - \hat{u}_{\lambda,\lambda_0}\|_{L^2(0,T;X)}.$$

We compute $\ln(\mathcal{E}_\lambda)$, with respect to $\ln(|\lambda - \lambda_0|)$ in both cases $\lambda \leq \lambda_0$ and $\lambda \geq \lambda_0$. We obtain, respectively, plots (a) and (b) in Figure 6.

We find that the evolution of this error is linear with respect to λ, and $\mathcal{E}_\lambda$ decreases with the distance $|\lambda - \lambda_0|$.

4.4.2. Evolution of the parametric ROM-POD error with respect to N

In what follows, we validate our *a priori* estimate (5) with respect to the POD modes number *N*. More precisely, we plot the evolution of $\ln(\mathcal{E}_\lambda^2 - \mathcal{E}_{\lambda_0}^2)$ with respect to ln (*N*), for different values of the parameters. More precisely, we consider the following two cases.

- $\lambda_0 = 10^{-2}$, $\lambda = 7.10^{-3}$, and 5.10^{-3}. We get plots (a) and (b) in Figure 7.
- $\lambda_0 = 1$, $\lambda = 0.7$, and 0.5. We get plots (c) and (d) in Figure 8.

5. Conclusion and perspectives

5.1. Conclusion

In this paper, we have considered the sensitivity study of ROMs by POD of a single-parameterised quasi-linear equation in general and the one-dimensional Burgers equation in particular. We have proved a mathematical result on the sensitivity of a ROM by a reference POD basis. This result shows the dependency of the parametric squared ROM-POD error on the squared distance between a parameter λ and the one of reference λ_0, and on the ROM dimension *N*. The decrease rate of this error depends on the one of the remainder of the primitives series of the reference POD basis vectors. An accuracy on this rate is strongly related to the solutions regularity: For a regularity of type $u_\lambda^h - u_{\lambda_0}^h \in L^2(0, T; H^m)$, it is a decrease rate whose behaviour is

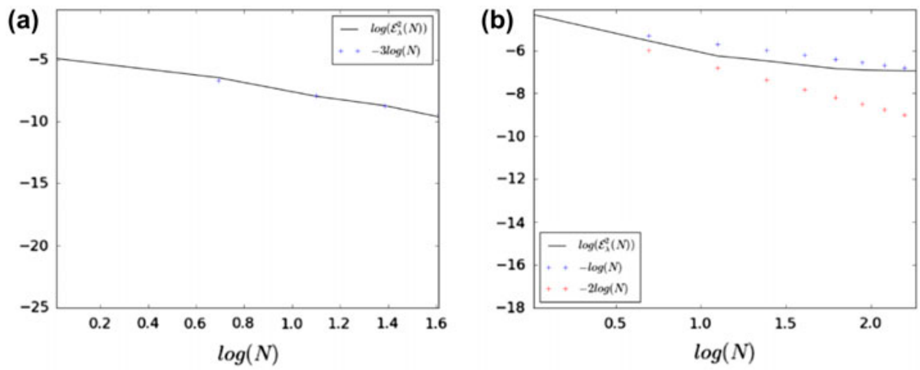

Figure 7. (a) Evolution of $\ln(\mathcal{E}^2_\lambda - \mathcal{E}^2_{\lambda_0})$ with respect to ln (N), for $\lambda_0 = 0.01$ and $\lambda = 0.007$. (b) Evolution of $\ln(\mathcal{E}^2_\lambda - \mathcal{E}^2_{\lambda_0})$ with respect to ln (N), for $\lambda_0 = 0.01$ and $\lambda = 0.005$.

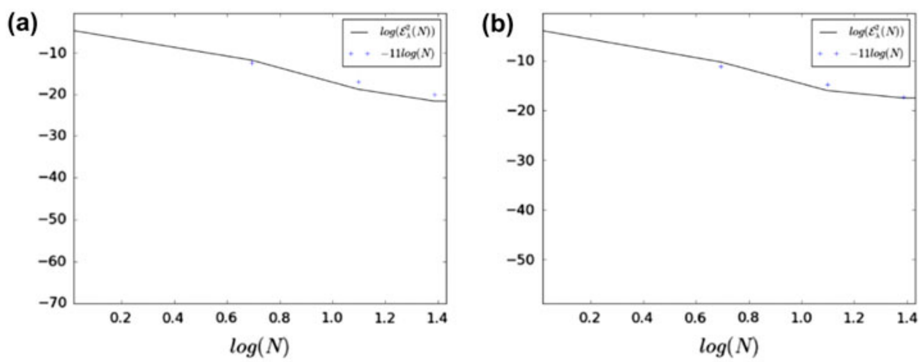

Figure 8. (a) Evolution of $\ln(\mathcal{E}^2_\lambda - \mathcal{E}^2_{\lambda_0})$ with respect to ln (N), for $\lambda_0 = 1$ and $\lambda = 0.7$. (b) Evolution of $\ln(\mathcal{E}^2_\lambda - \mathcal{E}^2_{\lambda_0})$ with respect to ln (N), for $\lambda_0 = 1$ and $\lambda = 0.5$.

similar to $\frac{1}{N^m}$. We have improved this result with respect to the parametric evolution, by the use of an expanded POD basis, associated with a reference solution and its first-order parametric derivative: Indeed, under this hypothesis, the *a priori* upper bound of the squared L^2-ROM-POD error is much better, because it provides a control of the parametric evolution depending only on the fourth power of the distance between a parameter λ and the one of reference λ_0.

Numerical tests were done to verify the sensitivity result on a reference POD basis, in the case of the Burgers equation. For a fixed POD modes number, we have verified the linear dependency of the ROM-POD error via the viscosity evolution. Moreover, we have retrieved estimates of the squared ROM-POD error by $\frac{1}{N^m}$ with respect to N.

5.2. Perspectives

Concerning the perspectives of this work, it is necessary first to obtain sharper estimates of the terms corresponding to the result 1, without supposing supplementary conditions on the solutions regularity, especially when dealing this time with the Navier–Stokes equations: In fact, we have shown in (Akkari, Hamdouni, Liberge, &

Jazar, in press) a numerical validation of result 1 relative to the power low of the ROM-POD error through the evolution of the viscosity parameter of an unsteady and incompressible fluid flow in a channel, around a circular cylinder. Moreover, we submitted a paper that presents a numerical validation of our theoretical estimates in results 1 and 2 relative to the power lows, in the case of an unsteady and incompressible fluid flow in a channel, around a circular cylinder (Akkari, Hamdouni, Liberge, & Jazar, in press). Indeed, we show that the slope of the logarithm of the numerical ROM-POD error by an expanded POD basis is stronger than one of the logarithm of the numerical ROM-POD error by a reference POD basis. And, the numerical parametric error in the first case is less than the one associated to the reduction by a reference POD basis.

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