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Fractional order generalised thermoelasticity to an infinite body with a cylindrical cavity and variable material properties

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This paper is concerned with the determination of thermoelastic displacement, stress and temperature produced in an infinite isotropic elastic body having a cylindrical cavity in the context of the *new consideration of heat conduction with fractional order generalised thermoelasticity with Lord–Shulman model (LS model) and Green–Naghdi model with energy dissipation (GN-III model)*. Here, the elastic parameters and the thermal conductivity are temperature dependent. The boundary of the cavity is subjected to time-dependent thermal and mechanical shocks. The *fractional order* generalised coupled thermoelasticity theories for the problem are combined into a unified formulation introducing the unified parameters. The governing equations of generalised thermoelasticity theory are obtained in the Laplace transform domain and are solved in that domain by finding out the roots by using the Laguerre's method. The inversion of the transform solution is carried out numerically by applying a method based on the Fourier series expansion technique. The numerical estimates for thermophysical quantities (displacement, temperature and stress) are obtained for copper-like material for weak, normal and strong conductivity and have been depicted graphically to estimate the effects of the fractional order parameter. The comparison of the results for different theories have been presented and the effects of temperature-dependent parameters are also discussed.

Keywords: generalised thermoelasticity; Lord–Shulman model; Green–Naghdi model; cylindrical cavity; fractional order

1. Introduction

Thermoelasticity theories which admit a finite speed of thermal signals (second sound) have aroused much interest in the last three decades. In contrast to the conventional coupled thermoelasticity theory based on a parabolic heat equation (Chadwick, 1960), which predicts an infinite speed of propagation of heat, these theories involve a hyperbolic heat equation and are referred to as generalised thermoelasticity theories. The extended thermoelasticity theory proposed by Lord and Shulman (1967) incorporates a flux-rate term into Fourier's law of heat conduction and formulates a generalised form that involves a hyperbolic-type heat transport equation admitting finite speed of thermal signals. The temperature rate-dependent generalised thermoelasticity theory developed by Green and Lindsay (1972) involving two relaxation times, where the Fourier law of heat conduction is left unchanged but the classical energy equation and stress–strain temperature relations are modified. The closed-form solutions for thermoelastic

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problems in generalised theory of thermoelsticity have been obtained by Hetnarski and Ignaczak (1993). Erbay and Suhubi (1986) studied longitudinal wave propagation in a cylinder. The results show that the relaxation times play a significant role in the cases involving shock wave propagation, laser technique, a rapidly propagation of crack tip, etc. On the experimental side, available evidence in support of the existence of finite thermal wave speed in solids is rather sparse, although an experimental study for second-sound propagation in dielectric solids and some related experimental observations were reported nearly four decades ago (Ackerman & Guyer, 1968; Rogers, 1971; Narayanmurti & Dynes, 1972). Most engineering materials such as metals possess a relatively high-rate thermal damping and thus are not suitable for use in experiments concerning second-sound propagation. But given the state of recent advances in material science, it may be possible in the foreseeable future to identify (or even manufacture for laboratory purpose)an idealised material for the purpose of studying the propagation of thermal waves at finite speed.

Green and Naghdi (1992a, 1992b, 1993) provided sufficient basic modifications in the contitutive equations that permit treatment of a much wider class of heat flow problems labelled as GN I, II and III. GN models include a term called 'thermal displacement gradient' among the independent constitutive variables. When the three theories are linearised, the heat transport equation of GN I is the same as the classical equation, whereas both GN II and III admit propagation of thermal signals of finite speeds (Green & Naghdi, 1993). An important feature of GN III theory is that it accommodates dissipation of thermal energy due to the presence of thermal damping term. More detailed discussion on the subject is available in the books of Hetnarski and Eslami (2009) and Ignaczak and Ostoja-Starzewski (2010). In the context of a linearised version of this theory (Green & Naghdi, 1992b, 1993), a theorem on uniqueness of solutions has been established by Chandrasekhariah (1996).

Thermoelastic interactions with energy dissipation in an infinite solid with distributed periodically varying heat sources have been studied by Banik, Mallik, and Kanoria (2007) and for functionally graded material without energy dissipation have been studied by Mallik and Kanoria (2006). The dynamic response in a functionally graded spherically isotropic hollow sphere with temperature-dependent elastic parameters has been studied by Ghosh and Kanoria (2010). Das, Kar, and Kanoria (2013) have studied magneto-thermoelastic response in a transversely isotropic hollow cylinder under thermal shock with three-phase-lag effect. Kar and Kanoria (2007) have analysed thermoelastic interactions with energy dissipation in an unbounded body with a spherical hole. Islam, Mallik, and Kanoria (2011) have discussed the study of dynamical response in a two-dimensional transversely isotropic thick plate with spatially varying heat sources and body forces. Generalised thermoelastic problem of a spherical shell under thermal shock has been solved by Kar and Kanoria (2009). Differential equations of fractional order have been the focus of many studies due to their frequent appearance in various application in fluid mechanics, viscoelasticity, biology, physics and engineering. The most important advantage of using fractional differential equations in these and other applications is their non-local property. It is well known that the integer order differential operator is a local operator but the fractional order differential operator is non-local. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. This is more realistic, and this is one reason why fractional calculus has become more and more popular (Caputo, 1967; Mainardi, 1997; Podlubny, 1999).

Fractional calculus has been used successfully to modify many existing models of physical processes. One can state that the whole theory of fractional derivatives and integrals was established in the second half of the nineteenth century. The first application of fractional derivatives was given by Abel (Gorenflo & Vessella, 1991), who applied fractional calculus in the solution of an integral equation that arises in the formulation of the Tautochrone problem. The generalisation of the concept of derivative and integral to a non-integer order has been subjected to several approaches, and some various alternative definitions of fractional derivatives appeared in (Debnath & Bhatta, 2007; Gorenflo & Mainardi, 1997; Hilfer, 2000; Zenkour & Abouelregal, 2014). In the last few years, fractional calculus was applied successfully in various areas to modify many existing models of physical processes, e.g. chemistry, biology, modelling and identification, electronics, wave propagation and viscoelasticity. Islam and Kanoria (2011) have studied the one-dimensional problem of a fractional order two-temperature thermopiezoelasticity. One can refer to Padlubny (1999) for a survey of applications of fractional calculus.

In this work, we have investigated the thermoelastic displacement, stress and temperature in an infinite isotropic elastic body having a cylindrical cavity with temperature-dependent material parameters under both time-dependent thermal and mechanical shocks in the context of *fractional order* generalised thermoelasticity. The *fractional order* generalised coupled thermoelasticity theories are combined into a unified formulation introducing the unified parameters. The Laplace transform technique has been used to solve the problem. Numerical inversion of the Laplace transform has been done by applying a method based on the Fourier series expansion technique. A complete and comprehensive analysis of the results has been presented for Lord–Shulman (LS) model and GN-III model *of generalised thermoelasticity, where the heat equation consists of some non-local fractional operator signifying not only the present state, but also the previous states due to sudden temperature change. The effect of fractional order parameter is also discussed.*

2. Development of fractional order theory

Recently, a considerable research effort is expended to study anomalous diffusion, which is characterised by the time-fractional diffusion-wave equation by Kimmich (2002) as follows

$$\rho c = \kappa I^{\xi} c_{,ii}\,, \tag{2.1}$$

where ρ is the mass density, c the concentration, κ the diffusion conductivity and i the coordinate symbol, which takes the value 1, 2, 3. The notation I^{ξ} is the Riemann–Liouville fractional integral, introduced as a natural generalisation of the well-known n-fold repeated integral $I^n f(t)$ written in a convolution-type form as in (Mainardi & Gorenflo, 2000), which is written as follows:

$$\begin{aligned} I^n f(t) &= \frac{1}{\Gamma(n)} \int_0^t (t-\tau)^{n-1} f(\tau)d\tau, \quad 0 < n \leq 2, \\ &= f(t), \quad n = 0. \end{aligned} \tag{2.2}$$

where $\Gamma(n)$ is the Gamma function.

According to Kimmich (2002), Equation (2.1) describes different cases of diffusion where $0 < \xi < 1$ corredsponds to weak diffusion (subdiffusion), $\xi = 1$ corresponds to normal diffusion, $1<\xi<2$ corresponds to strong diffusion (superdiffusion) and $\xi=2$ corresponds to ballistic diffusion.

It should be noted that the term diffusion is often used in a more generalised sense including various transport phenomena. Equation (2.1) is a mathematical model of a wide range of important physical phenomena, for example, the subdiffusive transport occurs in widely different systems ranging from dielectrics and semiconductors through polymers to fractals, glasses, porous and random media. Superdiffusion is comparatively rare and has been observed in porous glasses, polymer chain, biological systems, transport of organic molecules and atomic clusters on surface. One might expect the anomalous heat conduction in media where the anomalous diffusion is observed.

Fujita (1990) considered the heat wave equation for the case of $1\leq\xi\leq 2$

$$\rho c_v T = kI^{\xi} T_{,ii}\,, \tag{2.3}$$

where c_v is the specific heat, k is the thermal conductivity and the subscript ',' means the derivative with respect to the coordinate x_i. Equation (2.3) can be obtained as a consequence of the non-local constitutive equation for the heat flux components q_i in the form

$$q_i = -kI^{\xi-1} T_{,i} \quad 1<\xi \leq 2. \tag{2.4}$$

Povstenko (2011) used the Caputo heat wave equation defined in the form

$$q_i = -kI^{\xi-1} T_{,i} \quad 0<\xi\leq 2, \tag{2.5}$$

Cattaneo (1948) introduced a law of heat conduction to replace the classical Fourier law in the form

$$q_i + \tau_0 \dot{q}_i = -k\nabla T. \tag{2.6}$$

Sherief, El-Sayed, and Abd El-Latief (2010) introduced a formula of heat conduction as

$$q_i + \tau_0 \frac{\partial^{\xi} q_i}{\partial t^{\xi}} = -k\frac{\partial T}{\partial t}. \quad 0<\xi\leq 1, \tag{2.7}$$

where

$$\frac{\partial^{\xi}}{\partial t^{\xi}} f(y,t) = \begin{cases} f(y,t) - f(y,0) & \xi \to 0, \\ I^{\xi-1}\frac{\partial f(y,t)}{\partial t} & 0<\xi<1, \\ \frac{\partial f(y,t)}{\partial t} & \xi = 1 \end{cases} \tag{2.8}$$

and proved a uniqueness theorem and derived a reciprocity relation and a variational principle.

In the limit, as ξ tends to one, Equation (2.7) reduces to the well-known Cattaneo law used by Lord and Shulman (1967) to derive the equation of the generalised theory of thermoelasticity with one relaxation time.

Youssef (2010) introduced another formula of heat conduction and taking into consideration (2.4)–(2.6)

$$q_i + \tau_0 \frac{\partial q_i}{\partial t} = -kI^{\xi-1}\nabla T, \quad 0<\xi\leq 2, \tag{2.9}$$

and a uniqueness theorem has been proved.

Ezzat established a new model of fractional heat conduction equation by using the new Taylor series expansion of time-fractional order, developed by Jumarie (2010) as

$$q_i + \frac{\tau_0^{\xi}}{\xi!}\frac{\partial^{\xi} q_i}{\partial t^{\xi}} = -k\nabla T, \quad 0<\xi\leq 1, \tag{2.10}$$

El-Karamany and Ezzat (2011) introduced two general models of fractional heat conduction law for a non-homogeneous anisotropic elastic solid. Uniqueness and reciprocal theorems are proved, and the convolutional variational principle is established and used to prove a uniqueness theorem with no restriction on the elasticity or thermal conductivity tensors except symmetry conditions. The two-temperature dynamic coupled LS and fractional coupled thermoelasticity theories result as limit cases. For fractional thermoelasticity not involving two temperatures, El-Karamany and Ezzat (2011) established the uniqueness, reciprocal theorems and convolution variational principle. The dynamic coupled and Green–Naghdi (GN) thermoelasticity theories result as limit cases. The reciprocity relation in case of quiescent initial state is found to be independent of the order of differ integration (El-Karamany & Ezzat, 2011a, 2011b). Fractional order theory of a perfect conducting thermoelastic medium not involving two temperatures was investigated by Ezzat and El-Karamany (2011c). Fractional order two-temperature generalised thermoelasticity with finite wave speed was investigated by Sur and Kanoria (2012).

3. Basic equations

Equations of motion in absence of body forces are

$$\rho \ddot{u}_i = \sigma_{ij,j}. \tag{3.1}$$

The constitutive equations are

$$\sigma_{ij} = 2\mu e_{ij} + (\lambda e_{kk} - \gamma\theta)\delta_{ij} \tag{3.2}$$

where $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ and $\theta = T - T_0$ such that $\frac{|(T-T_0)|}{T_0} \ll 1$.

Heat equation corresponding to generalised thermoelasticity proposed by LS and GN in absence of heat source in unified form (Bagri & Eslami, 2007a, 2007b) is

$$I^{\xi-1}\left[\left(a + b\frac{\partial}{\partial t}\right)(K\theta_{,i})_{,i} + c(K^*\theta_{,i})_{,i}\right] = \left(a\frac{\partial}{\partial t} + d\frac{\partial^2}{\partial t^2}\right)\left(\frac{K}{\kappa}\theta + \gamma T_0 e\right) \tag{3.3}$$

where

$$a = 1, \quad b = 0, \quad c = 0, \quad d = \tau_0, \quad \xi = 1 \qquad \text{for LS model,}$$

$$a = 0, \quad b = 1, \quad c = 1, \quad d = 1, \quad \xi = 1 \qquad \text{for GN model III,}$$

and $\gamma = (3\lambda + 2\mu)\alpha_t$, α_t is the coefficient of linear thermal expansion, $e = e_{kk}$, τ_0 is the relaxation time for LS model, ρ is the density, T_0 is the reference temperature and c_v is the specific heat at constant strain.

The elastic parameters are assumed to be temperature dependent. Thus, we replace λ, μ, K and K^* by $\lambda_0 f(T)$, $\mu_0 f(T)$, $K_0 f(T)$ and $K_0^* f(T)$ respectively, where λ_0, μ_0, K_0 and K_0^* are assumed to be constants and $f(T)$ is a given non-dimensional function of temperature. Then, the equations corresponding to (3.1), (3.2) and (3.3) take the following form

$$\sigma_{ij} = f(T)[2\mu_0 e_{ij} + (\lambda_0 e_{kk} - \gamma_0\theta)\delta_{ij}], \tag{3.4}$$

$$\rho\ddot{u}_i = f(T)[2\mu_0 e_{ij} + (\lambda_0 e_{kk} - \gamma_0\theta)\delta_{ij}]_{,j} + f(T)_{,j}[2\mu_0 e_{ij} + (\lambda_0 e_{kk} - \gamma_0\theta)\delta_{ij}] \tag{3.5}$$

and

$$\begin{aligned} I^{\xi-1}\left[\left(a + b\frac{\partial}{\partial t}\right)[K_0 f(T)\theta_{,i}]_{,i} + c[K_0^* f(T)\theta_{,i}]_{,i}\right] \\ = \left(a\frac{\partial}{\partial t} + d\frac{\partial^2}{\partial t^2}\right)\left[f(T)\left(\frac{K_0}{\kappa}\theta + \gamma_0 T_0 e\right)\right], \end{aligned} \tag{3.6}$$

where $\gamma_0 = (3\lambda_0 + 2\mu_0)\alpha_t$.

When the elastic terms, the thermal conductivity and additional material constant for GN theories are temperature independent, then $f(T) = 1$ (Ezzat, Othman, & El-Karamany, 2001).

4. Formulation of the problem

We consider an infinite isotropic elastic body with a cylindrical cavity of radius R in an undisturbed state at initial temperature T_0. We use the cylindrical coordinate system (r, θ, z) with z-axis coincident with the axis of cylinder. Due to axial symmetry, the problem is one dimensional with all the considered functions depending on the radial distance r and the time t. Then, the displacement vector has the components $u_r = u(r, t)$, $u_\theta(r, t) = u_z(r, t) = 0$ and temperature T can be taken as $T(r, t)$.

We will assume that

$$f(T) = 1 - \alpha^* T,$$

where α^* is called the empirical material constant. Since $\frac{|(T-T_0)|}{T_0} \ll 1$, $f(T)$ may be approximated to

$$f(T) \simeq f(T_0) = 1 - \alpha^* T_0.$$

So, in the context of linear theory of the generalised coupled thermoelasticity, the equation of motion and heat equation can be written in their respective forms as:

$$\rho\ddot{u} = f(T_0)\left[(\lambda_0 + 2\mu_0)\frac{\partial e}{\partial r} - \gamma_0\frac{\partial\theta}{\partial r}\right], \tag{4.1}$$

$$I^{\xi-1}\left[K_0\left(a + b\frac{\partial}{\partial t}\right)\nabla^2\theta + cK_0^*\nabla^2\theta\right] = \left(a\frac{\partial}{\partial t} + d\frac{\partial^2}{\partial t^2}\right)\left(\frac{K_0}{\kappa}\theta + \gamma_0 T_0 e\right) \tag{4.2}$$

and e is the cubical dilatation given by

$$e = \frac{1}{r}\frac{\partial(ru)}{\partial r}, \tag{4.3}$$

and ∇^2 is the Laplacian, given by,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}.$$

Constitutive equations in the present case are

$$\sigma_{rr} = f(T_0)\left[2\mu_0\frac{\partial u}{\partial r} + \lambda_0 e - \gamma_0\theta\right], \tag{4.4a}$$

$$\sigma_{\theta\theta} = f(T_0)\left[2\mu_0\frac{u}{r} + \lambda_0 e - \gamma_{0\theta}\right], \tag{4.4b}$$

$$\sigma_{zz} = f(T_0)[\lambda_0 e - \gamma_0\theta], \tag{4.4c}$$

$$\sigma_{zr} = \sigma_{\theta r} = \sigma_{z\theta} = 0. \tag{4.4d}$$

Introducing the following non-dimensional variables

$$r' = \frac{r}{\kappa}\left(\frac{\lambda_0 + 2\mu_0}{\rho}\right)^{1/2}, \quad u' = \frac{u}{\kappa}\left(\frac{\lambda_0 + 2\mu_0}{\rho}\right)^{1/2}, \quad t' = \frac{t}{\kappa}\left(\frac{\lambda_0 + 2\mu_0}{\rho}\right),$$

$$\tau_0' = \frac{\tau_0}{\kappa}\left(\frac{\lambda_0 + 2\mu_0}{\rho}\right), \quad R' = \frac{R}{\kappa}\left(\frac{\lambda_0 + 2\mu_0}{\rho}\right), \quad \theta' = \frac{\theta}{T_0}, \quad \sigma' = \frac{\sigma}{\mu_0}$$

where $b_1 = \frac{\gamma_0 T_0}{\mu}$, $g = \frac{\gamma_0\kappa}{K}$, $\beta = \left(\frac{\lambda_0+2\mu_0}{\mu_0}\right)^{1/2}$, $a_1 = \frac{b_1}{\beta^2}$.

Preceding equations take the following forms (dropping the primes for convenience)

$$f(T_0)[\nabla^2 e - a_1\nabla^2\theta] = \frac{\partial^2 e}{\partial t^2}, \tag{4.5}$$

$$I^{\xi-1}\left[\left(a + bG\frac{\partial}{\partial t}\right)\nabla^2\theta + cGk^*\nabla^2\theta\right] = \left(a\frac{\partial}{\partial t} + dG\frac{\partial^2}{\partial t^2}\right)(\theta + ge), \tag{4.6}$$

$$\sigma_{rr} = f(T_0)\left[\beta^2\frac{\partial u}{\partial r} + (\beta^2 - 2)\frac{u}{r} - b_1\theta\right], \tag{4.7a}$$

$$\sigma_{\theta\theta} = f(T_0)\left[(\beta^2 - 2)\frac{\partial u}{\partial r} + \beta^2\frac{u}{r} - b_1\theta\right], \tag{4.7b}$$

$$\sigma_{zz} = f(T_0)[(\beta^2 - 2)e - b_1\theta], \tag{4.7c}$$

where $\frac{(\lambda_0 + 2\mu_0)}{\rho\kappa} = G$.

Taking the Laplace transform of both sides of Equations (4.5)–(4.7), we have

$$(\nabla^2 - \alpha s^2)\bar{e} = a_1\nabla^2\bar{\theta}, \tag{4.8}$$

$$\left[\frac{1}{s^{\xi-1}}(a + bGs + cGK^*)\nabla^2 - (as + dGs^2)\right]\bar{\theta} = (as + dGs^2)g\bar{e}, \tag{4.9}$$

$$\alpha\bar{\sigma}_{rr} = \beta^2\frac{\partial\bar{u}}{\partial r} + (\beta^2 - 2)\frac{\bar{u}}{r} - b_1\bar{\theta}, \tag{4.10}$$

$$\alpha\bar{\sigma}_{\theta\theta} = (\beta^2 - 2)\frac{\partial \bar{u}}{\partial r} + \beta^2 \frac{\bar{u}}{r} - b_1 \bar{\theta}, \tag{4.11}$$

$$\alpha\bar{\sigma}_{zz} = (\beta^2 - 2)\bar{u} - b_1 \bar{\theta}, \tag{4.12}$$

where $\alpha = \frac{1}{f(T_0)} = \frac{1}{1-\alpha^* T_0}$.

Eliminating $\bar{e}$ or $\bar{\theta}$ from (4.8) and (4.9), one gets

$$\left[\frac{1}{s^{\xi-1}}(a + bGs + cGK^*)\nabla^4 - \left\{\frac{\alpha s^2(a + bGs + cGK^*)}{s^{\xi-1}} + (1+\varepsilon)(as + dGs^2)\right\}\nabla^2 + \alpha s^2(as + dGs^2)\right]\{\bar{\theta}, \bar{e}\} = 0, \tag{4.13}$$

where $\varepsilon = a_1 g$.

The solution of Equations (4.8) and (4.9) bounded at infinity can be written as:

$$\bar{\theta} = \sum_{i=1}^{2} A_i(s)(p_i^2 - \alpha s^2)K_0(p_i r) \tag{4.14}$$

and

$$\bar{e} = \sum_{i=1}^{2} B_i(s)K_0(p_i r) \tag{4.15}$$

where $\pm p_1$ and $\pm p_2$ are the roots of the characteristic equation

$$\frac{(a + bGs + cGK^*)p^4}{s^{\xi-1}} - \left\{\frac{\alpha s^2(a + bGs + cGK^*)}{s^{\xi-1}} + (1+\varepsilon)(as + dGs^2)\right\} p^2 + \alpha s^2(as + dGs^2) = 0. \tag{4.16}$$

From Equation (4.8), we obtain

$$B_i(s) = a_1 A_i(s) p_i^2; \quad i = 1, 2. \tag{4.17}$$

Hence, we get

$$\bar{e} = \sum_{i=1}^{2} a_1 A_i(s) p_i^2 K_0(p_i r). \tag{4.18}$$

Taking the Laplace transform of Equation (4.3) and upon using (4.18) and integrating, we have

$$\bar{u} = -\sum_{i=1}^{2} a_1 A_i(s) p_i K_1(p_i r). \tag{4.19}$$

Now substituting $\bar{\theta}$ and $\bar{u}$ into the Equations (4.10)–(4.12), we obtain

$$\alpha\bar{\sigma}_{rr} = a_1 \sum_{i=1}^{2} A_i(s)\left[\beta^2 \alpha s^2 K_0(p_i r) + \frac{2}{r} p_i K_1(p_i r)\right], \tag{4.20}$$

$$\alpha\bar{\sigma}_{\theta\theta} = a_1 \sum_{i=1}^{2} A_i(s)\left[(\beta^2 s^2 \alpha - p_i^2)K_0(p_i r) - \frac{2}{r} p_i K_1(p_i r)\right], \tag{4.21}$$

$$\alpha\bar{\sigma}_{zz} = a_1 \sum_{i=1}^{2} (\beta^2 s^2 \alpha - 2)\, p_i^2 A_i(s)\, K_0(p_i r). \tag{4.22}$$

In order to evaluate the unknown parameters $A_i(s)$ $(i = 1,\ 2)$, we use the boundary conditions on the internal surface $r = R$, which is given by thermal boundary condition

$$\theta(R,t) = \theta_0 H(t) \tag{4.23}$$

and mechanical boundary condition

$$\sigma_{rr}(R,t) = \sigma_0 H(t), \tag{4.24}$$

where $H(t)$ is the Heaviside unit step function.

Taking the Laplace transform of the boundary conditions (4.23) and (4.24), and using the Equations (4.14)–(4.20), we get following system of two linear algebraic equations

$$\sum_{i=1}^{2} (p_i^2 - \alpha s^2)\, K_0(p_i r)\, A_i(s) = \frac{\theta_0}{s} \tag{4.25}$$

and

$$\sum_{i=1}^{2} \left[\beta^2 s^2 \alpha K_0(p_i r) + \frac{2 p_i K_1(p_i r)}{r} \right] A_i(s) = \frac{\alpha\sigma_0}{a_1 s}. \tag{4.26}$$

From the above two equations, the unknown parameters $A_1(s)$ and $A_2(s)$ can be determined. Hence, we obtain the solutions in the Laplace transform domain. The result agrees with that of Banik Mallik, and Kanoria (2009) for LS and GN-III model for normal conductivity.

5. Numerical results and discussions

To get the solution for thermal displacement, temperature and stress in space time domain, we have to apply the Laplace inversion formula to the Equatios (4.19), (4.14) and (4.20), respectively. This has been done numerically using a method based on the Fourier series expansion technique (Honig & Hirdes, 1984). To get the roots of the polynomial Equation (4.16) in the complex domain, we have used the Laguerre's method. The numerical code has been prepared using the *Fortran 77* programming language. For the purpose of illustration, we consider the Copper metal with material constants (Banik et al., 2009)

$$K_0 = 386\,\mathrm{Wm^{-1}deg^{-1}}, \quad \alpha_t = 1.78 \times 10^{-5}\ K^{-1}, \quad c_E = 3.831\ \mathrm{JKg^{-1}deg^{-1}},$$

$$\mu_0 = 3.86 \times 10^{10}\ \mathrm{Nm^{-2}}, \quad \lambda_0 = 7.76 \times 10^{10}\ \mathrm{Nm^{-2}}, \quad \rho = 8954\ \mathrm{kg\ m^{-3}}.$$

The other parameters are taken as (Banik et al., 2009)

$$\beta^2 = 4,\ \ T_0 = 293\mathrm{K},\ \ b_1 = .042,\ \ g = 1.61,\ \ \tau_0 = .02,\ \ R = 1,$$

$$a_1 = .0105,\ \ \alpha^* = .0005\ K^{-1}.$$

Also we have taken $t = .4,\ \theta_0 = 1,\ \sigma_0 = -.05.$ for computational purpose. In order to study the effect of temperature-dependent material parameters, we have taken $\alpha = 1$ and

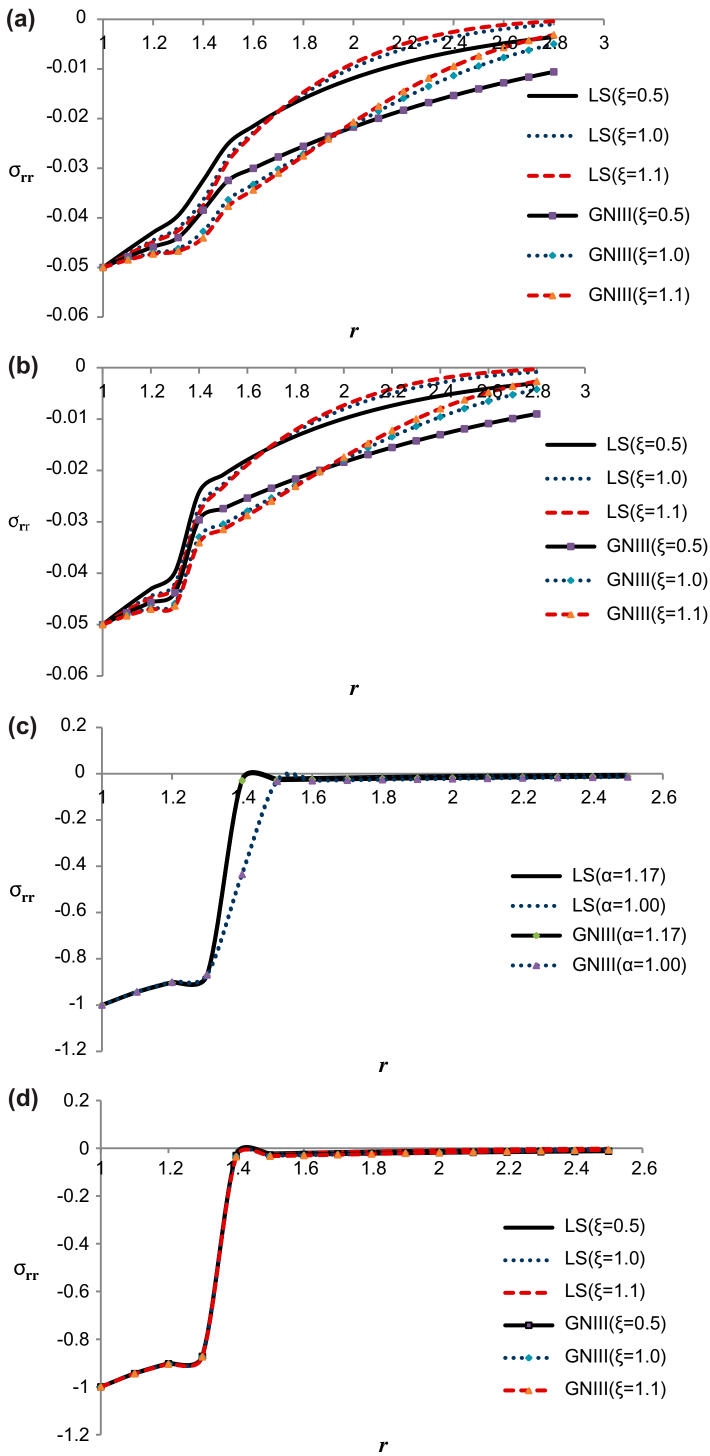

Figure 1. (a) Variation of radial stress σ_{rr} vs. r for $t = .4$ and $\alpha = 1.00$. (b) Variation of radial stress σ_{rr} vs. r for $t = .4$ and $\alpha = 1.17$. (c) Variation of radial stress σ_{rr} vs. r for $\xi = .5$. (d) Variation of radial stress component σ_{rr} vs. r for $t = .4$.

$\alpha = 1.17$. The variation of the stress distributions, temperature and displacement is presented graphically in a number of figures for weak conductivity ($\xi = .5$), normal conductivity ($\xi = 1.0$) and strong conductivity ($\xi = 1.1$).

Figure 1(a) and (b) represent the variation of the radial stress (σ_{rr}) against the radial distance r for the LS and GN-III models for weak ($\xi = .5$), normal ($\xi = 1.0$) and strong ($\xi = 1.1$) conductivity for $\alpha = 1.00$ and $\alpha = 1.17$, respectively. For both the figures, the boundary condition is satisfied at the boundary of the cavity ($r = 1$). *Here, the radial stress component is compressive in all the cases of conductivity for both the models*. It is seen that the magnitude of the stress σ_{rr} is gradually decreases with the increment of the radial distance r. It is also observed that the rate of decay of the *magnitude of the stress* is *high* in the case of high conductive material ($\xi = 1.1$) in comparison with that of the low conductive material ($\xi = .5$) for both the models. This is physically plausible. In this connection, it is noticed that the *magnitude of the radial stress* is large for the GN-III model compared to LS model for all the cases of conductivity.

Figure 1(c) shows the variation of radial stress σ_{rr} vs. the radial distance r for both the LS and GN-III models for $\xi = .5$, $t = .4$ and $\sigma_0 = -1$. For the small values of the stress σ_{rr} in comparison with $\sigma_0 = -1$, all the graphs for $\alpha = 1.00$ and $\alpha = 1.17$ are coincides. Also Figure 1(c) shows the same qualitative behaviour with those of Banik et al. (2009).

Figure 1(d) represents the variation of the radial stress vs. radial distance for both the LS model and GN-III model for all types of conductivity ($\xi = .5$, $\xi = 1.0$ and $\xi = 1.1$). The other parameters are taken as $t = .4$, $\sigma_0 = -1$ and $\alpha = 1.17$. Here, all the graphs are coincident. This phenomenon is obvious, since the numerical values of σ_{rr} are very small in comparison with the imposed stress ($\sigma_0 = -1$) on the boundary of the cavity. Therefore, the differences of the numerical values of σ_{rr} for $\xi = .5$, $\xi = 1.0$ and $\xi = 1.1$, and for fixed r are very small. Also Figure 1(d) shows the same qualitative behaviour with those of Banik et al. (2009) for $\xi = 1.0$.

Figure 2 is plotted to show the variation of the hoop stress ($\sigma_{\theta\theta}$) against the radial distance r for the LS and GN-III models for weak ($\xi = .5$), normal ($\xi = 1.0$) and strong ($\xi = 1\ .1$) conductivity for $\alpha = 1.17$. It is seen that the magnitude of the stress $\sigma_{\theta\theta}$ gradually decreases with the increment of the radial distance r. Also, the rate of decay of the magnitude of the stress is large for high conductivity ($\xi = 1.1$) compared with that of the low conductivity ($\xi = .5$) for both the models. It is also noticed that the magnitude

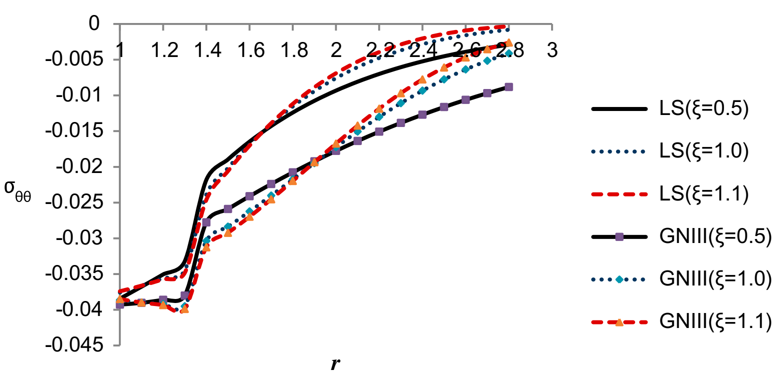

Figure 2. Variation of stress $\sigma_{\theta\theta}$ vs. r for $t = .4$ and $\alpha = 1.17$.

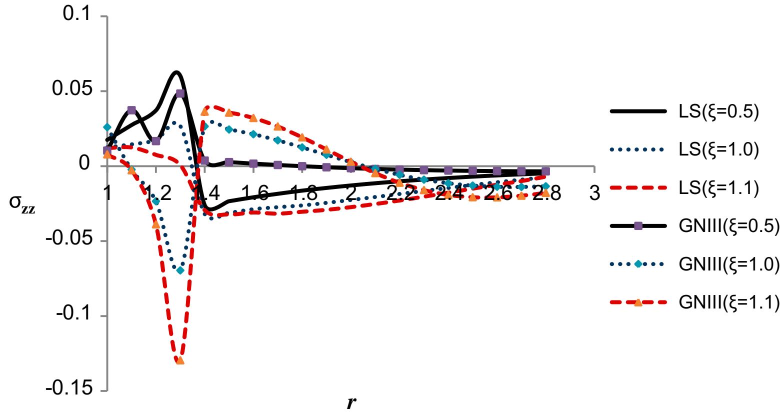

Figure 3. Variation of stress σ_{zz} vs. r for $t = .4$ and $\alpha = 1.17$.

of the hoop stress is large for the GN-III model compared to LS model for all the cases of conductivity.

Figure 3 shows the variation of the stress component σ_{zz} against the radial distance r for the time $t = .4$ and $\alpha = 1.17$. The magnitude of the stress σ_{zz} gradually decreases

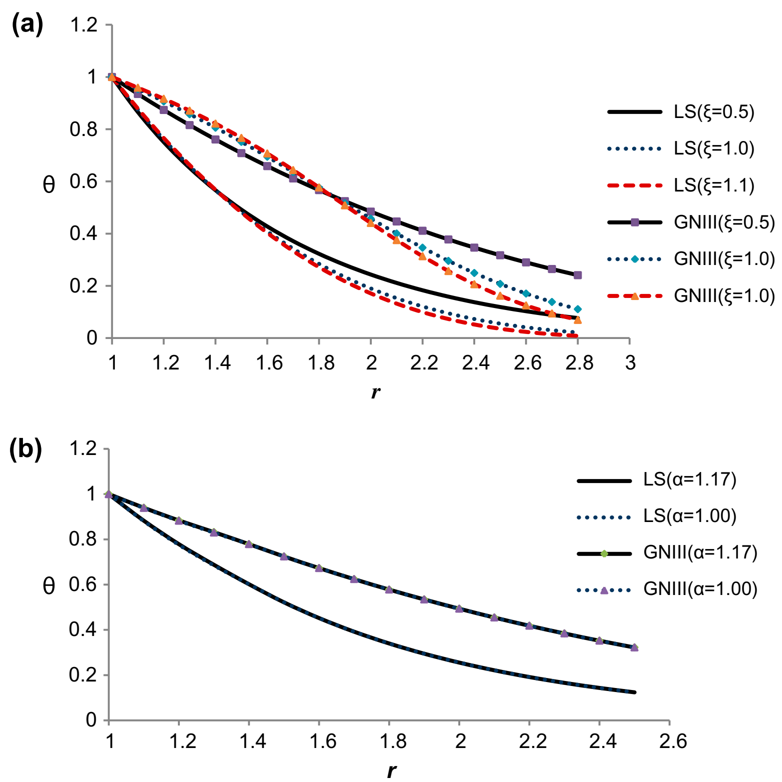

Figure 4. (a) Variation of temperature θ vs. r for $t = .4$ and $\alpha = 1.17$. (b) Variation of temperature component θ vs. r for $\xi = .5$.

after $r = 1.4$ in the increment of the radial distance r. The behaviour of the stress component σ_{zz} is completely different from the other two stress components σ_{rr} and $\sigma_{\theta\theta}$. The stress component σ_{zz} is somewhere compressive and somewhere reflexive. But, the other two stress components σ_{rr} and $\sigma_{\theta\theta}$ are compressive into the whole region.

Figure 4(a) represents the variation of the temperature θ vs. the radial distance r for the LS and GN-III models for weak ($\xi = .5$), normal ($\xi = 1.0$) and strong ($\xi = 1.1$) conductivity for $\alpha = 1.17$. From the figure it is seen that at the boundary $r = 1$, the magnitude of the temperature θ is 1, which agrees with the imposed boundary condition. The magnitude of the temperature θ is maximum at $r = 1$, where the step-input temperature is imposed and then the temperature θ gradually decreases with increasing r. This is physically plausible. Here, the temperature is reflexive. It is also observed that the rate of decay of the magnitude of the temperature is large in the case of high conductive material ($\xi = 1.1$) in comparison with that of the low conductive material ($\xi = .5$) for both the models.

Figure 4(b) shows the variation of the temperature θ against the radial distance r for the LS and GN-III model for weak conductivity $\xi = .5$ with $\alpha = 1.00$ and $\alpha = 1.17$. The boundary condition $\theta = 1$ is satisfied on the boundary for both the models. Here, the magnitude of the temperature is large for GN-III model in comparison with the LS

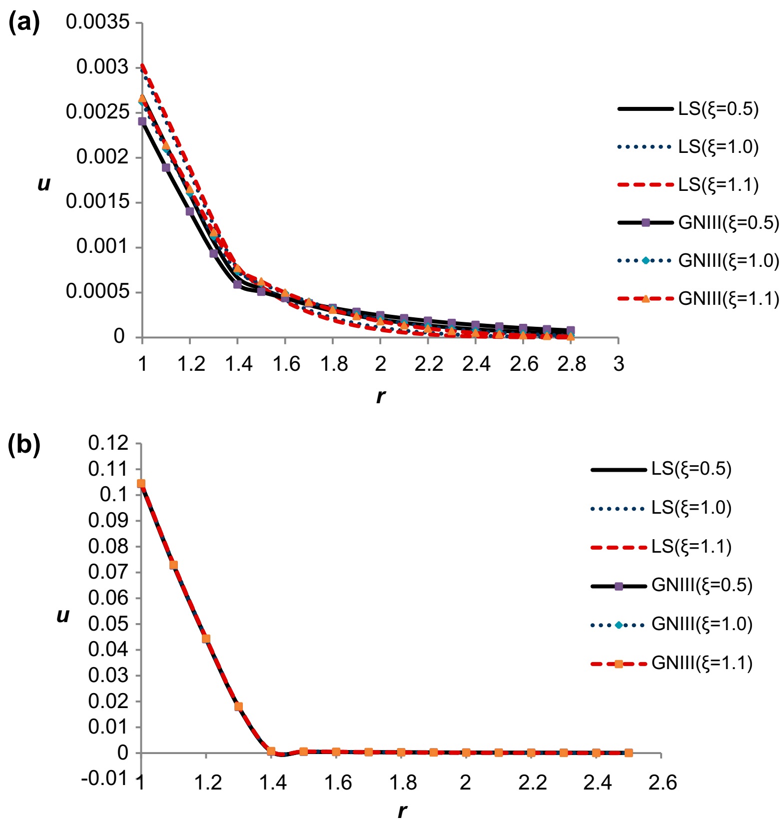

Figure 5. (a) Variation of displacement component u vs. r for $t = .4$ and $\alpha = 1.17$. (b) Variation of displacement component u vs. r for $t = .4$.

model. Due to the small changes in the magnitude of the temperature for fixed r, no effect is observed for $\alpha = 1.00$ and $\alpha = 1.17$. Figure 4(b) shows the same qualitative behaviour with those of Banik et al. (2009) for $\xi = 1.0$.

Figure 5(a) represents the variation of the displacement u against the radial distance r for the LS and GN-III models for weak ($\xi = .5$), normal ($\xi = 1.0$) and strong ($\xi = 1.1$) conductivity for $\alpha = 1.17$. It is seen that the magnitude of displacement rapidly decreases in the region $1 \leq r \leq 1.4$ and then slowly decreases in the region under consideration. All the graphs of the displacement are reflexive for all the cases under consideration.

Figure 5(b) represents the variation of the displacement u vs. the radial distance for both the LS and GN-III model for all types of conductivity ($\xi = .5$, $\xi = 1.0$ and $\xi = 1.1$). The other parameters are taken as $t = .4$, $\sigma_0 = -1$ and $\alpha = 1.17$. Here, all the graphs in the figure are coincident. This is due to the small changes of the numerical values of u in different values of r. Also Figure 5(b) shows the same qualitative behaviour with those of Banik et al. (2009) for $\xi = 1.0$.

6. Conclusions

The problem of investigating the stress, temperature and displacement in an infinite isotropic elastic body having a cylindrical cavity is studied in the light of LS and GN-III model in the context of fractional heat conduction equation. The method of Laplace Transform is used to write the basic equations which is then solved by Laguerre's method. The numerical inversion of Laplace Transform is computed by the method of Fourier series expansion technique. The analysis of the result permits some concluding remarks:

(1) It is observed that the rate of decay of the magnitude of the stress and temperature is high in the case of high conductive material ($\xi = 1.1$) in comparison with that of the low conductive material ($\xi = .5$) for both the models.
(2) It is noticed that the magnitude of the radial stress and hoop stress is large for the GN-III model compared to LS model for all the cases of conductivity.
(3) The effects of fractional order in the temperature equations are clearly observed in the figures.
(4) The results carried out in this paper can be used to design various instruments under fractional order thermoelasticity theory to meet engineering requirements.

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