Friction-induced vibration and stick-slip waves

Short survey and new results

Abdelbacet Oueslati

Université de Lille 1, UFR de mathématiques pures et appliquées
Laboratoire de Mécanique de Lille - CNRS UMR 8107
F-59655 Villeneuve d'Ascq cedex
abdelbacet.oueslati@univ-lille1.fr

ABSTRACT. *This paper presents a short review and new results about the self-excited responses under the form of stick-slip regimes. First, the Van-der Pol oscillator with one degree of freedom is considered. Then it is shown that it is possible to build semi-analytical and numerical (by the FEM.) solutions of stick-slip-separation waves for a brake-like system. Then, we present new results concerning the mechanical model composed of a rigid half space in frictional sliding with an elastic half-space. The method of solution, based on periodic complex Radoks potentials, is novel and differs from those in literature. Besides, in contrast with many works, we shall consider the longitudinal elongation which plays a crucial rule in the solution procedure. A unique and weakly singular solution is found and satisfies all stick-slip conditions except over a narrow zone at transition points which implies a crack-like behaviour at the stick-slip borders.*

RÉSUMÉ. *Cet article présente une étude bibliographique non exhaustive et des résultats nouveaux sur les vibrations auto-entretenues sous forme d'ondes stick-slip. On commence par établir la réponse stick-slip-separation d'un système mécanique simple à un seul degré de liberté puis pour un solide élastique borné formé de deux cylindres coaxiaux. On présente ensuite de nouveaux résultats concernant le glissement stationnaire entre un massif rigide sur un demi-espace élastique. La méthode de solution est basée sur les potentiels complexes et périodiques de Radok. Par ailleurs, contrairement à plusieurs travaux antérieurs, nous prenons en compte la déformation longitudinale et nous montrons qu'elle joue un rôle crucial dans la formulation et la résolution de problème. Une solution unique et faiblement singulière est construite. Elle vérifie toutes les conditions de contact unilatéral frottant sauf sur une petite zone extrêmement faible à la transition entre les régions de glissement et d'adhérence.*

KEYWORDS: *nonlinear dynamic, contact with dry friction, stick-slip waves.*

MOTS-CLÉS : *dynamique non linéaire, contact unilatéral avec frottement sec, ondes d'adhérence-glissement.*

DOI:10.3166/EJCM.20.167-188

1. Introduction

During the two last decades, an important issue emerges in the area of *elasto-dynamic* problems involving frictional contact, namely the friction-induced vibration resulting from the flutter instability in the spirit of Poincaré-Hopf bifurcation. The induced vibrations have the form of stick-slip or stick-slip-separation self-sustained oscillations propagating along the contact interface and generally accompanied with noise emission. Many examples are common in daily life such as creaking door, noise of chalk against a table, brake squeal, guitar sound, silo music (sound emission during the flow of granular materials through silos) are generally associated to stick-slip propagation. Study of such periodic regimes is relevant for break squeal (Moirot *et al.*, 2000; Nakai *et al.*, 1996), simulations of earthquakes and seismology analysis (Andrews *et al.*, 2005; Ben-Zion *et al.*, 1998; Cochard *et al.*, 1994), study of ultrasonic motors (Zharii, 1996), granular discharge from silos (Oueslati, 2004; Mukesh *et al.*, 2006), interpretation of Shallamach waves in sliding of rubber against a rigid substrate (Schallamach, 1971).

In literature, first studies of stick-slip focused on discrete systems, typically the Van-der Pol or the Klarbring oscillator composed of springs-mass assemblage in frictional contact with a rigid substrate moving with a constant velocity. For such mechanical systems, the formation of stick-slip motion is attributed to a static coefficient of friction higher than a kinematic one or to the decay of the kinematic coefficient with the sliding velocity. The transition toward friction induced vibration is numerically obtained in many works (Oestreich *et al.*, 1996; Oancea *et al.*, 1997). The construction of analytical stick-slip solutions for discrete mechanical systems does not present any particular difficulty (Feeny *et al.*, 1998; Moirot, 1998).

Recent investigations of the steady sliding contact between dissimilar elastic half-spaces or between a semi-infinite elastic solid and a rigid one showed that the steady state is dynamically unstable for a *constant* friction coefficient. For instance, Renardy (Renardy, 1992) explained the steady sliding contact between a rigid substrate and an incompressible half-plane within the framework of neo-Hooken constitutive behavior. He found that flutter instability occurs in the limit of elasticity for a coefficient of friction greater then 1. This same conclusion was established independently by Martins (Martins *et al.*, 1995a; Martins *et al.*, 1995b) and (Simoes *et al.*, 1998) after investigation of the dynamic response of an elastic and viscoelastic semi-infinite solid in contact with a rigid body. Further, they showed that the presence of viscous dissipation has the effect of increasing the minimum value of coefficient of friction required for existence of self-excited vibration. In the same spirit, surface instabilities in a Mooney-Rivlin half-space compressed against a rigid flat surface are studied by Désoyer and Martins (Désoyer *et al.*, 1998) and it is found that the problem is again ill-posed for sufficiently large coefficients of friction. In a series of interesting papers, (Adams, 1995; Adams, 1998; Adams, 2001) showed that the steady sliding between two dissimilar half-planes is ill-posed for a wide range of coefficient of friction, material combinations and relative sliding velocity. (Adams, 1995) suggested that the dynamic instability of the steady state is related to the destabilization of the so-called interfacial slip waves. Such waves exist in frictionless contact and are called also the "generalized Rayleigh waves" because they propagate with the Rayleigh wave celerity. Recall that slip waves are initially observed by (Weertman, 1963) and (Murty, 1975). Later (Ranjith *et al.*, 2001) demonstrated the connection between the ill-posedness of the Coulomb friction problem and slip waves. Precisely, it was shown that, for material combinations where the generalized Rayleigh wave exists, the steady sliding with Coulomb friction is dynamically unstable for an arbitrarily small values of the friction coefficient.

The paper begins by revisiting the elastodynamic of a simple discrete mechanical system in the presence of the unilateral contact with Coulomb friction. The second sections is concerned with an elastic and bounded continuum solid modeling a brake-like-system. For this solid, it is possible to construct analytically and numerically by the finite element method different families of stick-slip and stick-slip-separation-reverse slip waves. The last part is devoted to the investigation of stick-slip waves crossing the contact interface between an elastic half-space and and a rigid one. The contact between the solids is governed by unilateral constraints and Coulomb friction law and the deformable body is loaded by remote uniform stresses $\tau^*_{yy} < 0$, $\tau^*_{xy} > 0$ and τ^*_{xx}. The method of solution, based on periodic complex Radoks potentials, is novel and differs from those in literature, namely the series method and the Weertmans dislocation formulation. Besides, in contrast with many works, we shall consider the longitudinal elongation which plays a crucial rule in the solution procedure. The considered loading introduces an additional velocity V^* related to the longitudinal elongation ε^*_{xx} due to both the normal stress τ^*_{yy} and longitudinal stress τ^*_{xx}. We demonstrate that if V^* vanishes then there is no solution. If $V^* \neq O$, a unique singular solution is found and satisfies all stick-slips equations and inequalities, except one : the normal contact stress exhibits a positive singularity over a small zone which means a separation in a narrow zone.

2. Stick-slip response of 1-d.o.f system

Let us start by examining the simplest friction oscillator, namely the Van-der Pol system. It consists of an elastically mounted mass on a driving belt moving at the speed V as depicted in Figure 1.

The contact force is splitted in a normal component N and a tangential force F and the coefficient of friction f between the mass and the belt shall be modeled by a static coefficient f_s greater than the dynamical one f_d. Furthermore, we assume that $f_s > f_d$. The equation of motion of the mass writes

$$m\ddot{x} + k\,x = F(\dot{x} - V) \qquad [1]$$

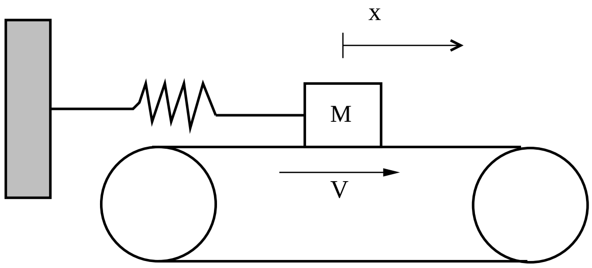

Figure 1. *The friction oscillator of Van-der Pol*

and the boundary conditions expressing the Coulomb friction law are given by

$$|F(0)| \leq f_s\, N \quad \text{stick} \qquad [2]$$

$$F\big((V-\dot{x}) \geq 0\big) = f_d\, N < f_s\, N \quad \text{positive slip} \qquad [3]$$

$$F\big((V-\dot{x}) \leq 0\big) = -f_d\, N > -f_s\, N \quad \text{negative slip} \qquad [4]$$

The Coulomb friction law Eq.(2,3,4) can be reduced to the following

$$|F| \leq f_s\, N \qquad [5]$$

$$(V-\dot{x})\, F - |(V-\dot{x})|\, f_d\, N = 0 \qquad [6]$$

The analytical determination of the trajectories does not raise a particular difficulty. On one hand, in the half space $\dot{x} \leq V$ one gets the positive slip state and the trajectories are complete or truncated ellipses centered in $x_e = \dfrac{f_d\, N}{k}$. On the other hand, in the half space $\dot{x} \geq V$, the state is a negative slip and the trajectories are ellipses centered in $-x_e$. The zone of stick state of the mass M is given by the segment $[(-x_a, V), (x_a, V)]$ with $x_a = \dfrac{f_s\, N}{k}$. It is easy to establish that there exists a unique position of stable equilibrium for any initial conditions in the domain S. It exists also an attractor limiting cycle for initial conditions outside of S. This cycle is composed of a stick phase during $\Delta t_a = \frac{2\,(f_s - f_d)\, N}{kV}$ and positive slip period during $\Delta t_g = \dfrac{2\pi Arctan(\Delta t_a w/2)}{w}$ where $w = \sqrt{\frac{k}{m}}$. The phase diagram of the mechanical system is plotted in Figure 2

Notice that there is a discontinuity of friction at the border of stick and slip zones resulting in a discontinuity of the acceleration.

Similar analysis may be derived for the same problem but by assuming that the dynamical friction coefficient is not constant but it decreases linearly with the relative velocity between the mass and the moving belt.

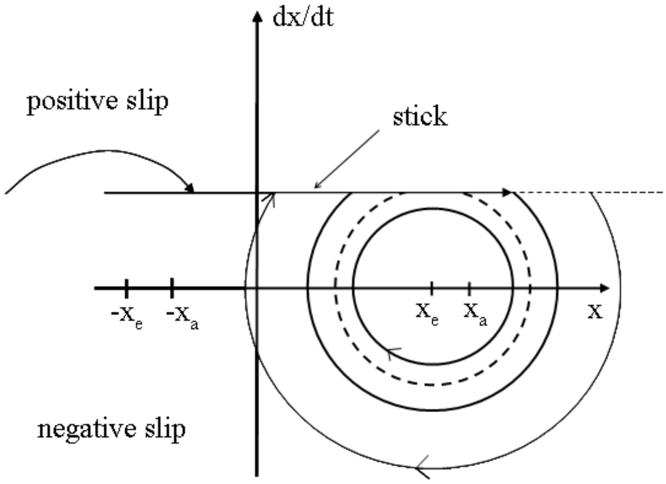

Figure 2. *The phase diagram of the stick-slip regime*

Equation governing the motion of the mass is also given by (1) while the Coulomb friction conditions write now as follows

$$|F(0)| \leq f_d\, N \quad \text{stick} \qquad [7]$$

$$F\big((V - \dot{x}) \geq 0\big) = f_d\, \big(1 - \tau(V - \dot{x})\big)\, N \quad \text{positive slip} \qquad [8]$$

$$F\big((V - \dot{x}) \leq 0\big) = -f_d\big(1 - \tau(V - \dot{x})\big)\, N \quad \text{negative slip} \qquad [9]$$

In the half space $\dot{x} \leq V$ we have a positive slip and the trajectories are divergent spirals centered in $x_e^+ = \frac{fd(1-\tau V)N}{k}$ (it is supposed that $(1 >> \tau V)$). In the half space $\dot{x} \geq V$ there is a negative slip and the trajectories are still divergent spirals centered in $x_e^- = \frac{fd(1+\tau V)N}{k}$. The stick zone zone is the segment $[(-x_a, V), (x_a, V)]$ with $x_a = \dfrac{f_s\, N}{k}$. A unique equilibrium position x_e is found. It easy to show that this equilibrium is unstable by flutter in the spirit of the Hopf bifurcation. Observe that the decrease of the coefficient of friction with respect to the relative velocity is equivalent to introduce a negative damping in the equation of motion $(m\ddot{x} - f_d\tau N\dot{x} + kx = f_d(1 - \tau V)N)$. Besides, there exists a limiting cycle which for any initial condition as shown in Figure 3.

3. Stick-slip waves for a brake-like system

Consider a brake-like system composed of an elastic annular tube with internal radius R and external radius R^* in frictional contact with a rotating rigid shaft of

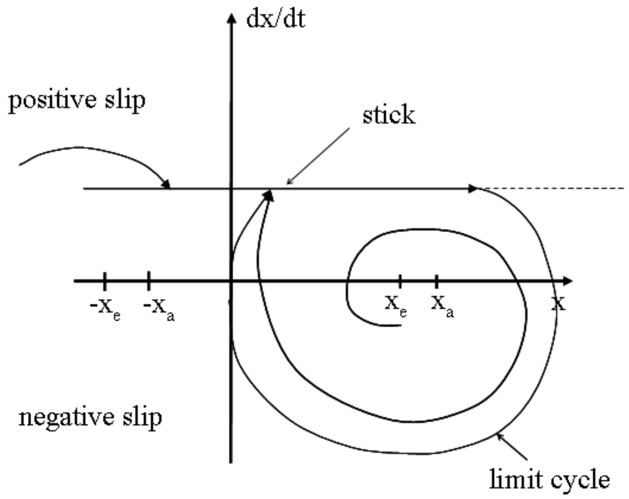

Figure 3. *The phase diagram of the mass*

radius $R + d, (d \geq 0)$ and of angular velocity Ω, figure 4. The elastic cylinder is fixed at its outer surface and the frictional model is the Coulomb's law with a *constant* coefficient f. The mismatch d is considered as a load parameter controlling the normal contact pressure.

Within the framework of linear small elastic plane strain, the dynamic equations of the motion with the corresponding boundary and unilateral frictional contact read :

$$\mathbf{u} = \frac{\overline{\mathbf{u}}}{R},\ \sigma = \frac{\overline{\sigma}}{E},\ r = \frac{\overline{r}}{R},\ \gamma = \frac{\rho R^2 \Omega^2}{E},\ \xi = \frac{R^*}{R},\ \delta = \frac{d}{R},\ t = \Omega \overline{t},\ \dot{\mathbf{u}} = \frac{d\mathbf{u}}{dt}$$

$$\left\{ \begin{array}{l} div\sigma = \gamma \ddot{\mathbf{u}} \\ E\,\sigma = \lambda tr(\epsilon)\mathbf{I} + 2\mu\epsilon \\ \epsilon = grad_s \mathbf{u} \\ u(\xi,\theta,t) = v(\xi,\theta,t) = 0 \\ \sigma_{rr}(1,\theta,t) = -p(\theta,t),\ \sigma_{r\theta}(1,\theta,t) = -q(\theta,t) \\ u \geq \delta,\ p \geq 0,\ p(u-\delta) = 0 \\ |q| \leq fp, \quad q(1-\dot{v}) - fp|1-\dot{v}| = 0 \end{array} \right. \qquad [10]$$

The steady sliding solution is given by :

$$\left\{ \begin{array}{l} u_e = \delta \frac{1}{\xi^2-1}(\frac{\xi^2}{r} - r),\ v_e = \delta f \frac{1}{\xi^2-1}(\frac{\xi^2}{r} - r)(1 + \frac{1}{\xi^2(1-2\mu)}) \\ p_e = \delta \frac{1}{1-\xi^2} \frac{1}{1+\mu}(\xi^2 + \frac{1}{1-2\mu}),\ q_e = f p_e \end{array} \right. \qquad [11]$$

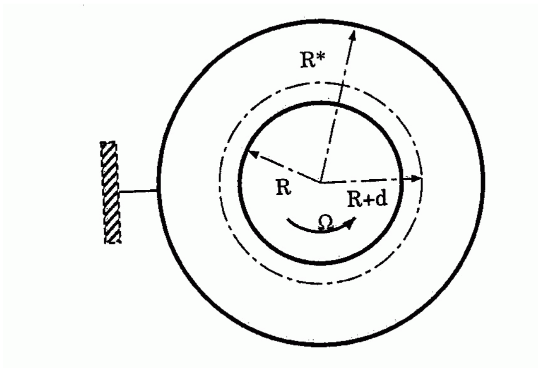

Figure 4. *The problem of coaxial cylinders in frictional contact*

It is shown in (Moirot, 1998; Moirot *et al.*, 2002) that this steady state is unstable which results in the apparition of stick and/or separation zones on the contact interface. An interesting simplification of the problem is obtained when the displacement is sought in the form

$$u(r,\theta,t) = X(r)U(\theta,t), \quad v(r,\theta,t) = X(r)V(\theta,t), \quad X(r) = \frac{1}{\xi^2-1}(\frac{\xi^2}{r} - r)$$

In this approximation, the following local equations are obtained from the virtual work equation when admissible displacements are restricted to the considered expressions

$$\left\{ \begin{array}{l} \ddot{U} - bV'' - DV' + gU = P, \\ \ddot{V} - aU'' + DU' + hV = Q, \\ P \geq 0, \; U - \delta \geq 0, \; P(u - \delta) = 0, \\ |Q| \leq fP, \; Q(1 - \dot{V}) - fP|1 - \dot{V}| = 0 \end{array} \right. \tag{12}$$

where $'$ denotes the derivative with respect θ and

$$\left\{ \begin{array}{l} a = \frac{\tilde{a}A}{\gamma B}, \; b = \frac{\tilde{b}A}{\gamma B}, \; g = \frac{2\tilde{a}+2(\xi^2-1)\tilde{b}}{\gamma B}, \; h = \frac{2\xi^2\tilde{b}}{\gamma B}, \\ \tilde{a} = \frac{1-\nu}{(1+\nu)(1-2\nu)}, \; \tilde{b} = \frac{1}{2(1+\nu)}, \\ A = -\frac{2\xi^2 ln\xi}{\xi^2-1} + \frac{1+\xi^2}{2} > 0, \; B = \frac{\xi^4 ln\xi}{\xi^2-1} + \frac{1-3\xi^2}{4} > 0, \\ D = \frac{aC_1-bC_2}{A}, \; C_1 = \frac{2\xi^2 ln\xi}{\xi^2-1} - 1 > 0, \; C_2 = -\frac{2\xi^2 ln\xi}{\xi^2-1} - 1 + 2\xi^2 > 0 \end{array} \right. \tag{13}$$

A periodic solution is sought in the form of a wave propagating at constant velocity :

$$U = U(\Phi), \; V = V(\Phi), \quad \text{où } \Phi = \theta - ct \tag{14}$$

The governing equations of such a wave follow from [12] is therefore given by

$$\begin{cases} (c^2 - b)U'' - DV' + gU = P \\ (c^2 - a)V'' + DU' + hV = Q \\ P \geq,\ U \geq \delta,\ P(U - \delta) = 0 \\ |Q| \leq fP,\ Q(1 - \dot{V}) - fP|1 - \dot{V}| = 0 \end{cases} \qquad [15]$$

The steady state is given by

$$\begin{cases} U_e = \delta, \quad V_e = \frac{\delta f g}{h}, \\ P = P_e, \quad Q_e = fP_e. \end{cases} \qquad [16]$$

The boundary conditions correspondent to different conditions of the dry friction are given as follows :

– in the stick zone

$$U = \delta,\ \dot{V} = 1,\ P > 0,\ |Q| < f\ P \qquad [17]$$

– in the positive slip zone

$$U = \delta,\ \dot{V} < 1,\ P > 0,\ Q = f\ P \qquad [18]$$

– in the negative slip zone

$$U = \delta,\ \dot{V} > 1,\ P > 0,\ Q = -f\ P \qquad [19]$$

– in the separation zone

$$U > \delta,\ P = Q = 0 \qquad [20]$$

The reduced Equation [12] combined with the boundary conditions listed in [17,18,19] and [20] enable one to construct different families of stick-slip and stick-slip-separation waves. For example, Figure 5 represents the phase diagrams for stick-slip waves for different normal contact pressure and Figure 6 shows some phase diagrams for stick-slip-separation wave in the plane of the normal velocity and normal displacement.

3.1. *Numerical results by the FEM*

In order to have a better idea on the validity of the reduced equations and on the repartition of different stick, slip and separation zones, a numerical solution of the initial equations by the finite element method and by direct time-integration is now considered in complement to the semi-analytical approach.

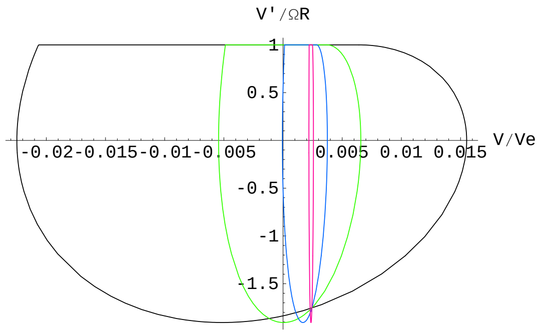

Figure 5. *The phase diagram* (V, V') *for the stick-slip regime for different angular velocity* Ω

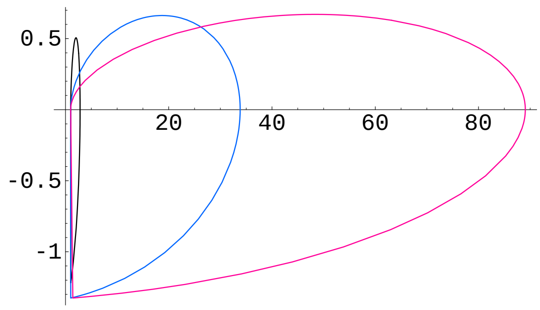

Figure 6. *The phase diagram* (V, V') *for the stick-slip regime for different angular velocity* Ω

An explicit scheme using Lagrange multipliers, as proposed in (Carpenter *et al.*, 1991) for frictional contact, is applied. The discretized equations are written as

$$\left\{ \begin{array}{l} \mathbf{M}\ddot{\mathbf{u}}_m + \mathbf{K}\mathbf{u}_m + \mathbf{C}^T_{m+1}\boldsymbol{\lambda}_m = \mathbf{F}_m \\ \mathbf{C}_{m+1}\left(\mathbf{u}_{m+1} + \mathbf{X}\right) = 0 \end{array} \right. \qquad [21]$$

where F_m denotes the given forces, G_{m+1} are the contact constraints and λ_m are associated Lagrange multipliers representing normal and tangential reactions. At any time step, the velocity and acceleration vectors, $\dot{u}_m$ and $\ddot{u}_m$, are related to displacements and time-increment h following the well known β-method,

$$\left\{ \begin{array}{l} \dot{\mathbf{u}}_m = \frac{1}{1+2\beta_1}\{\dot{\mathbf{u}}_{m-1} + \Delta t(1-\beta_1)\ddot{\mathbf{u}}_{m-1} + \frac{2\beta_1}{\Delta t}(\mathbf{u}_{m+1} - \mathbf{u}_m)\} \\ \ddot{\mathbf{u}}_{m+1} = \frac{2}{\Delta t^2}(\mathbf{u}_{m+1} - \mathbf{u}_m - \Delta t\dot{\mathbf{u}}_m) \end{array} \right. \qquad [22]$$

with $0.5 \leq \beta_2 \leq 1$. The new coordinates are given by$X + u$ of a boundary nodal point of the deformable solid are first computed under the assumption of null reactions. This prediction step is followed by a correction step when the non-penetration condition is not satisfied. This correction step consists of re-evaluating the nodal reactions in order to ensure the contact unilateral condition and of re-writing the new constraints G_{m+1} :

$$\left\{ \begin{array}{l} \boldsymbol{\lambda}_m = [\Delta t^2 \mathbf{C}_{m+1} \mathbf{M}^{-1} \mathbf{C}_{m+1}^T]^{-1} \mathbf{C}_{m+1} (\mathbf{u}_{m+1}^* + \mathbf{X}) \\ \mathbf{u}_{m+1}^c = -\Delta t^2 \mathbf{M}^{-1} \mathbf{C}_{m+1}^T \boldsymbol{\lambda}_m \\ \mathbf{u}_{m+1} = \mathbf{u}_{m+1}^* + \mathbf{u}_{m+1}^c \end{array} \right. \quad [23]$$

Internal iterations are then performed in order to satisfy Coulomb's law. A contact node is in stick regime if $\lambda_{mn} > 0$ and if $|\lambda_{mt}| < f\ \lambda_{mt}$ while it is in slip regime if $\lambda_{mn} > 0$ and $|\lambda_{mt}| = f\ \lambda_{mt}$.

Numerical simulations have been performed with $\beta_2 = 0.9$. The numerical damping induced by this value of β_2 is not a nuisance in the computation of the limiting cycle since the energy loss of the system is compensed continuously by the rotating cylinder. However, this damping accelerates artificially the convergence rate to the limit response. It has been checked that the convergence rate is practically the same for $0.6 \leq \beta_2 \leq 0.9$ and slower for $0.5 \leq \beta_2 \leq 0.6$ (Oueslati *et al.*, 2003).

Numerical simulations with various initial data have been performed in order to study the transition to a limit regime which can be a stick-slip or stick-slip-separation wave. It has been found that the limit regime may be different for two different initial conditions. The stick-slip regime occurs if the the normal contact pressure is hight enough or for a small coefficient of friction. For example, choosing $f = 0.2$, $\delta = 0.004$ and $\Omega = 1rd/sec$ the limit cycle results as a stick-slip wave with 4 modes as plotted in Figure 7. Furthermore, it is found that the stick-slip-separation wave may be obtained when the mismatch is small enough or when the friction is high enough. For example, a stickslipseparation wave with 8 modes is obtained for $\Omega = 50$ rad/s, $\delta = 0.001$ and $f = 0.7$, cf. Figure 10. The result for radial displacement is shown in Figure 10 where the propagation phenomenon is clearly seen. It is worth noting that for a very hight coefficient of friction ($f > 1.5$), a hight angular velocity Ω and small mismatch δ, we may obtain a regime of stick-separation wave. Figure 8 presents an example of stick-separation wave with 2 modes obtained with $f = 2$, $\delta = 0.004$ and $\Omega = 80rd/sec$.

3.2. *Stick-slip-separation-reverse slip waves*

It is worth noting that another type of self-excited regime, namely a stick-slip-separation-reverse slip response were pointed out for a simple friction oscillator with a more complicated dry friction law in (Teufel *et al.*, 2005).

We notice that this regime was not obtained by using the finite element method. However, by solving numerically the nonlinear system [12] by the program Boundsco based on the multiple shooting method (Oberle *et al.*, 1989), this new regime is obtained. It is worth noting that the used algorithm is able to compute the switching points

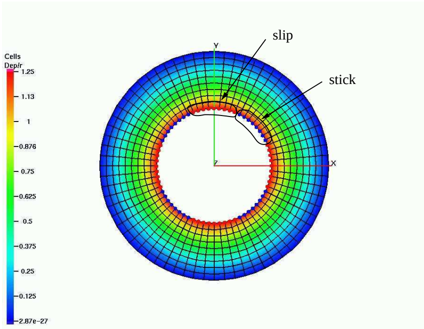

Figure 7. *Stick-slip wave with 4 modes*

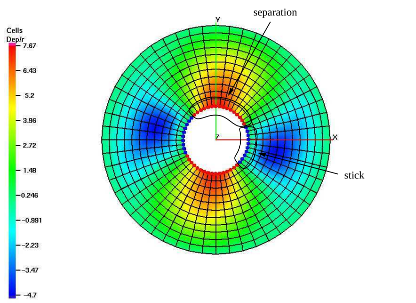

Figure 8. *Stick-separation wave with 2 modes*

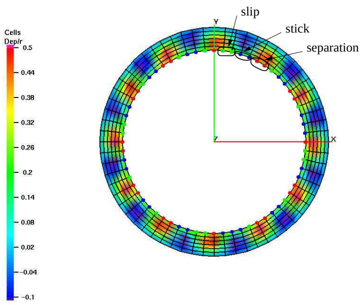

Figure 9. *Stick-slip-separation wave with 8 modes*

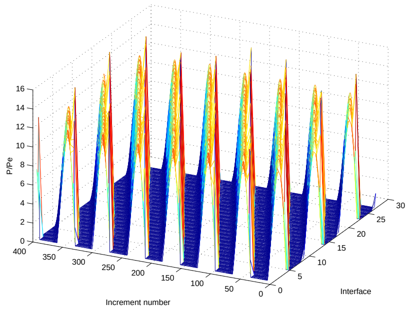

Figure 10. *Normal contact pressure for stick-slip-separation wave*

between different regions automatically. For instance, at the borders between different zones appropriate switching conditions must be satisfied, e.g. $P = 0$ at the start of the separation zone and $U = \delta$ and U' jumps back to 0 at the end. For example, Figure 11, shows the radial displacement for different values of d and the loci of the switching points. Observe that if the mismatch d becomes very small, the switching points and coalesce and the slip region right of the separation zone vanishes and one obtains the stick-slip-separation state (Nguyen *et al.*, 2008). In Figure 12 a phase plane plot for the traveling wave in mode-8 with reverse slip (overshooting) is depicted.

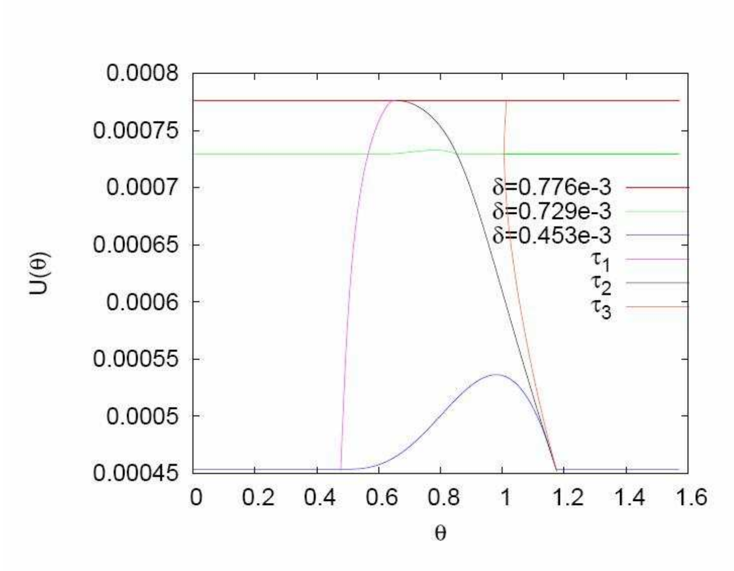

Figure 11. *Radial displacement showing small separation for small contact pressure*

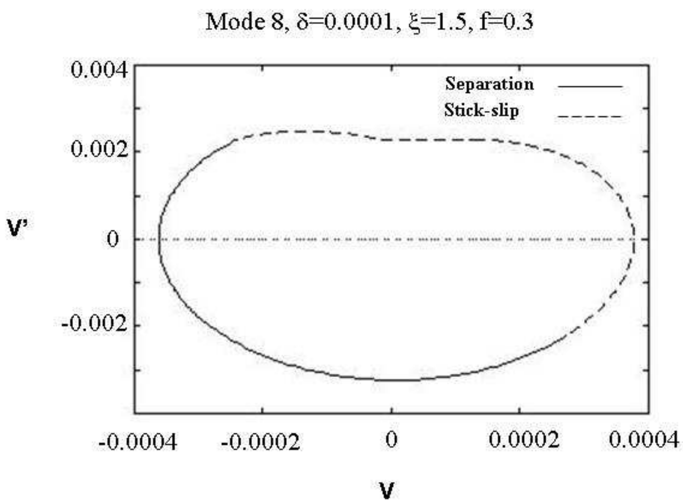

Figure 12. *Phase plane plot of the tangential displacement of a traveling wave with reverse slip (overshooting)*

4. Stick-slip waves between an elastic half-space and a rigid block

4.1. *Description of the problem*

Consider an elastic solid, with shear modulus G and waves velocities c_1, c_2 occupying the lower half space Ω^- and sliding against the upper rigid half-space Ω^+ which moves to the right with velocity V, as shown in Figure 13, in plane strain conditions. Unilateral contact and Coulomb friction with constant friction coefficient μ are assumed. It is emphasized that the *interfacial* or *local* coefficient of friction μ is the ratio of shear to normal contact pressure at the interface which would cause local slipping to occur (Adams, 2001). The elastic body is subjected to remote constant stresses $\tau^*_{yy} < 0$, $\tau^*_{xy} > 0$ such that $\tau^*_{xy} = -\mu^*\ \tau^*_{yy}$, with $\mu^* < \mu$.

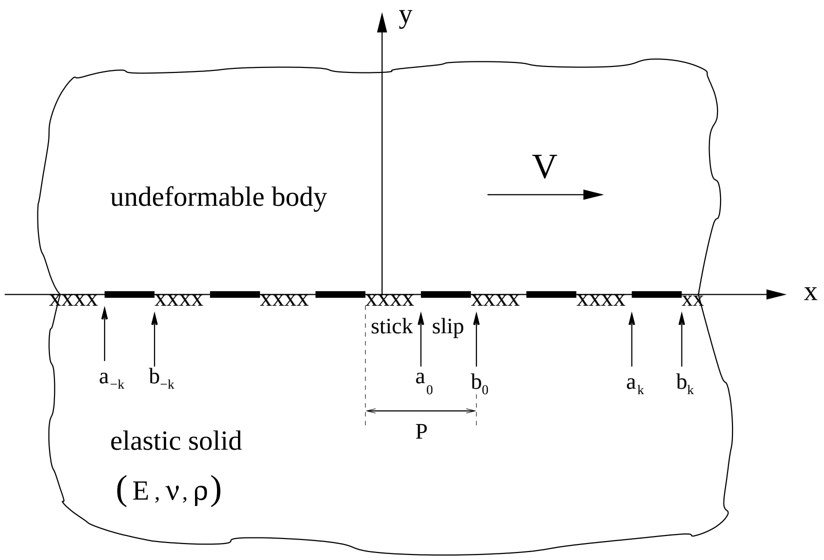

Figure 13. *A rigid body sliding on an elastic half-space*

We investigate here the possibility of relative motion of the two bodies, due to the existence of periodic stick and slip regions which propagate along the interface with some wave speed c. In this case the quantity μ^* may be interpreted as the apparent coefficient of friction, since sliding occurs with that ratio of applied shear to normal traction (Adams, 1995).

We assume that the periodic stick-slip wave consists in a stick region plus a slip one indefinitely repeated as shown in Figure 13. Let us denote by SL the set of the slip segments

$$SL = \cdots [a_{-k}, b_{-k}] \bigcup \cdots [a_0, b_0] \bigcup \cdots [a_k, b_k] \bigcup \cdots, \quad k \in \mathbb{N}$$

The remainder part of contact interface is the stick zone and will be refereed by ST. The infinite axis $y = 0$ is oriented in the direction of increasing x and the following notation will be frequently used : $SL^+ = lim_{y \to 0^+} SL$ and $SL^- = lim_{y \to 0^-} SL$. In the same way, one defines ST^+ and ST^-.

If we fix the origin of coordinate frame at $b_{-1} = 0$, we can simply denote any particular slip zone $[a_i, b_i]$ by its generic segment $[a \equiv a_0, b \equiv b_0]$ and any stick zone $[b_{k-1}, a_k]$ by its generic segment $[b_{-1} = 0, a \equiv a_0]$.

Together with a fixed frame coordinates (X, Y), we shall use also moving coordinates $x = X - c\,t$, $Y = y$ where c is the wave velocity. Material derivative in steady state case is denoted by a dot, $\dfrac{dg}{dt} \equiv \dot{g} = -c\, g_{,x}$.

Let the material velocity in the elastic body at interface be decomposed into the sum

$$\dot{U} = V^* + \dot{u} = V^* - c\, u_{x,x} \qquad [24]$$

where V^* is some velocity defined hereafter. From elastic law, the elongation in the x-direction is

$$\varepsilon^*_{xx} = u^*_{x,x} = -\frac{\nu(1+\nu)}{E}\,\tau^*_{yy} > 0 \qquad [25]$$

The velocity V^* is defined hereafter by

$$V^* = -c\, u^*_{x,x} = c\,\frac{\nu(1+\nu)}{E}\,\tau^*_{yy} \qquad [26]$$

Note that the velocity V^* is positive because $c < 0$ as it will be shown hereafter. Equation [26] establishes a link between τ^*_{yy}, c and V^* so that the unknowns reduced to c and b, for prescribed stresses and V^*. The additional stresses correspondent to the perturbed stick-slip motion are denoted by σ_{xy}, σ_{yy} and σ_{xx}.

Let us now recall the *steady* elastodynamic equations within the framework of homogenous and isotropic elasticity under the plane strain hypothesis. In the moving frame (Oxy) attached to the propagating stick-slip wave ($x = X - ct\,, \quad y = Y$), one introduces the Radok complex variables

$$z_1 = x + i\beta_1 y \quad ; \quad z_2 = x + i\beta_2 y$$

where $\beta_1 = \sqrt{1 - \dfrac{c^2}{c_1^2}}$, $\beta_2 = \sqrt{1 - \dfrac{c^2}{c_2^2}}$, $i = \sqrt{-1}$ is the imaginary unit number and c_1, c_2 stand for the celerity of longitudinal and shear waves defined respectively by

$$c_1 = \sqrt{\frac{\lambda + 2G}{\rho}}\,, \quad c_2 = \sqrt{\frac{G}{\rho}}$$

G is the shear modulus, λ denotes the Lamé's coefficient and ρ is the mass density. The displacement and stress fields are given in terms of two complex potentials ϕ_1 and ϕ_2 as follows (Radok, 1956)

$$u_x = -\frac{1}{G}\,\Re e\Big(\phi_1(z_1) + \frac{1+\beta_2^2}{2}\phi_2(z_2)\Big) \quad [27]$$

$$u_y = \frac{1}{G}\,\Im m\Big(\beta_1\phi_1(z_1) + \frac{1+\beta_2^2}{2\beta_2}\phi_2(z_2)\Big) \quad [28]$$

$$\sigma_{xx} = -2\,\Re e\Big(\frac{2\beta_1^2-\beta_2^2+1}{2}\phi_1'(z_1) + \frac{1+\beta_2^2}{2}\phi_2'(z_1)\Big) \quad [29]$$

$$\sigma_{yy} = (1+\beta_2^2)\,\Re e\Big(\phi_1'(z_1) + \phi_2'(z_2)\Big) \quad [30]$$

$$\sigma_{xy} = 2\,\Im m\Big(\beta_1\phi_1'(z_1) + \frac{(1+\beta_2^2)^2}{4\beta_2}\phi_2'(z_2)\Big) \quad [31]$$

where $\Re e(Z)$ and $\Im m(Z)$ represent respectively the real and the imaginary part of the complex number Z.
Here we focus our attention on the subsonic waves i.e. $c < c_2$. Hence, $\beta_1 > 0$ and $\beta_2 > 0$.

4.2. *Method of solution*

Following Bui (2006), the construction method of the solution is based on the *displacement continuation.* From the condition $u_y = 0$ along the contact interface (real axis where $z_1 = z_2 = z = x + i0$) and Equation [28] one obtains

$$\beta_1\,\phi_1(z) + \frac{(1+\beta_2^2)}{2\beta_2}\,\phi_2(z) = 0 \quad [32]$$

This equation suggests the following definition for the function ϕ_2

$$\phi_2(z_2) := -\frac{2\beta_1\beta_2}{(1+\beta_2^2)}\,\phi_1(z_2) \quad [33]$$

Substitution of [33] into [27-31] results in the following equations on the interface

$$u_x = -\frac{(1-\beta_1\beta_2)}{G}\,\Re e\Big(\phi_1(z)\Big) \quad [34]$$

$$\sigma_{yy} = (1+\beta_2^2-2\beta_1\beta_2)\,\Re e\Big(\phi_1'(z)\Big) \quad [35]$$

$$\sigma_{xx} = \Big((-1+\beta_2^2) + 2\beta_1(\beta_2-\beta_1)\Big)\,\Re e\Big(\phi_1'(z)\Big) \quad [36]$$

$$\sigma_{xy} = \beta_1(1-\beta_2^2)\,\Im m\Big(\phi_1'(z)\Big) \quad [37]$$

Hence, all mechanical fields are determined through the knowledge of the function ϕ_1 and its derivative ϕ_1'.

In the sequel, unless stated otherwise, the notation $\Phi(z) := \phi_1'(z)$ will be used.

We shall search for a complex solution under the form of a Cauchy integral with a distribution $f(t)$

$$\Phi(z) = \frac{1}{2i\pi}\int_a^b \frac{f(t)}{t-z}dt \qquad [38]$$

It is useful to underline that if $f(t)$ is real then the conjugate function of Φ is given by

$$\overline{\Phi}(z) = -\frac{1}{2i\pi}\int_a^b \frac{f(t)}{t-z}dt = -\Phi(z) \qquad [39]$$

Note that [39] may remains valid for some complex functions $f(t)$.

Stick-slip solution

The sliding state may be written as

$$\sigma_{xy} + \tau^*_{xy} = -\mu(\sigma_{yy} + \tau^*_{yy}) \Rightarrow \sigma_{xy} + \mu\sigma_{yy} = -\tau^*_{xy} - \mu\tau^*_{yy} := T^* \qquad [40]$$

By setting $\gamma_1 = \beta_1(1-\beta_2^2) > 0$ and $\gamma_2 = 1+\beta_2^2 - 2\beta_1\beta_2$, Eq.(40) becomes

$$-g\Phi(z^-) + \Phi(z^+) = \frac{2iT^*}{\gamma_1 - i\mu\gamma_2} := f_2(t) \qquad [41]$$

where

$$g = -\frac{\gamma_1 + i\mu\gamma_2}{\gamma_1 - i\mu\gamma_2} \qquad [42]$$

Following Mushkhelishvili (Muskhelishvili, 1953), the appropriate solution of[41] has the form

$$\Phi^{slip}(z) = \frac{f_2}{2i\pi}X(z)\Big\{\int_{SL^+}\frac{1}{X(t^+)}\frac{dt}{t-z} + C_2\Big\} \qquad [43]$$

where C_2 is an arbitrary constant.

Now, equating the slip velocity of stick zone to $V - V^*$, $u_{x,x} = \frac{\partial u_x}{\partial x} = -\frac{\left(V-V^*\right)}{c}$ we obtain the density function $f_1(t)$ by

$$f_1(t) = \Phi(t^+) - \Phi(t^-) = -\frac{2G(V-V^*)}{c(1-\beta_1\beta_2)} \qquad [44]$$

Thus, the expression of complex potential in the stick zone is given by

$$\Phi^{stick}(z) = \frac{f_1}{2i\pi} X(z) \Big\{ \int_{ST+} \frac{dt}{X(t^+)(t-z)} + C_1 \Big\} \quad [45]$$

where C_1 is constant.

The stick-slip potential is obtained by sum the two complex solutions Φ^{stick} and Φ^{slip}. We obtain the following Cauchy integral with discontinuous density

$$\Phi(z) = \frac{1}{2i\pi} X(z) \Big\{ f_1 \int_{ST+} \frac{dt}{X(t^+)(t-z)} + f_2 \int_{SL+} \frac{dt}{X(t^+)(t-z)} + C_0 \Big\} \quad [46]$$

where C_0 is an arbitrary constant. Moreover, since the stress field vanishes at infinity then $C_0 = 0$.

Explicit calculation gives the following analytic expression of the stick-slip potential splitted in its real and imaginary parts

$$\Phi(z \in SL) = \Big\{ -\frac{f_1}{2} \big(1 - |X(z)|\big) + \frac{T^*}{\gamma_1} cos(\pi\alpha)\, |X(z)| \Big\} + \\ i \Big\{ -\frac{f_1}{2}\, \mu\, \frac{\gamma_2}{\gamma_1} \left(|X(z)| - 1\right) + \frac{T^*}{\gamma_1} \big(1 - |X(z)| sin(\pi\alpha)\big) \Big\} \quad [47]$$

and

$$\Phi(z \in ST) = -\frac{f_1}{2} - i \Big\{ -\frac{f_1}{2} \mu \frac{\gamma_2}{\gamma_1} + \frac{f_1}{2\cos(\pi\alpha)} |X(z)| + \frac{T^*}{\gamma_1} \big(1 - |X(z)|\big) \Big\} \quad [48]$$

Complex functions given by (47) and (48) are weakly singular (i.e. square integrable) which avoids the problem of unbounded energy sources or sinks (Freund, 1978) at switching boundaries between stick and slip segments.

Moreover, since Φ must behave as $O(1/z^2)$ at infinity then terms of order $O(1/z)$ must be canceled. After some cumbersome calculus this condition writes

$$f_1(\gamma_1 + i\mu\gamma_2) e^{-i\pi\alpha} - 2T^*(\delta - 1) = 0 \quad [49]$$

where $\delta = b/a$.

Equation (49) is apparently a complex one in the form $F_1(\delta, c) + iF_2(\delta, c) = 0$, where F_1 and F_2 are real, so that it should be splitted into two real equations. However, exact calculus shows that $F_2 \equiv f_1(-\gamma_1\ sin(\alpha\pi) + \mu\ \gamma_2\ cos(\alpha\pi) = 0$

because $tan(\alpha\pi) = 0$. Equation (49) is thus reduced to $F_1(\delta, c) = 0$ which provides the first relation between the wave velocity c and δ

$$\frac{G(V - V^*)}{c(1 - \beta_1\beta_2)}(\gamma_1\ cos(\alpha\pi) + \mu\ \gamma_2\ sin(\alpha\pi)) + T^*(\delta - 1) = 0 \qquad [50]$$

The second relation between c and δ, which enables complete determination of these unknowns will be set further by examining the continuity of the tangential displacement (the normal component $u_y = 0$ along the contact interface). After some cumbersome calculus one gets

$$-\frac{V}{c}\delta + \Big\{\frac{(V - V^*)}{c} - (1 - \beta_1\beta_2)\ cos(\pi\alpha)\frac{T^*}{2G\gamma_1}\Big\}\pi(\frac{1}{2} - \alpha)\frac{(\delta - 1)}{cos(\pi\alpha)} = 0 \qquad [51]$$

In conclusion, the unknowns of this stick-slip problem (c, δ) for given loading conditions (T^*, V, V^*) are completely determined by the set of the two equations (50) and (51). Physical quantities involved in the problem may be scaled as follows

$$a = 1,\ \delta = \frac{b}{a} \to b > 1,\ \frac{T^*}{G} \to T,\ \frac{V}{c_2} \to v,\ \frac{V^*}{c_2} \to v^*,\ \frac{c_1}{c_2} \to \overline{c}_1,\ \frac{c}{c_2} \to \overline{c}$$

We obtain the following final equations

$$-\frac{v}{\overline{c}}b + \Big\{\frac{(v - v^*)}{\overline{c}} - (1 - \beta_1\beta_2)\ cos(\pi\alpha)\frac{T}{2\gamma_1}\Big\}\pi(\frac{1}{2} - \alpha)\frac{(b - 1)}{cos(\pi\alpha)} = 0 \qquad [52]$$

and

$$\frac{(v - v^*)}{\overline{c}(1 - \beta_1\beta_2)}(\gamma_1\ cos(\alpha\pi) + \mu\ \gamma_2\ sin(\alpha\pi)) + T(b - 1) = 0 \qquad [53]$$

5. Results and discussion

The stick-slip problem $(\overline{c}, b)$, for given loading conditions (T, v, v^*) is solved by the set of the two equations (52) and (53). This problem is highly nonlinear in the wave velocity c and the uniqueness of the solution cannot be easily proved. Moreover its numerical solution by Newton's method for example, is inaccurate and fails because of singularities. Let us propose a simpler method to solve it. Instead of finding $(\overline{c}, b)$ for a given loading condition (v, v^*, T) (or (v, v^*, τ^*_{xy})), we search the solution of the following *inverse problem* :

Find (v, T) *for any given pair* $(\overline{c}, b)$ *and given* v^*.

Therefore, one obtains a linear algebric system

$$\Big[\pi(\frac{1}{2} - \alpha)\frac{(b - 1)}{cos(\pi\alpha)} - b\Big]\frac{v}{\overline{c}} - \Big[\pi(\frac{1}{2} - \alpha)(b - 1)\frac{(1 - \beta_1\beta_2)}{2\gamma_1}\Big]T \quad = \quad (\frac{1}{2} - \alpha)\frac{\pi(b - 1)}{cos(\pi\alpha)}\frac{v^*}{\overline{c}} \qquad [54]$$

and

$$\Big[2\frac{(\gamma_1\ cos(\alpha\pi)+\mu\ \gamma_2\ sin(\alpha\pi))}{(1-\beta_1\beta_2)}\Big]\frac{v}{\overline{c}}+[b-1]T \quad = \qquad 2\frac{(\gamma_1\ cos(\alpha\pi)+\mu\ \gamma_2\ sin(\alpha\pi))}{(1-\beta_1\beta_2)}\frac{v^*}{\overline{c}} \qquad [55]$$

The system (54-55) is mathematically well-posed in the sense that we have two equations for two unknowns (v, T). Note that once T is known the stresses τ^*_{yy} and τ^*_{xy} are easily obtained.

The solution is straightforward and depends on the determinant of the linear system

$$D(b,\overline{c}) = \frac{(b-1)}{\gamma_1\,\overline{c}}H(b,\overline{c}) \qquad [56]$$

where

$$H(b,\overline{c}) = -\gamma_1 b+\gamma_1\pi(\frac{1}{2}-\alpha)\frac{(b-1)}{cos(\pi\alpha)}+\pi(\frac{1}{2}-\alpha)\big(\gamma_1\ cos(\alpha\pi)+\mu\ \gamma_2\ sin(\alpha\pi)\big) \qquad [57]$$

It is easy to establish that $H(b,\overline{c}) > 0\ \forall b > 1$ and $\forall\overline{c} \in]-1,0[$. Thus the determinant $D(b,\overline{c})$ is strictly negative $\forall b > 1$ and $\forall\overline{c} \in]-1,0[$.

Notice that if $v^* = 0$ then Equations [54-55] are reduced to a homogenous algebric system. The later have non trivial solutions if and only if the determinant $D(b,\overline{c})$ vanishes. However, we have checked that $D(b,\overline{c}) < 0$ in the domain $-1 < \overline{c} < 0$ and $b > 1$. Therefore, only the trivial solution $v = 0$ (statics) and $T = 0$ (the loading at the Coulomb limit) exists.
We conclude that no solution exists in the case $v* = 0$.

For the case $v^* \neq 0$ the solution is straightforward for v and T

$$v = \frac{v^*}{1+A(b,\overline{c})} \qquad [58]$$

where

$$A(b,\overline{c}) = \frac{4b\gamma_1 cos(\alpha\pi)}{\pi(-1+2\alpha)\Big(\mu\gamma_2 sin(2\pi\alpha)+\gamma_1(-1+2b+cos(2\pi\alpha))\Big)}$$

and

$$T = -\frac{4b\gamma_1(\gamma_1+\gamma_1 cos(2\alpha\pi)+\mu\gamma_2 sin(2\alpha\pi))}{(b-1)(-1+\beta_1\beta_2)}\times \frac{v^*}{\overline{c}\ \Big(4b\gamma_1 cos(\alpha\pi)+\pi(-1+2\alpha)\big(\mu\gamma_2 sin(\pi\alpha)+\gamma_1(-1+2b+cos(\pi\alpha))\big)\Big)}$$

Note that the solution pair (v, T) exists and is unique for any wave velocity such that $-1 < \bar{c} < 0$, any $b > 1$ and for a given v^*.

As can be noted in Equations [34] and [35], the gradient $u_{x,x}$ and the stress σ_{yy} are of opposite signs. It is easy to observe that the slip velocity has a negative singularity at $x = b$. Therefore, the normal stress shows a positive singularity as seen in Figure 14 (where Σ_{yy}/G is plotted) which implies a separation near the singular transition.

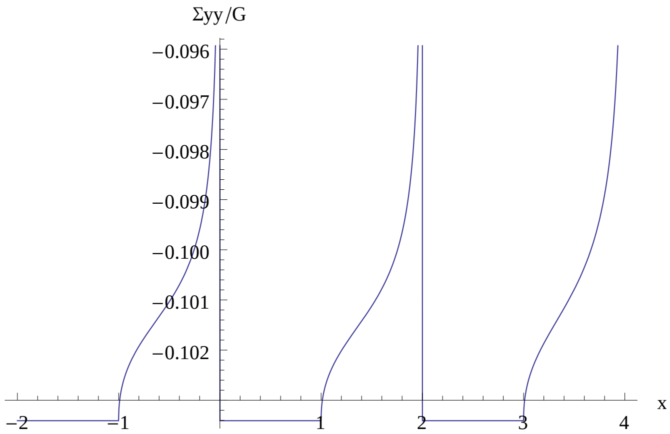

Figure 14. *The normalized stress Σ_{yy}/G (with $E = 2.1\,10^{11} Pa$, $\rho = 7850 Kg/m^3$, $\nu = 0.3$, $\mu = 0.5$, $\mu^* = 0.4$, $v^* = 0.001$ and $\bar{c} = -0.7$) over stick-slip segments exhibits a positive singularity at the transition points $SL \to ST$*

6. Conclusion

In this paper we attempt to give insight into the mechanism of the stick-slip waves induced by the flutter instability of the dry friction. A discrete system of friction oscillator and a bounded solid with a constant coefficient of friction have been revisited. New results concerning the possibility of existence of stick-slip waves along the interface between an elastic half-space and a rigid one have been presented. Further research are needed since a global stability analysis of these travelling waves and the general question of transition to a limit cyclic response still remain open problems.

Acknowledgements

The author would like to thank Professor H.D. Bui for collaboration and fulfill discussions.

7. References

Adams G., « Self-excited oscillations of two elastic half-spaces sliding with a constant coefficient of friction », *J. Appl. Mech.*, vol. 62, p. 867-872, 1995.

Adams G., « Steady sliding of two elastic half-spaces with friction reduction due to interface stick-slip », *J. Appl. Mech.*, vol. 65, p. 470-475, 1998.

Adams G., « An intersonic slip pulse at a frictional interface between dissimilar materials », *J. Appl. Mech.*, vol. 68, p. 81-86, 2001.

Andrews D. J., Ben-Zion Y., « Wrinkle-like slip pulse on a fault between different materials », *PAMM . Proc. Appl. Math. Mech.*, vol. 5, p. 139140, 2005.

Ben-Zion Y., Andrews D. J., « Properties and implications of dynamic rupture along a material interface », *Bull. Seismol. Soc. Amer.*, vol. 88, p. 1085-1094, 1998.

Carpenter N. J., Taylor R. L., Katona M. G., « Lagrange constraints for transient finite element surface contact », *Int. J. Numer. Methods Engrg.*, vol. 32, p. 103128, 1991.

Cochard A., Madariaga R., « Dynamic faulting under rate-dependent friction », *Pure Appl. Geophys.*, vol. 142, p. 419-445, 1994.

Désoyer T., Martins J. A. C., « Surface instabilities in a Mooney-Rivlin body with frictional boundary conditions », *Int. J. Adhesion and adhesives*, vol. 18, p. 413-419, 1998.

Feeny B., Guran A., Hinrichs N., Popp K., « A historical review on dry friction and stick-slip phenomena », *Appl. Mech. Rev.*, vol. 105, p. 321-340, 1998.

Martins J. A. C., Guimaraes J., Faria L. O., « Dynamic surface solutions in linear elasticity and viscoelasticity with frictional boundary conditions », *J. Vib. Acoust*, vol. 117, p. 445-451, 1995a.

Martins J. A. C., Guimaraes J., Faria L. O., « Dynamic surface solutions in linear elasticity and viscoelasticity with frictional boundary conditions », *J. Vib. Acoust*, vol. 117, p. 445-451, 1995b.

Moirot F., Etude de la stabilité d'un équilibre en présence de frottement de Coulomb : application au crissement des freins à disques, Thèse de doctorat, Ecole Polytechnique, 1998.

Moirot F., Nguyen Q. S., « Brake squeal : A problem of flutter instability of the steady sliding solution ? », *Arch. Mech.*, vol. 52, p. 645-662, 2000.

Moirot F., Nguyen Q. S., Oueslati A., « An example of stick-slip and stick-slip-separation waves », *Eur. J. Mech. A/Solids*, vol. 22, p. 107-118, 2002.

Mukesh L. D., Jonnalagadda K. K., Kandikatla R. K., Kesava R. K., « Silo music : Sound emission during the flow of granular materials through tubes », *Powder Technology*, vol. 167, p. 55-71, 2006.

Murty G. S., « Wave propagation at an unbonded interface between two elastic half-spaces », *J. Acoustic soc. America*, vol. 58, p. 1094-1095, 1975.

Muskhelishvili N. I., *Some basic problems of the mathematical theory of elasticity*, Noordhoff, Groningen, 1953.

Nakai M., Yokoi M., « Band brake squeal », *J. Vib. Acoustics*, vol. 118, p. 190-197, 1996.

Nguyen Q. S., Oueslati A., Steindl A., Teufel A., Troger H., « Travelling interface waves in a brake-like system under unilateral contact and Coulomb friction », *C.R. Mecanique*, vol. 336, p. 203-209, 2008.

Oancea V., Laursen T. A., « Stability analysis of state-dependent dynamic frictional sliding », *J. Non-linear Mechanics*, vol. 32, p. 837-853, 1997.

Oberle H. J., Grimm W., Berger E., A program for the numerical solution of optimal control problems, Rapport de recherche n° 515 der DFVLR, Universität Hamburg, 1989.

Oestreich M., Hinrichs N., Popp K., « Bifurcation and stability analysis for a nonsmooth friction oscillator », *Arch. Appl. Mech.*, vol. 66, p. 301-314, 1996.

Oueslati A., Ondes élastiques de surface et fissures dinterface sous contact unilatéral et frottement de Coulomb, Thèse de doctorat, Ecole Polytechnique, 2004.

Oueslati A., Nguyen Q. S., Baillet L., « Stick-slip-separation waves in unilateral and frictional contact », *C. R. Mecanique*, vol. 66, p. 133-140, 2003.

Ranjith K., Rice J. R., « Slip dynamics at an interface between dissimilar materials », *J. Mech. Phys. Solids*, vol. 49, p. 341-361, 2001.

Renardy M., « Ill-posedness at the boundary for elastic solids sliding under Coulomb friction », *J. Elasticity*, vol. 27, p. 281287, 1992.

Schallamach A., « How does rubber slide ? », *Wear*, vol. 17, p. 301-312, 1971.

Simoes F. M., Martins J. A. C., « Instability and ill-posedness in some friction problems », *J. Engn. Science*, vol. 36, p. 1265-1293, 1998.

Teufel A., Steindl A., Troger H., « On nonsmooth bifurcations in a simple friction oscillator », *J. Geophys. Res.*, vol. 102, p. 139-140, 2005.

Weertman J., « Dislocations moving uniformly on the interface between isotropic media of different elastic properties », *J. Mech. Phys. Solids*, vol. 11, p. 197-204, 1963.

Zharii O., « Frictional contact between the surface wave and a rigid strip », *J. Appl. Mech*, vol. 63, p. 15-20, 1996.