Higher-order accurate compact difference solutions for vibration problems of one dimensional continuous systems

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ABSTRACT. *In this paper, we present a fourth-order accurate and a seventh-order accurate, one-step compact difference methods. These methods can be used to solve initial or boundary-value problems which can be modeled by a first-order linear system of differential equations. It is then shown in detail how these methods can be used to solve vibration problems of one-dimensional continuous systems. Natural frequencies of a cantilever beam in transverse vibrations are computed and the results are compared to analytical ones to prove the high accuracy and efficiency of both methods. A comparison was also made to a finite element solution and the results have shown that both compact-difference methods yield more accurate values even with a reduced number of intervals.*

RÉSUMÉ. *Dans cet article nous présentons deux méthodes numériques basées sur les différences compactes, une précise au quatrième ordre et une autre au septième ordre. Ces deux méthodes peuvent servir à la résolution de problèmes à valeur initiale ou à valeurs limites, modélisables par un système d'équations différentielles du premier ordre. Nous montrons en détail comment ces méthodes sont appliquées au calcul des fréquences propres de systèmes unidirectionnels continus. Les résultats obtenus sont confrontés à des valeurs analytiques et la haute précision des deux méthodes est mise en évidence. Une deuxième comparaison avec des valeurs obtenues par la méthode des éléments finis a montré que les méthodes proposées sont plus précises.*

KEYWORDS: *fourth-order and seventh-order accuracy, compact differences, boundary-value problems, eigenvalues, vibrations of continuous systems, tapered beams, shear deformation, rotary inertia.*

MOTS-CLÉS : *différences compactes, précision au quatrième et septième ordre, valeurs propres, poutre effilée, inertie rotationnelle, déformation due au cisaillement.*

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1. Introduction

The study of the vibrations of beams and other continuous one-dimensional mechanical systems is of major importance in aeronautical engineering and other fields. Missiles, aircraft wings, fuselages, propeller blades, rotor blades… are all examples of so called aerospace beams. The complexity of the structure and geometry of these beams makes it often impossible to obtain analytical solutions. The improvement of existing numerical methods, and the devising of new ones, remains therefore the concern and objective of many research efforts.

There exist already different methods that can deal with the vibration problems of continuous one-dimensional systems. The finite element method, the Rayleigh-Ritz method and to a lesser extent the finite difference method, are among the most commonly used methods. Fourier transforms have also been used (Karlson, 1985).The differential quadrature method (Bert *et al.*, 1996), the boundary characteristic orthogonal polynomials (Liew *et al.*, 1995) and the pseudo spectral method (Lee *et al.*, 2004) have been used in recent years. The Adomian decomposition method has also been applied to beam vibration problems (Hsin *et al.*, 2008). A pure boundary element method has been applied to study the torsional vibrations of composite bars (Sapountzakis, 2005).

The compact difference methods, presented here, have both the accuracy of integral methods and the relative simplicity of the finite difference method. They are called compact because the approximation of derivatives is made over only two consecutive nodes. Nevertheless, the approximation error is $O(h^4)$ for the fourth-order accurate method and $O(h^7)$ for the seventh-order one, h being the interval separating the two consecutive nodes. To obtain this kind of accuracy when approximating the first derivative by classical finite difference approach one would need to use five consecutive equally-spaced nodes for the fourth-order accurate method and eight for the seventh-order accurate one. A higher number may be required if the nodes are not equally spaced (Rubin *et al.*, 1976).

One other characteristic of the method introduced here is that it carries out a global search, yielding a large number of eigenvalues and their corresponding eigenvectors in one run of the computer code. This is an advantage very few other methods present.

Fourth-order accurate compact-difference methods have been used by (Malik, 1988) and (Yahiaoui, 1993) for the linear stability studies of boundary layers. In view of their higher accuracy and relative simplicity, efforts should be made to extend the compact-difference approach to all engineering science problems where it can be applied.

In this paper, we introduce two compact-difference methods: one is fourth-order and the second is seventh-order accurate. We then show in detail how these methods can be applied to all types of vibration problems of one-dimensional continuous systems. A comparison with analytical solutions and to values obtained by a finite element solution using ABAQUS has proven the very high accuracy associated with the proposed methods.

2. General formulation of a seventh-order accurate method

This method is based on the one-step integration formula (Abramowitz and Stegun, 1988, p. 897) which we apply for a vector of functions:

$$\vec{Z}_n = \vec{Z}_{n-1} + (h_n/2)\left(\vec{Z}'_n + \vec{Z}'_{n-1}\right) - (h_n^2/10)\left(\vec{Z}''_n - \vec{Z}''_{n-1}\right) \\ +(h_n^3/120)\left(\vec{Z}'''_n + \vec{Z}'''_{n-1}\right) + O(h_n^7) \qquad [1]$$

The "primes" indicate derivatives of with respect to the independent variable and the indices refer to node numbers.

The mathematical problem we consider here is a first-order system of differential equations of the form:

$$\vec{Z}' = A\vec{Z}$$

where A is a square matrix of order m. It is in general a function of the independent variable. The domain is an interval $[a, b]$ which is divided into N small, generally unequal, intervals such that:

$$a = x_0 < x_1 < \cdots < x_n < \cdots < x_N = b \quad ; \quad h_n = x_n - x_{n-1}$$

We then write the second and third derivatives of the vector $\vec{Z}$ as matrix transformations of the form:

$$\vec{Z}'' = B\vec{Z}$$

$$\vec{Z}''' = C\vec{Z}$$

with:

$$B = A' + A^2$$

$$C = B' + BA \\ = A'' + A'A + AA' + (A' + A^2)A \\ = A'' + 2A'A + AA' + A^3$$

Equation [1] becomes:

$$\left(I + \frac{h_n}{2}A_{n-1} + \frac{h_n^2}{10}B_{n-1} + \frac{h_n^3}{120}C_{n-1}\right)\vec{Z}_{n-1} - \left(I - \frac{h_n}{2}A_n + \frac{h_n^2}{10}B_n - \frac{h_n^3}{120}C_n\right)\vec{Z}_n = 0$$

where I is the identity matrix of the same order as matrix A.

Let:

$$Q_{n-1} = I + \frac{h_n}{2}A_{n-1} + \frac{h_n^2}{10}B_{n-1} + \frac{h_n^3}{120}C_{n-1}$$

$$R_n = -I + \frac{h_n}{2}A_n - \frac{h_n^2}{10}B_n + \frac{h_n^3}{120}C_n$$

to obtain the compact form:

$$Q_{n-1}\vec{Z}_{n-1} + R_n\vec{Z}_n = 0 \quad , \quad 1 \leq n \leq N \qquad [2]$$

Equation [2] is the basis for the solution to different problems. It certainly can be used as a time-marching method for initial-value problems just like the methods of Runge-Kutta, Adams, etc. But marching methods are generally not convenient for boundary-value problems unless they are combined with other methods such as the shooting method for example. The advantage of the method developed here, in addition to its higher accuracy, is that it can directly handle boundary conditions at two ends, and thereby allows for the solution of some types of boundary-value problems such as the vibrations of continuous one-dimensional mechanical systems.

3. Application to vibrations of continuous one-dimensional systems

Among boundary-value problems which can be treated with this compact-difference method are eigenvalue problems dealing with the vibration (axial, flexural, torsional, coupled, etc.) of one-dimensional continuous mechanical systems (strings, beams, rods, transmission shafts, etc.).

For such a problem, Equation [2] along with appropriate boundary conditions can be cast in the generalized eigenvalue-problem form:

$$G\vec{Z} = \lambda F\vec{Z} \qquad [3]$$

This more standard form makes it possible to take advantage of readily available eigenvalue solvers.

Since matrices Q_{n-1} and R_n are linear combinations of matrices *I*, *A*, *B*, and C, Equation [3] can be obtained by simply writing matrices A, B, and C in the form:

$$A = A_1 - \lambda A_2$$

$$B = B_1 - \lambda B_2$$

$$C = C_1 - \lambda C_2$$

As we will see later, the decomposition of the system dynamics matrix *A* is possible for all vibration problems of one-dimensional mechanical systems that do not include viscous damping. We will show that this is true in the case of the flexural vibrations of beams with and without rotary inertia and shear deformation effects. Such a case is representative of all vibration problems of one-dimensional continuous systems.

We now try to write matrices B and C in the desired form:

$$\begin{aligned} B &= A' + A^2 = (A_1 - \lambda A_2)' + (A_1 - \lambda A_2)(A_1 - \lambda A_2) \\ &= A_1' + A_1^2 - \lambda(A_2' + A_1 A_2 + A_2 A_1) + \lambda^2 A_2^2 \end{aligned}$$

We safely assume that:

$$A_2^2 = 0 \qquad [4]$$

We will see that this is true in the case of flexural vibration of a beam, where matrix A_2 is highly sparse. In fact, this matrix has only one nonzero entry for most cases of one-dimensional continuous systems. This entry comes from the acceleration term in the equations of motion. The lateral vibration of a beam including the effects of shear deformation and rotary inertia yields two nonzero entries. But even in this case, the conditions on A_2 still holds. It follows that:

$$B_1 = A_1' + A_1^2$$

$$B_2 = A_2' + A_1 A_2 + A_2 A_1$$

As for matrix C, we have:

$$\begin{aligned} C &= B' + BA = B_1' - \lambda B_2' + (B_1 - \lambda B_2)(A_1 - \lambda A_2) \\ &= B_1' + B_1 A_1 - \lambda(B_2' + B_1 A_2 + B_2 A_1) + \lambda^2 B_2 A_2 \end{aligned}$$

We also assume that:

$$B_2 A_2 = 0 \qquad [5]$$

This will also turn out to be the case for most, but not all vibration problems of one-dimensional continuous systems. As a result, we get:

$$C_1 = A_1'' + 2A_1A_1' + A_1'A_1 + A_1^3$$

$$C_2 = A_2'' + 2A_1'A_2 + 2A_2'A_1 + A_1A_2' + A_2A_1' + A_1^2A_2 + A_1A_2A_1 + A_2A_1^2$$

We now have a more conveniently linear eigenvalue problem. But even if conditions [4, 5] are not satisfied, the solution of the non linear eigenvalue problem is still possible as shown in the work (Bridges *et al.*, 1984), who presents different methods of solving problems where the eigenvalue appears in a non linear form.

Matrices Q and R of Equation [2] can now be written:

$$Q_{n-1} = (Q_1)_{n-1} - \lambda(Q_2)_{n-1}$$

$$R_n = (R_1)_n - \lambda(R_2)_n$$

where:

$$(Q_1)_{n-1} = I + \frac{h_n}{2}(A_1)_{n-1} + \frac{h_n^2}{10}(B_1)_{n-1} + \frac{h_n^3}{120}(C_1)_{n-1}$$

$$(Q_2)_{n-1} = \frac{h_n}{2}(A_2)_{n-1} + \frac{h_n^2}{10}(B_2)_{n-1} + \frac{h_n^3}{120}(C_2)_{n-1}$$

$$(R_1)_n = -I + \frac{h_n}{2}(A_1)_n - \frac{h_n^2}{10}(B_1)_n + \frac{h_n^3}{120}(C_1)_n$$

$$(R_2)_n = \frac{h_n}{2}(A_2)_n - \frac{h_n^2}{10}(B_2)_n + \frac{h_n^3}{120}(C_2)_n$$

We now have a generalized eigenvalue problem in the form of Equation [3], *i.e.*:

$$G\vec{Z} = \lambda F\vec{Z}$$

This represents a system of mN equations in $m(N+1)$ unknowns, *m* being the order of matrix A and *N* the number of intervals. Matrices G and F, and the global vector $\vec{Z}$ are given by:

$$G = \begin{pmatrix} (Q_1)_0 & (R_1)_1 & 0 & \cdots & 0 & 0 \\ 0 & (Q_1)_1 & (R_1)_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & (Q_1)_{N-2} & (R_1)_{N-1} & 0 \\ 0 & 0 & 0 & 0 & (Q_1)_{N-1} & (R_1)_N \end{pmatrix}$$

$$F = \begin{pmatrix} (Q_2)_0 & (R_2)_1 & 0 & \cdots & 0 & 0 \\ 0 & (Q_2)_1 & (R_2)_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & (Q_2)_{N-2} & (R_2)_{N-1} & 0 \\ 0 & 0 & 0 & 0 & (Q_2)_{N-1} & (R_2)_N \end{pmatrix}$$

$$\vec{Z} = \begin{pmatrix} \vec{Z}_0 \\ \vec{Z}_1 \\ \vdots \\ \vec{Z}_N \end{pmatrix}$$

In order to complete the formulation, *m* boundary conditions are needed. These are specific to the problem to be solved. Each boundary condition provides an additional equation and therefore increases the number of rows of matrices G and *F* by one. Once all *m* boundary conditions are applied, we get a square system of equations of order $N_s = m(N+1)$.

3.1. *A typical problem: lateral vibrations of a beam*

The free transverse vibrations of a homogeneous beam are governed by the equation:

$$\frac{\partial^2}{\partial x^2}\left[I_{zz}(x)\frac{\partial^2 v}{\partial x^2}\right] + \frac{\rho}{E}S(x)\frac{\partial^2 v}{\partial t^2} = 0 \qquad [6]$$

where ρ is the material density and E is Young's modulus. For more generality, we have allowed for variable quadratic moment I_{zz} and cross-sectional area S.

Assuming harmonic oscillations, we write:

$$v(x,t) = f(x)\, sin(\omega t + \phi)$$

As a result, Equation [6] becomes:

$$\frac{d^2}{dx^2}\left[I_{zz}(x)\frac{d^2 f(x)}{dx^2}\right] - \omega^2\frac{\rho}{E}S(x)f(x) = 0 \qquad [7]$$

We then introduce the following non dimensional quantities:

$$\bar{x} = x/L$$

$$\bar{f} = f/L$$

$$\bar{S} = S/S_0$$

$$\bar{I}_{zz} = I_{zz}/I_0$$

Area S_0 and quadratic moment I_0 are reference quantities that will be defined for each specific problem. For a tapered cantilever beam for instance, we chose the values at the root section as references. Equation [7] becomes:

$$\frac{1}{L^2}\frac{d^2}{d\bar{x}^2}\left(I_0\bar{I}_{zz}\frac{1}{L^2}\frac{d^2(L\bar{f})}{d\bar{x}^2}\right) = \omega^2\frac{\rho}{E}S_0\bar{S}L\bar{f}$$

Or:

$$\frac{d^2}{d\bar{x}^2}\left(\bar{I}_{zz}\frac{d^2\bar{f}}{d\bar{x}^2}\right) = \frac{\rho S_0 L^4}{E I_0}\omega^2\bar{S}\bar{f} \qquad [8]$$

We let:

$$\lambda = \frac{\rho S_0 L^4}{E I_0}\omega^2$$

so the frequencies are given by:

$$\omega = \frac{\sqrt{\lambda}}{L^2}\sqrt{\frac{E I_0}{\rho S_0}}$$

and equation [8] becomes:

$$\left(\bar{I}_{zz}\bar{f}''\right)'' = \lambda\bar{S}\bar{f}$$

which we rewrite in the form:

$$\bar{f}^{(4)} = \lambda\frac{\bar{S}}{\bar{I}_{zz}}\bar{f}_n - \frac{\bar{I}''_{zz}}{\bar{I}_{zz}}\bar{f}''_n - 2\frac{\bar{I}'_{zz}}{\bar{I}_{zz}}\bar{f}'''_n$$

We then let:

$$\vec{Z}_n = \begin{pmatrix} \bar{f}_n \\ \bar{f}_n' \\ \bar{f}_n'' \\ \bar{f}_n''' \end{pmatrix}$$

It follows that:

$$\vec{Z}_n' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda\dfrac{\bar{S}}{\bar{I}_{zz}} & 0 & -\dfrac{\bar{I}_{zz}''}{\bar{I}_{zz}} & -2\dfrac{\bar{I}_{zz}'}{\bar{I}_{zz}} \end{bmatrix} \vec{Z}_n$$

Matrices A_1 and A_2 and their first derivatives are therefore given by:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\dfrac{\bar{I}_{zz}''}{\bar{I}_{zz}} & -2\dfrac{\bar{I}_{zz}'}{\bar{I}_{zz}} \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\dfrac{\bar{S}}{\bar{I}_{zz}} & 0 & 0 & 0 \end{bmatrix}$$

$$A_1' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{\bar{I}_{zz}'\bar{I}_{zz}'' - \bar{I}_{zz}\bar{I}_{zz}'''}{\bar{I}_{zz}^2} & 2\dfrac{(\bar{I}_{zz}')^2 - \bar{I}_{zz}\bar{I}_{zz}''}{\bar{I}_{zz}^2} \end{bmatrix}$$

$$A_2' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \dfrac{\bar{S}\bar{I}_{zz}' - \bar{S}'\bar{I}_{zz}}{\bar{I}_{zz}^2} & 0 & 0 & 0 \end{bmatrix}$$

and the nonzero entries of matrices $A_1''$and $A_2''$are:

$$A_1''(4,3) = \frac{\left[(\bar{I}_{zz}'')^2 - \bar{I}_{zz}\bar{I}_{zz}^{(4)}\right]\bar{I}_{zz} - 2(\bar{I}_{zz}'\bar{I}_{zz}'' - \bar{I}_{zz}\bar{I}_{zz}''')\bar{I}_{zz}'}{\bar{I}_{zz}^3}$$

$$A_1''(4,4) = 2\,\frac{(\bar{I}_{zz}'\bar{I}_{zz}'' - \bar{I}_{zz}\bar{I}_{zz}''')\bar{I}_{zz} - 2[(\bar{I}_{zz}')^2 - \bar{I}_{zz}\bar{I}_{zz}'']\bar{I}_{zz}'}{\bar{I}_{zz}^3}$$

$$A_2''(4,1) = \frac{[\bar{S}\bar{I}_{zz}'' - \bar{S}''\bar{I}_{zz}]\bar{I}_{zz} - 2(\bar{S}\bar{I}_{zz}' - \bar{S}'\bar{I}_{zz})\bar{I}_{zz}'}{\bar{I}_{zz}^3}$$

It can easily be checked that the previously set conditions on matrix A_2 [4] and on its product with matrix B_2 [5] hold. This gives the conveniently linear eigenvalue problem we have hoped for.

We consider in particular the case of a cantilever beam. Our choice here is arbitrary since we could have chosen a *free-free*, a *clamped-simply supported* or any other combination of classical boundary conditions. Non classical boundary conditions such as lumped masses somewhere along the beam span or at beam ends can easily be handled by this method.

To be more specific, we consider the example of a cantilever beam with a lumped mass at its free end. The beam is tapered in width and height (Figure 1) and has a rectangular cross-section whose width and height are given by:

$$w(x) = w_0[1 - (1 - \sigma_w)\,x/L]$$

$$h(x) = h_0[1 - (1 - \sigma_h)\,x/L]$$

where σ_w and σ_h are the taper ratios for the width and height, respectively.

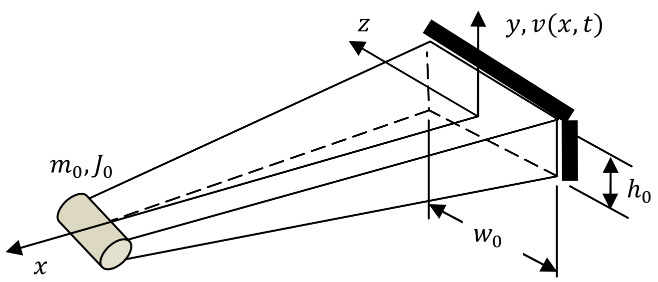

Figure 1. *A tapered cantilever beam with a lumped mass at its free end*

We let:

$$\overline{w}(\bar{x}) = \frac{w(\bar{x})}{w_0} = 1 - (1 - \sigma_w)\bar{x}$$

$$\bar{h}(\bar{x}) = \frac{h(\bar{x})}{h_0} = 1 - (1 - \sigma_h)\bar{x}$$

and define the reference area and quadratic moment to be:

$$S_0 = w_0 h_0$$

$$I_0 = \frac{w_0 h_0^3}{12}$$

Therefore:

$$\bar{S}(\bar{x}) = \overline{w}(\bar{x})\bar{h}(\bar{x})$$

$$\bar{I}(\bar{x}) = \overline{w}(\bar{x})\bar{h}^3(\bar{x})$$

The boundary conditions on $v(x,t)$ at the clamped end of the beam are:

$$v(0,t) = 0$$

$$\frac{\partial v}{\partial x}(0,t) = 0$$

The corresponding boundary conditions on $\bar{f}(\bar{x})$ are:

$$\bar{f}(0) = 0$$

$$\bar{f}'(0) = 0$$

These imply that the only nonzero entries in the first two rows of matrices G and F are:

$$G(1,1) = 1$$

$$G(2,2) = 1$$

The boundary conditions corresponding to a lumped mass m_0 at the free end are obtained by summing forces in the y-direction and summing moments about the center of gravity of the lumped mass:

$$-V_y(L,t) = m_0 \frac{\partial^2 v(L,t)}{\partial t^2} \quad [9]$$

$$-M_z(L,t) + V_y(L,t)d = J_0 \frac{\partial^2 \theta}{\partial t^2}(L,t) \quad [10]$$

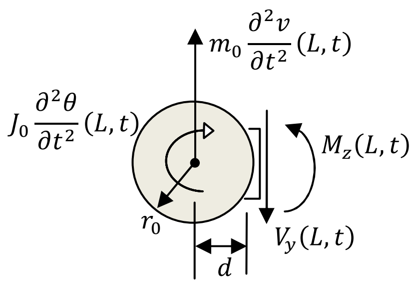

Figure 2. *Fee-body diagram of a lumped mass at the beam free end*

where J_0 is the moment of inertia of the lumped mass about its center of gravity and d is the distance from that point to the beam tip (Figure 2).

Using the fact that the sheer force and bending moment are given by:

$$M_z = EI_{zz} \frac{\partial^2 v}{\partial x^2}$$

$$V_y = -\frac{\partial M_z}{\partial x} = -E\left(\frac{\partial I_{zz}}{\partial x}\frac{\partial^2 v}{\partial x^2} + I_{zz}\frac{\partial^3 v}{\partial x^3}\right)$$

and expressing in terms of the previously defined non dimensional quantities, Equations [9] and [10] can respectively be written:

$$\bar{I}'_{zz}(1)\bar{f}''(1) + \bar{I}_{zz}(1)\bar{f}'''(1) = -\lambda \frac{m_0}{\rho S_0 L}\bar{f}(1)$$

$$\left(\bar{I}_{zz}(1) + \frac{d}{L}\bar{I}'_{zz}(1)\right)\bar{f}''(1) + \frac{d}{L}\bar{I}_{zz}\bar{f}'''(1) = \lambda \frac{J_0}{\rho S_0 L^3}\bar{f}'(1)$$

It follows that the only nonzero entries in the last two rows of matrices G and F are:

$$G(N_s-1,N_s-1)=\bar{I}'_{zz}(1)$$

$$G(N_s-1,N_s)=\bar{I}_{zz}(1)$$

$$F(N_s-1,N_s-3)=-\frac{m_0}{\rho S_0 L}$$

$$G(N_s,N_s-1)=\bar{I}_{zz}+\frac{d}{L}\bar{I}'_{zz}(1)$$

$$G(N_s,N_s)=\frac{d}{L}\bar{I}_{zz}(1)$$

$$F(N_s,N_s-2)=\frac{J_0}{\rho S_0 L^3}$$

N_s being the order of the system.

In order to check the accuracy our method, we consider the case of a cantilever beam of constant cross-section ($\sigma_h=\sigma_w=1$) and we set the mass and moment of inertia of the lumped mass to zeros. Our choice of this case is motivated by the fact that analytical values for ω exist and are given by:

$$\omega=\left(\frac{\beta}{L}\right)^2\sqrt{\frac{EI_{zz}}{\rho S}}$$

Values of the frequency parameter β are solutions to the characteristic equation:

$$\cosh\beta\cos\beta+1=0 \qquad [11]$$

and can be obtained by different method of solving nonlinear algebraic equations.

A *FORTRAN* code was written to implement the present sixth-order accurate compact difference scheme. Eigenvalues and eigenvectors were obtained using subroutine "*rg*" from the IMSL FORTRAN library.

The results obtained (Table 1) confirm the very low error associated with the present compact-difference method. The maximum relative error for the first ten frequencies is highly minimal (it is about 0.005% for N=25 at the tenth mode). With 100 intervals, the results are practically identical to the analytical values.

Table 1. *Frequencies of the first ten vibration modes of a prismatic cantilever beam (w=30 cm; h =5cm; L=2m; ρ=2800 Kg/m^3; E=72GPa)*

$\omega(rad/s)$			
mode	Seventh-order accurate compact difference method N=25	Seventh-order accurate compact difference method N=100	Analytical solution
1	64.3364946	64.3364946	64.3364917
2	403.1899295	403.1899291	403.1899111
3	1128.9434978	1128.9434764	1128.9434259
4	2212.2789523	2212.2786370	2212.2785380
5	3657.0574446	3657.0551019	3657.0549379
6	5463.0186748	5463.0070805	5463.0068334
7	7630.1942450	7630.1504495	7630.1500975
8	10158.6207557	10158.4843433	10158.4838548
9	13048.3764625	13048.0088283	13048.0081514
10	16299.6090863	16298.7239413	16298.7229849

In Table 2 we show the effect of a lumped mass at the free end of the cantilever beam on its first five frequencies. The lumped mass was taken as a solid circular cylinder whose length (l_c) is equal to the beam width at the free end and whose mass is expressed in percentage of that of the beam:

$$m_0 = K_m m_b$$

Its radius and moment of inertia are given by:

$$r_0 = \sqrt{m_0/(\pi\rho_c l_c)}$$

$$J_0 = m_0 r_0^2/2$$

The mass density of the cylinder (ρ_c) was taken to be the same as that of the beam and the parameter K_m was varied from 5 to 100%.

The results show that, as one might intuitively expect, a lumped mass at the free end lowers all first five frequencies of a cantilever beam. The relative change in the first frequency varies from 2.4% when m_0 equals 5% of the beam mass to 29.6% when the lumped mass is equal to the beam mass.

Table 2. *Effect of a lumped mass at the free end on the first five frequencies (m_0 is given in % of beam mass; N=25)*

$m_0(\%)$	$\omega_1(\mathrm{rad/s})$	$\omega_2(\mathrm{rad/s})$	$\omega_3(\mathrm{rad/s})$	$\omega_4(\mathrm{rad/s})$	$\omega_5(\mathrm{rad/s})$
0	64.3365	403.1899	1128.9435	2212.2790	3657.0574
5	62.7836	393.8370	1103.6321	2164.2955	3580.1669
10	61.3341	385.7498	1083.1201	2127.7059	3524.6023
25	57.5060	366.9868	1039.9088	2056.0410	3420.2623
50	52.4318	346.7485	998.3105	1988.6940	3314.5495
100	45.2951	324.3688	952.1189	1894.43	3107.6979

We now apply the method to see the effect of taper on the natural frequencies of the same cantilever beam. Figure 3 shows the separate effects of width taper and height taper on the frequencies of the first three modes.

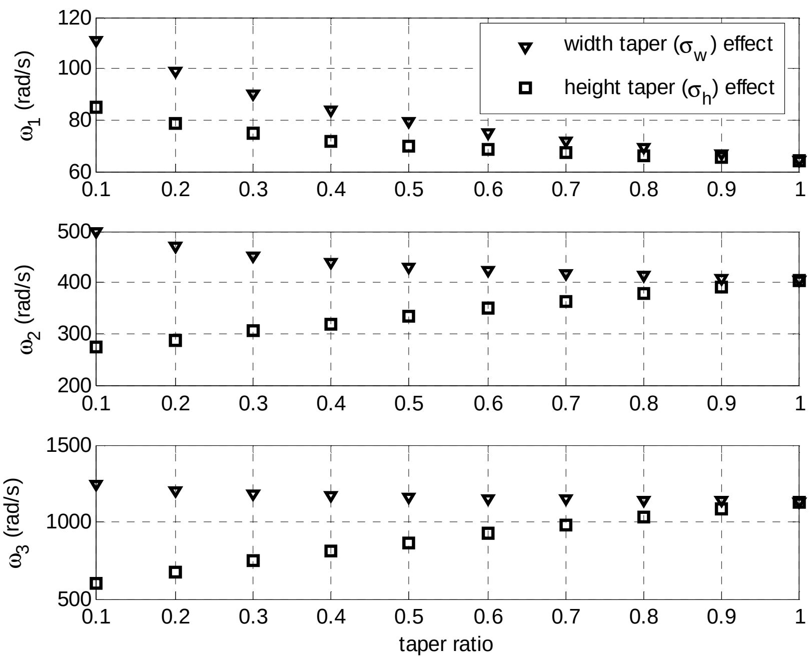

Figure 3. *Effect of width taper (σ_w) and height taper (σ_h) on the first three vibration frequencies*

It is seen that tapering the width by a factor σ_w causes a much greater increase in the first frequency than tapering the height by an equal factor. We also note on the same figure that the effect of tapering the height on the higher modes is reversed. This is confirmed by Table 3 where it can be observed that tapering the width increases all ten frequencies while tapering the height increases the first frequency but lowers the other nine.

Table 3. *Effect of 50% width and height taper on the first ten frequencies of a cantilever beam (N=100)*

Mode	No taper	$\sigma_w = 0.5$	$\sigma_h = 0.5$
1	64.3364946	78.9595345	69.9681013
2	403.1899291	430.3583484	335.1715714
3	1128.9434764	1156.4269332	864.8578225
4	2212.2786370	2240.3809436	1655.0743579
5	3657.0551019	3685.4948060	2708.1547749
6	5463.0070805	5491.6698541	4024.1913879
7	7630.1504495	7658.9708607	5603.2785078
8	10158.4843433	10187.4220586	7445.4551134
9	13048.0088283	13077.0372148	9550.7418726
10	16298.7239413	16327.824500	11919.1506089

3.2. *Extension to other one-dimensional continuous systems*

Most vibration problems of one-dimensional continuous systems are governed by equations similar to Equation [6] but with lower order spatial derivatives. Axial vibrations of a rod of variable cross-section are governed by the equation:

$$\frac{\partial}{\partial x}\left(S\frac{\partial u}{\partial x}\right) = \frac{\rho S}{E}\frac{\partial^2 u}{\partial t^2}$$

where u is the axial displacement and S is the variable cross-section area.

The torsional vibrations of a rod of variable cross-section are governed by a similar equation:

$$\frac{\partial}{\partial x}\left(I_p\frac{\partial \theta}{\partial x}\right) = \frac{\rho I_p}{G}\frac{\partial^2 \theta}{\partial t^2}$$

Where θ is the torsion angle, I_p is the polar moment of inertia of the cross-sectional area and G is shear modulus.

These equations have one acceleration term only. Matrix A_2 will have one nonzero entry, just like in the case of transverse vibrations of a beam. These equations can also be solved using our seventh-order accurate compact difference method.

4. A fourth-order accurate method

Although condition [4] can be satisfied for most, if not all, vibration problems one-dimensional continuous systems, that on the product of *B2* and *A2* [5] is not valid for some specific problems. We will see that this is true in the example of lateral beam vibrations including rotary inertia and shear deformation effects. For this problem there are two acceleration terms in the coupled equations of motion, resulting in two nonzero entries in matrix A_2. The product of A_2 with matrix B_2 will not be zero and we cannot apply the seventh-order accurate method.

For such problems we propose the following fourth-order accurate compact difference method, which is based on the Euler-Maclaurin summation formula (Isaacson *et al.*, 1966):

$$\sum_{i=0}^{n-1} f(x+ih) = \frac{1}{h}\int_x^{x+nh} f(\xi)d\xi - \frac{1}{2}[f(x+nh) - f(x)]$$
$$+\frac{h}{12}[f'(x+nh) - f'(x)] - \frac{h^3}{120}[f'''(x+nh) - f'''(x)] + O(h^5) \qquad [12]$$

We set $n = 1$ and take the derivative of Equation [12] with respect to x:

$$f'(x) = \frac{1}{h}\int_x^{x+h} f'(\xi)d\xi - \frac{1}{2}[f'(x+h) - f'(x)]$$
$$+\frac{h}{12}[f''(x+h) - f''(x)] - \frac{h^3}{120}\left[f^{(4)}(x+h) - f^{(4)}(x)\right] + O(h^5)$$

This can be written in the following form:

$$f(x+h) = f(x) + \frac{h}{2}[f'(x+h) + f'(x)]$$
$$-\frac{h^2}{12}[f''(x+h) - f''(x)] + O(h^4)$$

This truncated series, applied to a vector of variables, gives an equation similar to Equation [1] but with a truncation error that is of the order of h_n^4 instead of h_n^7. The resulting discretized equation is:

$$\vec{Z}_n = \vec{Z}_{n-1} + \frac{h_n}{2}\left(\vec{Z}'_n + \vec{Z}'_{n-1}\right) - \frac{h_n^2}{12}\left(\vec{Z}''_n - \vec{Z}''_{n-1}\right) + O(h_n^4)$$

Which can be put in the same form as Equation [2], *i.e.*:

$$\left(I + \frac{h_n}{2}A_{n-1} + \frac{h_n^2}{12}B_{n-1}\right)\vec{Z}_{n-1} + \left(-I + \frac{h_n}{2}A_n - \frac{h_n^2}{12}B_n\right)\vec{Z}_n = 0$$

The generalized eigenvalue problem remains of the same form as in the previously implemented seventh-order accurate method, with the following changes in the basic matrices:

$$(Q_1)_{n-1} = I + \frac{h_n}{2}(A_1)_{n-1} + \frac{h_n^2}{12}(B_1)_{n-1}$$

$$(Q_2)_{n-1} = \frac{h_n}{2}(A_2)_{n-1} + \frac{h_n^2}{12}(B_2)_{n-1}$$

$$(R_1)_n = -I + \frac{h_n}{2}(A_1)_n - \frac{h_n^2}{12}(B_1)_n$$

$$(R_2)_n = \frac{h_n}{2}(A_2)_n - \frac{h_n^2}{12}(B_2)_n$$

Since there is no matrix C as in the seventh-order accurate method, condition [5] is no longer needed and the method is less restrictive on the type of problems that can be solved.

To check for accuracy, the first ten frequencies of the same uniform cantilever beam are recalculated using the fourth-order accurate method. As seen in table 4, the values obtained are highly accurate. The relative difference at the tenth mode between the frequency given by this method and the one given by the analytical solution is about 0.5% when the number of intervals is equal to 25 and down to 0.002% when taking 100 intervals.

Table 4. *Comparison of the frequencies given by the fourth-order accurate method to analytical ones*

mode	Fourth-order method N=25	Fourth-order method N=100	Analytical solution
1	64.3365002	64. 3364946	64.3364917
2	403.1913182	403. 1899346	403.1899111
3	1128.9738584	1128. 9435957	1128.9434259
4	2212.5060088	2212. 2795345	2212.2785380
5	3658.0748824	3657. 0591542	3657.0549379
6	5466.3771012	5463. 0205800	5463.0068334
7	7639.2403303	7630. 1872015	7630.1500975
8	10179.6942462	10158. 5709938	10158.4838548
9	13092.4035199	13048. 1922560	13048.0081514
10	16384.1215861	16299. 0810378	16298.7229849

4.1. *Beam vibration including shear deformation and rotary inertia effects*

We will now apply the fourth-order accurate method to the lateral vibration problem of a beam including the effect of shear deformation and rotary inertia (Figure 4).

Let J be the rotary moment of inertia per unit length of the beam. The equations of motion are:

$$\partial V/\partial x = \rho S\, \partial^2 v/\partial t^2 \qquad [13]$$

$$\partial V/\partial x + V = J\, \partial^2 \theta/\partial t^2 \qquad [14]$$

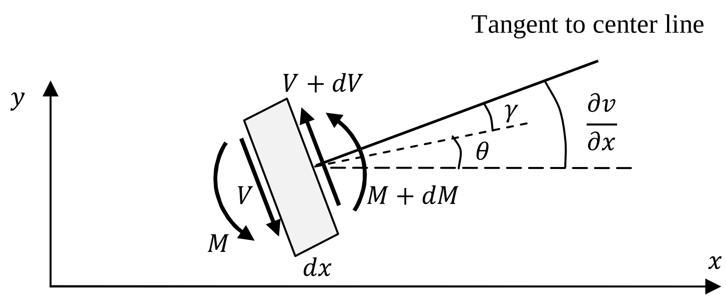

Figure 4. *Element of beam subject to shear deformation and rotary inertia effects*

The shear force V can be related to the shear deformation angle γ through the fundamental relations:

$$V = kSG\left(\frac{\partial v}{\partial x} - \theta\right) \qquad [15]$$

where G is the shear modulus, v is the deflection of the center line, θ is the angle due to bending and k is a factor depending on the shape of the cross section ($k = 2/3$ for a rectangular cross section, for example).

Bending moment M is given by:

$$M = EI_{zz}\frac{\partial \theta}{\partial x} \qquad [16]$$

Substituting Equations [15] and [16] into [13] and [14], we get the final coupled equations of motion:

$$\frac{\partial}{\partial x}\left(EI_{zz}\frac{\partial \theta}{\partial x}\right) + kSG\left(\frac{\partial v}{\partial x} - \theta\right) = J\frac{\partial^2 \theta}{\partial t^2} \qquad [17]$$

$$\frac{\partial}{\partial x}\left(kSG\left(\frac{\partial v}{\partial x} - \theta\right)\right) = \rho S\frac{\partial^2 y}{\partial t^2} \qquad [18]$$

Again assuming harmonic motion, Equations [17] and [18] can be written in the form of a system of first-order differential equations:

$$\frac{d}{dx}\begin{bmatrix}\theta\\ v\\ M\\ V\end{bmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{EI_{zz}} & 0\\ 1 & 0 & 0 & -\frac{1}{kSG}\\ -J\omega^2 & 0 & 0 & 1\\ 0 & \rho S\omega^2 & 0 & 0\end{pmatrix}\begin{bmatrix}\theta\\ v\\ M\\ V\end{bmatrix}$$

Matrices A_1 and A_2 and heir first derivatives are:

$$A_1 = \begin{pmatrix} 0 & 0 & \frac{1}{EI_{zz}} & 0\\ 1 & 0 & 0 & -\frac{1}{kSG}\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{pmatrix} \quad ; \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -J & 0 & 0 & 0\\ 0 & \rho S & 0 & 0\end{pmatrix}$$

$$A_1' = \begin{pmatrix} 0 & 0 & -\dfrac{I_{zz}'}{EI_{zz}^2} & 0 \\ 0 & 0 & 0 & \dfrac{S'}{kGS^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad A_2' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -J' & 0 & 0 & 0 \\ 0 & \rho S' & 0 & 0 \end{pmatrix}$$

It can be easily checked that condition [4] on matrix A_2 is satisfied. Matrices B_1 and B_2 can now be calculated and thereof matrices G and F of the generalized eigenvalue problem can be constructed.

The first ten frequencies of the same uniform cantilever beam are recalculated using the fourth-order accurate method and taking into consideration shear deformation and rotary inertia effects. The frequencies obtained are plotted below (Figure 5). It is found that omitting these effects causes a 3% overestimation of the fifth frequency and 11.5% of the tenth one. As it generally the case, these effects of are negligible for the lower frequencies of slender beams but not for thick ones (Ferreira *et al.*, 2006).

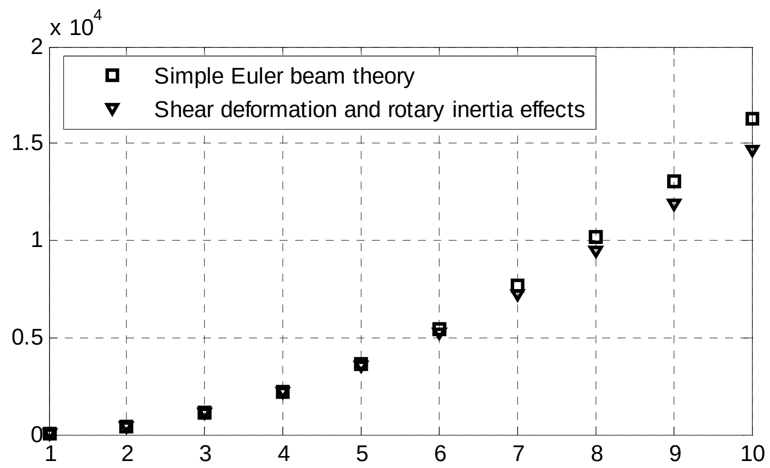

Figure 5. *Effect of shear deformation and rotary inertia on the first ten frequencies of a cantilever uniform beam with a rectangular cross section*

4.2. *Comparison to a finite element solution*

By all means, the purpose of this article was not to study the effect of taper or that of shear deformation and rotary inertia on beams but to show how the highly accurate compact difference methods can be applied to the solution of vibration problems of one-dimensional continuous systems.

To further demonstrate the precision of the proposed methods, we conduct a solution of the same cantilever beam vibration problem by a finite element method using ABAQUS. The B22 beam type element was used in the analysis. It is a quadratic three-node element that can handle bending, stretching and shear deformation.

In Table 5 we present the frequencies and the corresponding relative errors (with reference to the analytical values given previously) obtained by the compact-difference methods and the finite element method solution. It is clear that error associated with the compact-difference approach is much lower than that associated with the FEM solution. The CPU time for the tabulated values was 500ms for the FEM solution, 156ms for the fourth-order method and 187ms for the sixth-order method.

Table 5. *Comparison of the compact-difference methods with an FEM solution*

mode	fourth-order method (N=25)		Sixth-order method (N=25)		FEM solution using ABAQUS (100 B22 elements)	
	ω (rad/s)	% error	ω (rad/s)	% error	ω (rad/s)	% error
1	64.3365	0.000	64.3365	0.000	64.305	0.049
2	403.1913	0.000	403.1899	0.000	401.82	0.340
3	1128.9739	0.003	1128.9435	0.000	1119.8	0.610
4	2212.5060	0.010	2212.2790	0.000	2179.7	1.473
5	3658.0749	0.028	3657.0574	0.000	3572.5	2.312
6	5466.3771	0.062	5463.0187	0.000	5282.7	3.301
7	7639.2403	0.119	7630.1942	0.001	7290.1	4.457
8	10179.6942	0.209	10158.6208	0.001	9577.3	5.721
9	13092.4035	0.340	13048.3765	0.003	12123.	7.089
10	16384.1216	0.524	16299.6091	0.005	14907	8.539

5. Conclusions

In this work, we have introduced two compact difference methods, one is fourth-order and the other is seventh-order accurate. These methods are suitable for the solution of initial and boundary-value problems that can modeled by a first-order system of linear differential equations. Their main advantage is that they can handle, in a direct manner, some important boundary-value problems such as the vibrations of one-dimensional continuous mechanical systems. This has been shown by considering the typical problem of free lateral vibrations of a tapered cantilever beam, with and without the effects of shear deformation and rotary inertia effects. The case of a lumped mass at the free end of the beam was also treaded We note here that the method is one of global eigenvalue search, yielding an important number of frequencies and mode shapes in one run of the FORTRAN code implementing these methods.

The first ten frequencies of a cantilever prismatic beam have been computed using both methods. Their high accuracy has been demonstrated by comparing the obtained results to their analytical counterparts. For the seventh-order method, the relative error did not exceed 0.005% at the tenth mode when taking only 25 intervals. As for the fourth-order method, the error was about 0.5% at the tenth mode when taking 25 intervals and down to 0.002% when the number of intervals is increased to 100.

We then applied the seventh-order method to study the effect of tapering the width and height on the natural frequencies of a cantilever beam. The results show that tapering the width increases all ten frequencies while tapering the height increases the first frequency but lowers the other nine.

We have also shown that, whenever the seventh-order method cannot be applied to some particular problems (mainly problems with more than one acceleration term in the governing equations of motion), the fourth-order accurate method, being less restrictive, can be used. The transverse vibration of a beam including shear deformation and rotary inertia effects is such a problem. The frequencies obtained by the fourth-order method confirm that neglecting these effects has minimal impact on the values of the lower frequencies of slender beams.

Finally, we have made a comparison of the values of the frequencies obtained by the compact-difference methods to those obtained by a finite element method solution using ABAQUS. The results have strongly confirmed the higher accuracy of the compact-difference solutions.

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