RPCM: a strategy to perform reliability analysis using polynomial chaos and resampling

Application to fatigue design

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ABSTRACT. *Using stochastic finite elements, the response quantity can be written as a series expansion which allows an approximation of the limit state function. For computational purpose, the series must be truncated in order to retain only a finite number of terms. In the context of reliability analysis, we propose a new approach coupling polynomial chaos expansions and confidence intervals on the generalized reliability index as truncating criterion.*

RÉSUMÉ. *La méthode des éléments finis stochastiques permet d'exprimer la réponse d'un système sous forme d'une série polynomiale appelée chaos polynomial. Numériquement, cette série doit être tronquée pour ne retenir qu'un nombre fini de termes. Dans le cadre de l'analyse de fiabilité, nous proposons une méthode adaptative utilisant les intervalles de confiance sur l'indice de fiabilité généralisé comme critère de troncature du chaos.*

KEYWORDS: *Bootstrap, confidence intervals, reliability analysis, polynomial chaos, fatigue.*

MOTS-CLÉS : *Bootstrap, intervalles de confiance, analyse de fiabilité, chaos polynomial, fatigue.*

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Main notations

$c.o.v$	Cœfficient of variation
$\underline{X}$	Physical input random vector
$\underline{\xi}$	Standard Gaussian independent input vector
T	Isoprobabilistic transformation
G	Response of the system, performance function in the physical space
$\widetilde{G}$	Approximation of G using polynomial chaos
g_j	Chaos coefficient
$\underline{g}$	Vector of chaos coefficients
ψ_j	Hermite polynomials
p	Chaos order
P - $P^{(k)}$	Chaos number of coefficients - Size of the chaos of order k
M	Number of random variables
$E[.]$	Mathematical expectation
φ_M	Uncorrelated multinormal probability density function
N	Number of mechanical evaluations
$\underline{\xi}^{(i)}$	Vector of the i^{th} experiments in the Gaussian standard space
$\underline{G}$	Vector of mechanical evaluations
P_f - $\widetilde{P}_f$	Probability of failure - Probability of failure estimation using Monte Carlo simulation onto the polynomial chaos
P_f^{MC}	Probability of failure estimation using Monte Carlo simulation on the true mechanical model
β - $\widetilde{\beta}$	Reliability index - Reliability index estimation using Monte Carlo simulation onto the polynomial chaos
β^{MC}	Reliability index estimation using Monte Carlo simulation on the true mechanical model
Φ	Standard Gaussian cumulative density function
$N^{(p)}$	Size of the Design Of Experiments number p
B	Number of bootstrap loops
α	Confidence level
$[\widetilde{\beta}^{k,p}_{\mathtt{min}}, \widetilde{\beta}^{k,p}_{\mathtt{max}}]$	Confidence interval on the reliability index computed from chaos of order k on the DOE number p
$\widetilde{\beta}^{k,p}$	Punctual estimation of reliability index based on the DOE number p from chaos of order k

1. Introduction

Nowadays, it is a common practice in the industry to use the finite element method in order to predict the structural behavior. Particularly, fatigue lifetime predictions are primordial. According to the ASTM (American Society for Testing and Materials), the fatigue life is defined as the number of stress cycles N_f that a specimen sustains before the failure occurs. The fatigue process of mechanical components under service loading is random in essence. The randomness comes mainly from the loading process and the fatigue resistance of material. An appropriate modeling technique is required to include the variability of both material properties and external loadings. (Liu *et al.*, 2007) introduced a general methodology for stochastic fatigue lifetime prediction under variable amplitude loading in the case of uniaxial loading. They combine a stochastic S-N curve approach with Karhunen-Loève expansion technique. (Sain *et al.*, 2008) proposed a statistical framework in order to assess the fatigue lifetime of plain concrete beams (based on a linear elastic fracture mechanics crack propagation theory) through Monte Carlo simulations with Latin Hypercube Sampling technique. In the case of fatigue crack growth, (Bigerelle *et al.*, 2006) used Lambda distributions and Bootstrap techniques to obtain accurate estimations of the PDF (Probability Density Function) of the coefficients of the Paris-Erdogan relationship. According to (Walz *et al.*, 2006), the use of confidence bounds provide *important information on the accuracy of failure probability predictions in the light of the uncertainties of the available data base*. This proves that taking randomness into account in fatigue prediction is critical and of great interest. Besides the authors established a probabilistic fracture mechanic model for the failure of turbine disks. This model is based on reliability analysis coupled with Monte Carlo simulations and bootstrap confidence intervals on the probability of failure.

However, in all the previous cited references the randomness is simulated through Monte Carlo simulations. Even if Monte Carlo methods are universal, they are time computing expansive and sometimes impossible to use. A viable approach to overcome the computational burden is to substitute the true model for a response surface or a meta-model. Several kinds of surrogate meta-models have been developed, including polynomial response surface (Rajashekhar *et al.*, 1993; Zheng *et al.*, 2000), Kriging meta-modeling (Kaymaz, 2005; Echard *et al.*, 2009; Echard *et al.*, 2010), Gaussian meta-models (Marrel *et al.*, 2009), artificial neural networks (Papadrakakis *et al.*, 1996; Elhewy *et al.*, 2006; Cheng *et al.*, 2008), radial basis functions and support vector machines (Hurtado, 2007; Li *et al.*, 2010). Polynomial-based response surface is a widely used surrogate model due to its simplicity and effectiveness. The response surface method uses least-squares regression analysis to fit low-order polynomials to a set of experimental data. Besides, the model function is typically chosen to be first- or second-order polynomials which is likely awkward when it is used for representing multi-modalities and non-linearity commonly appearing in complex engineering problem (Giunta *et al.*, 1998). Therefore, in the case of reliability analysis, it requires the use of techniques such as FORM and SORM to find the reliability index which implies an iterative procedure (Gayton *et al.*, 2003). To interpolate the limit-state, another solution is to use the polynomial chaos expansion (Ghanem *et al.*, 1991; Soize *et al.*, 2004). It corresponds to a response surface in a particular basis (Hermite polynomials for example) which yields an intrinsic representation of the random response of the model in terms of its polynomial chaos expansion. This approximation on an explicit functional basis is of great interest and well suited to straightforward post-processing. The following quantities may be derived from the polynomial chaos expansion: statistical moment analysis (Malliavin, 1997), global sensitivity analysis (Sudret, 2008), reliability analysis (Sudret *et al.*, 2002) and the probability density function of response quantities. Thus, the scope of the present work paper is focus on such approach. Support Vector Machine (SVM) provides a novel approach to the two-category classification problem (operating or failed) with connections to the underlying statistical learning theory (Shawe-Taylor *et al.*, 2000). SVM methods have gained considerations with their great effectiveness for reliability studies especially when combined with subset simulation (Rocco *et al.*, 2002; Deheeger *et al.*, 2007). Following this classification approach, (Echard *et al.*, 2009; Echard *et al.*, 2010) proposed to used Kriging in reliability analysis (based on the work by (Kaymaz, 2005)). The idea is to develop an iterative method in order to optimize the number of experiments adding points in areas with high variance. These classification approaches are very efficient but if the reliability level is modified then all computations have to be done again.

More precisely, in the context of structural mechanics (Ghanem *et al.*, 1991) proposed the Spectral Stochastic Finite Element Method. In this setup, the inputs are represented by Gaussian random fields that are discretized using the Karhunen-Loève expansion. The model response is expanded onto a particular basis of the probability space called the polynomial chaos (Ghanem *et al.*, 1991; Soize *et al.*, 2004). The solution is computed by a Galerkin minimization scheme in the random dimension which makes the approach *intrusive*: modifications of the deterministic model and of the computer code are required. As the present work is aimed at addressing industrial problems, we focus our attention on the so-called *non intrusive* approaches (Isukapalli, 1999; Puig *et al.*, 2002; Berveiller *et al.*, 2006) which do not required an adapting of the governing equations. The deterministic model is considered to be a black-box (see (Keese, 2004) for example). According to (Pellissetti *et al.*, 2006), "*the interaction with third party codes is a key asset*" to solve large-scale problems but there are some limitations such as the consideration of a high number of random variables and low probability of failure. Applications of the stochastic finite elements method in structural reliability have been shown in (Sudret *et al.*, 2002; Choi *et al.*, 2004a). The reader can find a more detailed discussion on general purpose software for structural reliability in (Lemaire *et al.*, 2006; Pellissetti *et al.*, 2006; Reh *et al.*, 2006). In (Heiermann *et al.*, 2005), a strategy for the assessment of uncertainty, through confidence intervals, in the estimation of the failure probability of ceramic components due to the scatter of material data is presented. Confidence intervals for the failure probability are obtained by means of stochastic resampling methods and in order to save computation time, neural networks are used.

The objective of this paper is to demonstrate the contribution of resampling techniques for reliability analysis with polynomial chaos based approaches. Even if polynomial chaos based approaches are now commonly used, it remains two difficulties: firstly, the choice of the polynomial approximation order and secondly the selection of the best design of experiments. The polynomial chaos can be seen as a response surface. On that account, it is build from a design of experiments containing more or less data's (depends on the case study). But, in the industry, new data's can be very expensive to obtain. The use of resampling techniques permits to explore and evaluate the variability of results without new computations (Efron *et al.*, 1993; Gayton *et al.*, 2003): taking the initial design of experiments into account, the validity of the results can be checked. That is the reason why we use bootstrap techniques to compute confidence intervals on the reliability index and pilot polynomial chaos based approaches to find the best chaos order and the best DOE for the considered problem.

Section 2 deals with literature review on the various methods involved. Then, in section 3 the principles of the RPCM method are explained and detailed. Finally, some possibilities of the approach are investigated in sections 4 and 5 through academic and industrial applications.

2. Literature review

2.1. *Stochastic finite elements and reliability*

Following the historic work by Wiener (Wiener, 1938) on the *Homogeneous Chaos*, the Stochastic Finite Element Method has been developed by (Ghanem *et al.*, 1991). It is based on the discretization of the input random fields and on the expansion of the mechanical response onto the polynomial chaos (Ghanem *et al.*, 1991; Soize *et al.*, 2004). The input random vector $\underline{X}$ is transformed into a Gaussian random vector $\underline{\xi}$ with independent components using an isoprobabilistic transform T: $\underline{\xi} \mapsto T\left(\underline{X}\right)$ (Ditlevsen *et al.*, 2007; Lemaire, 2009). Thus, the interesting response quantity (performance function in a reliability context) G is approximated by a truncated series expansion:

$$G(\underline{X}) = G\left(T^{-1}(\underline{\xi})\right) \approx \widetilde{G}\left(T^{-1}(\underline{\xi})\right) = \sum_{j=0}^{P-1} g_j \psi_j(\underline{\xi}) \qquad [1]$$

p is the chaos order, P is the number of chaos coefficients defined by $P = \frac{(M+p)!}{M!p!}$ and $\{\psi_j, j = 0, \ldots, P-1\}$ are multivariate Hermite polynomials (even if other possibilities are available: see (Xiu *et al.*, 2002)) based on the M random variables, whose degree is less or equal than p (p is fixed *a priori*). The chaos coefficients g_j are evaluated either by projection or regression. Recently, a stepwise regression technique has been proposed by (Blatman *et al.*, 2008) to build up a *sparse* PC expansion in which only a small number of significant basis functions are retained in the response PC approximation.

The projection method. This method is used by (Puig *et al.*, 2002; Xiu *et al.*, 2002) and is based on the orthogonality of the Hermite polynomials with respect to the Gaussian measure. It comes from Equation [1]:

$$g_j = \frac{E[G.\psi_j]}{E[\psi_j^2]} = \frac{1}{E[\psi_j^2]} \int_{\mathbb{R}^M} G\left(T^{-1}(\underline{\xi})\right) \psi_j(\underline{\xi}) \varphi_M(\underline{\xi}) d\underline{\xi} \qquad [2]$$

where $\varphi_M(\underline{\xi})$ is the uncorrelated multinormal probability density function of size M. The integral can be evaluated by simulation (Monte Carlo, Latin Hypercube Sampling) or by a quadrature scheme (Field, 2002; LeMaître *et al.*, 2002; Keese, 2004). Nevertheless, this projection schemes might lead to prohibitive computational cost because of:

– a large set of realizations of the output variable $G\left(T^{-1}(\underline{\xi})\right)$ is required by simulation techniques;

– the cost of the quadrature technique strongly increases with the number of input parameters (for K integrating points and M random variables K^M calculations are required).

Despite, the projection method aims at computing the exact g_j coefficients when the goal of regression techniques is to find the best compromise for a given truncation order of Equation [1]. In consequence, the regression technique is used.

The regression method. This method is based on a least square minimization between the exact solution and the solution approximated using the polynomial chaos (Isukapalli, 1999; Berveiller *et al.*, 2006). Considering N experiments, the regression method consists in finding the set of coefficients $\underline{g}$ that minimizes the difference:

$$\Delta G = \sum_{i=1}^{N} \left(G\left(T^{-1}\left(\underline{\xi}^{(i)}\right)\right) - \tilde{G}\left(T^{-1}\left(\underline{\xi}^{(i)}\right)\right)\right)^2 \qquad [3]$$

$\underline{\xi}^{(i)}$ correspond to the vector of experiments coordinates in the standard space. The classical solution reads:

$$\underline{g} = \left(\Psi^T \Psi\right)^{-1} \Psi^T \underline{G} \qquad [4]$$

The matrix Ψ is defined as follow:

$$\mathbf{\Psi} = \begin{pmatrix} \psi_0(\underline{\xi}^{(1)}) & \cdots & \psi_{P-1}(\underline{\xi}^{(1)}) \\ \vdots & \ddots & \vdots \\ \psi_0(\underline{\xi}^{(n)}) & \cdots & \psi_{P-1}(\underline{\xi}^{(n)}) \end{pmatrix} \quad [5]$$

and $\underline{G} = \left\{G^{(1)}, \ldots, G^{(N)}\right\}^T$ corresponds to N mechanical evaluations. The $P \times P$ matrix $\Psi^T \Psi$ may be evaluated once and for all. The crucial point in this approach is to properly select the N regression points: $N \geqslant P$ is required so that a solution exits (system well-conditioned) but how many points must be taken into account? Which polynomial chaos order p must be chosen *a priori* when the smoothness of the model or function to approximate is unknown? These two problems are treated through confidence intervals on the reliability index β.

Reliability analysis. Structural reliability is usually defined in terms of the probability of failure P_f:

$$\begin{aligned} P_f &= \text{Prob}\left[G(\underline{X}) \leqslant 0\right] \\ &= \int_{G(\underline{X}) \leqslant 0} f_{\underline{X}}(\underline{x}) d\underline{x} \end{aligned} \quad [6]$$

$f_{\underline{X}}(\underline{x})$ is the joint probability density function of the random variables $\underline{X}$ and $G(\underline{X})$ defines the performance function:

– $G(\underline{X}) = 0$: defines the limit-state function,

– $G(\underline{X}) \leqslant 0$: defines the failure domain.

If possible, the reference value of the probability of failure is obtained through crude Monte Carlo simulation needing a large N_{MC} mechanical evaluations and expressed as:

$$P_f^{MC} = \frac{N_{G \leqslant 0}}{N_{MC}} \quad [7]$$

$N_{G \leqslant 0}$ being the number of time the performance function is lower than zero. As a stochastic finite element analysis is performed, the function G is replaced by its approximation $\widetilde{G}$ onto the chaos. The approximation $\tilde{P}_f$ of P_f is given by:

$$\tilde{P}_f = \frac{N_{\tilde{G} \leqslant 0}}{N_{MC}} \quad [8]$$

Finally, the general reliability index $\tilde{\beta}$ is defined as follow:

$$\tilde{\beta} = -\Phi^{-1}(\tilde{P}_f) \qquad [9]$$

where Φ is the standard Gaussian cumulative density function.

2.2. *Resampling techniques*

One of the goals of resampling techniques is to compute confidence interval on a statistic. The reliability index $\tilde{\beta}$ is a statistic which depends on the DOE see (Gayton *et al.*, 2003). Confidence intervals will be computed on the reliability index $\widetilde{\beta}$.

The bootstrap. The concept of the bootstrap was first introduced by Efron in 1979 (Efron, 1979). It is an improvement of the jackknife procedure. Efron's bootstrap has been set to estimate not only the standard error but also the distribution of a statistic. Let's consider an original set $\underline{G} = \{G_1, \ldots, G_N\}^T$ of N evaluations of a performance function G through mechanical computation. Let β be a real-value function of the distribution (such as its mean value for example). $\widetilde{\beta}$ is the value of β estimated from the data $\underline{G}$. The bootstrap principle is to create B bootstrap samples (also called bootstrap resamples) $\underline{G}^{(k)}, k = \{1, \ldots, B\}$ of size N_B build by sampling with replacement from the original data $\underline{G}$: bootstrap resampling do not required extra mechanical computations. For each $\underline{G}^{(k)}$, the corresponding value $\widetilde{\beta}$ is estimated. According to Efron and Tibshirani (Efron *et al.*, 1993) the number of resamples *B* is often between 30 and 200 for estimating standard errors and 1000 for confidence intervals involving percentiles estimation. However, for accurate percentiles estimations it could be more than 100000 (Hesterberg, 2007). Compared to the usual simulation techniques (Monte Carlo and LHS), the Bootstrap avoids to choose *a priori* a PDF for the set of data points. This way, there is no need to first identify the data's PDF before sampling which makes the Bootstrap suitable for industrial data's. However, due to the Bootstrap principle, the Bootstrap dataset is composed of elements from the original data points appearing zero, one, twice (or more) time which modifies the level of influence of each data point on the final parameter estimates.

Confidence intervals. A confidence interval gives an estimated range of values which is likely to include an unknown population parameter, the estimated range being calculated from a given set of data. Confidence intervals are usually calculated so that this confidence level α is 0.95, but other values (0.90, 0.99, 0.999, ...) can be produced. The size of the confidence interval gives us some idea about how uncertain we are about the unknown parameter. A very wide interval may indicate that more data's should be collected before anything very definite can be said about the parameter. We propose to use the Bias Corrected and accelerated (BCa) intervals as recommended in the literature (DiCiccio *et al.*, 1996; Walz *et al.*, 2006) which corrects the percentile interval for bias and skewness. Let's consider B bootstrap samples $\underline{G}^{(k)}, k = \{1, \ldots, B\}$ and their corresponding parameter $\widetilde{\beta}$. The confidence limits $[\widetilde{\beta}_{\mathtt{min}}, \widetilde{\beta}_{\mathtt{max}}]$ are the percentiles of the $\widetilde{\beta}$ distribution corresponding to the values of the normal distribution Φ at points u_1 and u_2. More precisely, u_1 and u_2 are the endpoints that is to say the points of the normal distribution where the percentiles will be computed. The definition of the points u_1 and u_2 includes the bias-correction u_{perc} as the acceleration a (DiCiccio *et al.*, 1996):

$$\begin{aligned} u_1 &= u_{perc} + \frac{u_{perc} + u_{\alpha/2}}{1 - a(u_{perc} + u_{\alpha/2})} \\ u_2 &= u_{perc} + \frac{u_{perc} + u_{1-\alpha/2}}{1 - a(u_{perc} + u_{1-\alpha/2})} \end{aligned} \quad [10]$$

u_{perc} is the $p-$percentile of Φ defined from the proportion p of bootstrap values of $\widetilde{\beta}$ less than the value computed on the whole $\underline{G}$ sample (which clearly corresponds to the bias-correction). The parameter a is called *acceleration* and is linked to the variation's rate of the standard error of $\widetilde{\beta}$ when the parameter β varies. According to (DiCiccio *et al.*, 1996), the acceleration a is defined as follow using a Jackknife procedure:

$$a = \frac{1}{6} \sum_{i=1}^{n} \left(\widetilde{\beta}_J - \widetilde{\beta}_{(-i)} \right)^3 \Big/ \left[\sum_{i=1}^{n} \left(\widetilde{\beta}_J - \widetilde{\beta}_{(-i)} \right)^2 \right]^{3/2} \quad [11]$$

In this expression, $\widetilde{\beta}_{(-i)}$ is the estimation of the parameter β from the initial set of data's $\underline{G}$ removing the i^{th} observation. $\widetilde{\beta}_J$ is the mean of all the $\widetilde{\beta}_{(-i)}$ values.

3. RPCM - Presentation of the proposed method

For computational purpose, the series defined by Equation [1] is truncated to only retain a finite number of terms. The convergence rate of this truncated PC depends on the smoothness of the function to approximate. (Field *et al.*, 2004) propose various metrics for evaluating *a posteriori* the accuracy of truncated PC expansions. However, in our context the smoothness of the model is not known *a priori* and our goal aims at minimizing the number of model evaluations: such measures cannot be directly computed. This difficulty is overcame using confidence intervals. The goal is to answer to the following questions:

– Which chaos order must be chosen?

– When should we stop to add extra data's?

These two questions leads to an adaptive procedure with a learning loop and an enrichment loop. There is one convergence test which is based on the confidence interval width. For each DOE, all the possible chaos orders are tested: this is the learning loop. Once all possible orders have been tested, if there is still no convergence then the DOE does not contain enough information and more data have to be provided: this is the enrichment loop. In the same way, if for the same chaos order, there is no convergence when increasing the size of the DOE, then the chaos order must be increased. The confidence interval are computed using Bootstrap resampling. The general procedure is detailed Figure 1. A focus is provided on the learning loop Figure 2. Each step is detailed hereafter.

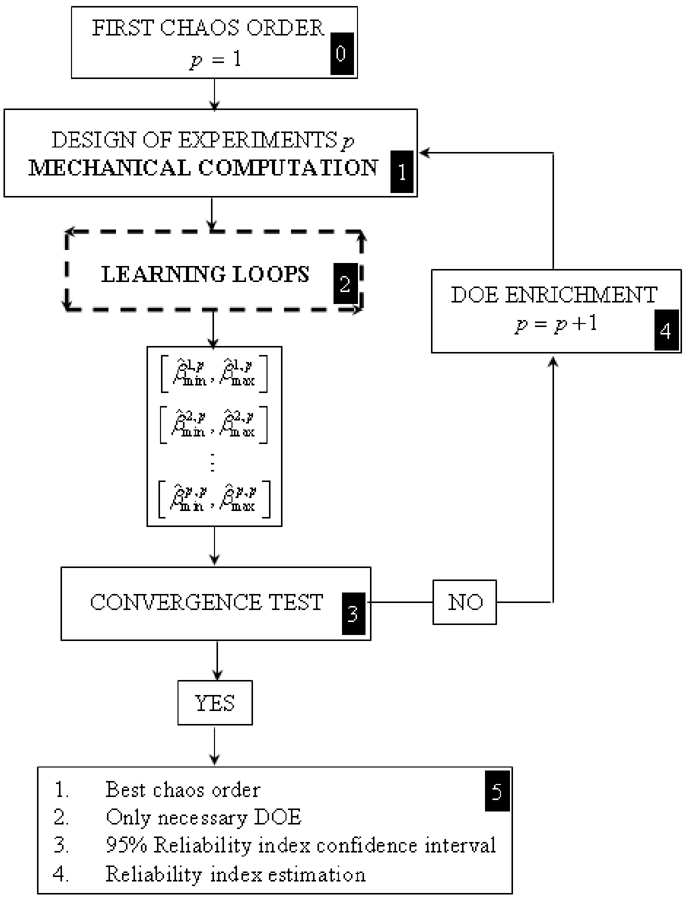

Figure 1. *Resampling Polynomial Chaos Method procedure*

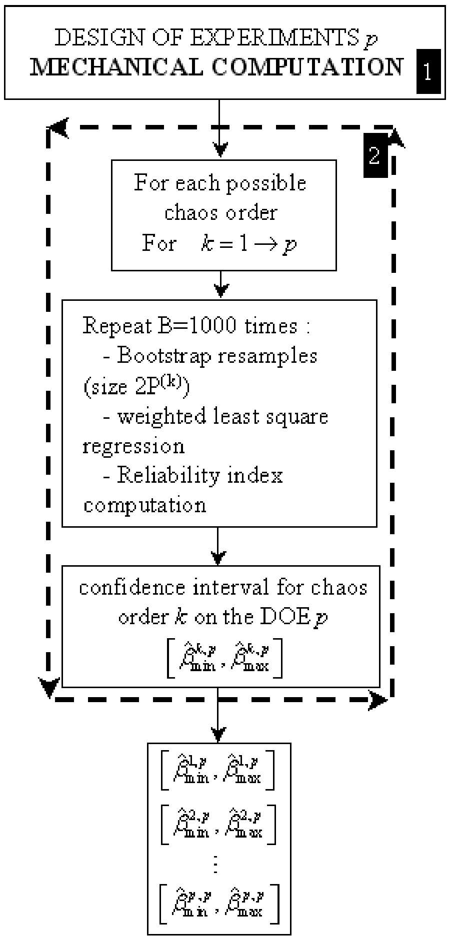

Figure 2. *Details on the learning loop (Step 2)*

Step 0 - First chaos order

Without any knowledge on the mechanical behavior, a linear chaos $p = 1$ (the less expansive chaos order regarding the required number of mechanical evaluations) is firstly set:

$$\tilde{G}(\underline{\xi}) = g_0 + g_1\xi_1 + ... + g_M\xi_M \quad [12]$$

It needs at least $M + 1$ mechanical computations of G to assess the g_i coefficients.

Step 1 - Design of experiments and mechanical computations

Each Design Of Experiments (DOE) is numbered in relation with the chaos order it permits to reach. For example, the DOE number p allows the computation of the chaos expansion of order up to p. The size $N^{(p)}$ of the DOE p is consequently linked to the number of coefficients $P^{(p)}$ of the chaos of order p. The proposed rule is (M is the number of random inputs):

$$N^{(p)} = 2P^{(p)} = 2\frac{(M+p)!}{M!p!} \qquad [13]$$

This rule is linked to data's replication due to the Bootstrap use. In order to be well-conditioned, the regression system needs $P+1$ different terms. Based on numerical tests, one way to satisfy this condition is to take a DOE of size $2P$. For each DOE, data are created using Latin Hypercube Sampling (McKay *et al.*, 2000). The Latin Hypercube Sampling (LHS) is a stratified sampling technique which ensures a better uniformity over $[0,1]^M$. It is recommended by (Novak *et al.*, 2001; Olsson *et al.*, 2002; Choi *et al.*, 2004b; Blatman *et al.*, 2007) to improve computational efficiency while reducing simulation cost. For the first DOE, the polynomial chaos order is set to 1 which implies that DOE number 1 has a size $N^{(1)} = 2(M+1)$.

Step 2 - Learning loop

The details of the learning loop are given in figure 2. All the possible chaos orders are tested; that is to say that for the DOE number p, chaos order k for $k = 1$ to $k = p$ is tested. For each chaos order k, a bootstrap resample of $2P^{(k)}$ experiments is randomly chosen among the $2P^{(p)}$ available experiments with replacement. The use of Bootstrap implies data's replication which modifies the level of influence of the data points. In a stochastic FEM scheme, this could modify the conditioning of the least square regression system. The use of weighted least squares regression (Carroll *et al.*, 1988; Ryan, 1997; Gayton *et al.*, 2003) ensures that each data point has an appropriate level of influence. Equation [3] is modified as follow:

$$\Delta G = \sum_{i=1}^{n} \omega_i \left(G\left(T^{-1}\left(\underline{\xi}^{(i)}\right)\right) - \tilde{G}\left(T^{-1}\left(\underline{\xi}^{(i)}\right)\right)\right)^2 \qquad [14]$$

ω_i is the weight of the i^{th} experiment and Equation [4] reads:

$$\underline{g} = \left[(W\mathbf{\Psi})^T (W\mathbf{\Psi})\right]^{-1} (W\mathbf{\Psi})^T (W\underline{G}) \qquad [15]$$

which is equivalent to replace Ψ by $W\Psi$ and the vector $\underline{G}$ by $W\underline{G}$. W is a diagonal matrix of weights ω_i for the regression problem. All weights are positive. A weight of k is equivalent to have replicated that data point k times during the bootstrap step. From the generated chaos, the probability of failure is calculated by sampling the polynomial chaos expansion using crude Monte Carlo simulations. However, in order to avoid the Monte Carlo bias, the same sample is used to evaluate each probability P_f. The size of the sample is chosen *a priori*: for a probability of order 10^{-n}, 10^{n+3} samples are used. Finally, the reliability index is deduced from Equation [9]. By generating $B = 1000$ bootstrap resamples, a sample of B reliability indexes is generated and a 95% confidence interval $[\tilde{\beta}^{k,p}_{\mathtt{min}}, \tilde{\beta}^{k,p}_{\mathtt{max}}]$ is obtained. This confidence interval is the confidence interval on the reliability index computed on a chaos of order k from the DOE number p. At the end of the learning loop, p reliability index confidence intervals are obtained from the DOE number p with all the possible chaos orders. The middle value $\tilde{\beta}^{k,p} = \frac{1}{2}(\tilde{\beta}^{k,p}_{\mathtt{min}} + \tilde{\beta}^{k,p}_{\mathtt{max}})$ of each confidence interval is used as a point estimation of the reliability index.

Step 3 - Convergence test

The convergence test is based on the confidence interval length. It is an indicator of the quality of the meta-model; the smallest the interval is, the best is the quality of the approximation onto the polynomial chaos. It is defined as follow:

$$\frac{|\tilde{\beta}^{k,p}_{\mathtt{max}} - \tilde{\beta}^{k,p}_{\mathtt{min}}|}{\tilde{\beta}^{k,p}} \leqslant \epsilon_\beta \qquad [16]$$

ϵ_β is defined *a priori* according to the level of accuracy (1% for example).

Step 4 - DOE enrichment

When $k = p$, the maximum chaos order for DOE number p has been reached. If there is still no convergence, the DOE needs enrichment. It is done using LHS. $N^{(p+1)} - N^{(p)}$ data points are added in order to guarantee that the chaos of order $p + 1$ can be computed.

Step 5 - Final conclusions

If the convergence test is successful the procedure stops. In the context of reliability analysis, according to the RPCM procedure, the best chaos order has been reached. Finally, the RPCM procedure provides the following information:

1) The best chaos order to reach the reliability objective with the required confidence interval,

2) The minimal DOE to build the chaos approximation and to reach the reliability objective,

3) The 95% reliability index confidence interval,

4) An estimation of the reliability index.

REMARK. — An increase of the coefficient of variation will increase the impact of the non linearity of the limit-state on the width of the confidence interval and on the final reliability index. This effect could mean an increase of the PCE order to fit properly the limit-state in the domain of variation of the random variables.

4. Academic examples for validation

The method has been implemented in Matlab. The accuracy of the method is first assessed with three academic examples: two analytical limit-state functions and one finite element model. Then, an industrial example is considered. Except for the two finite element models, the numerical results are compared to a reference solution based on MC simulation due to the study of the ratio $\tilde{\beta}^{k,p}/\beta_{MC}$. Indeed, we assume that in the case of FE simulations, a MC simulation is inconceivable.

4.1. *Preliminary check*

Before going further, a preliminary check is done using a trivial performance function in order to guarantee that the method is able to find the right chaos order. Let us consider a quadratic performance function:

$$G(x_1, x_2) = x_1^2 - \frac{x_2^2}{2} + 2 \qquad [17]$$

x_1 and x_2 are supposed to be standard Gaussian variables. The reference value is $\beta_{MC} = 1.2591$ and we focus on the ratio $\tilde{\beta}^{k,p}/\beta_{MC}$. Numerical results are given Figure 3 and Table 1. As we can see, the RPCM approach gives the right chaos order for the G function and for the reliability index as the confidence interval width is 0 for a chaos of order 2.

4.2. *Exponential limit-state*

This first academic example is an exponential limit-state function:

$$G(x_1, x_2) = \exp(0.2x_1 + 1.4) - x_2 \qquad [18]$$

There are two random variables, x_1 and x_2 which are supposed to be standard Gaussian variables. The reference value, $\beta_{MC} = 3.388$, is deduced from crude Monte Carlo sampling (1,000,000 samples). RPCM is initialized with a chaos of order $p = 1$, a confidence level $\alpha = 0.95$ and $B = 1000$ resamples. The size of the first DOE is

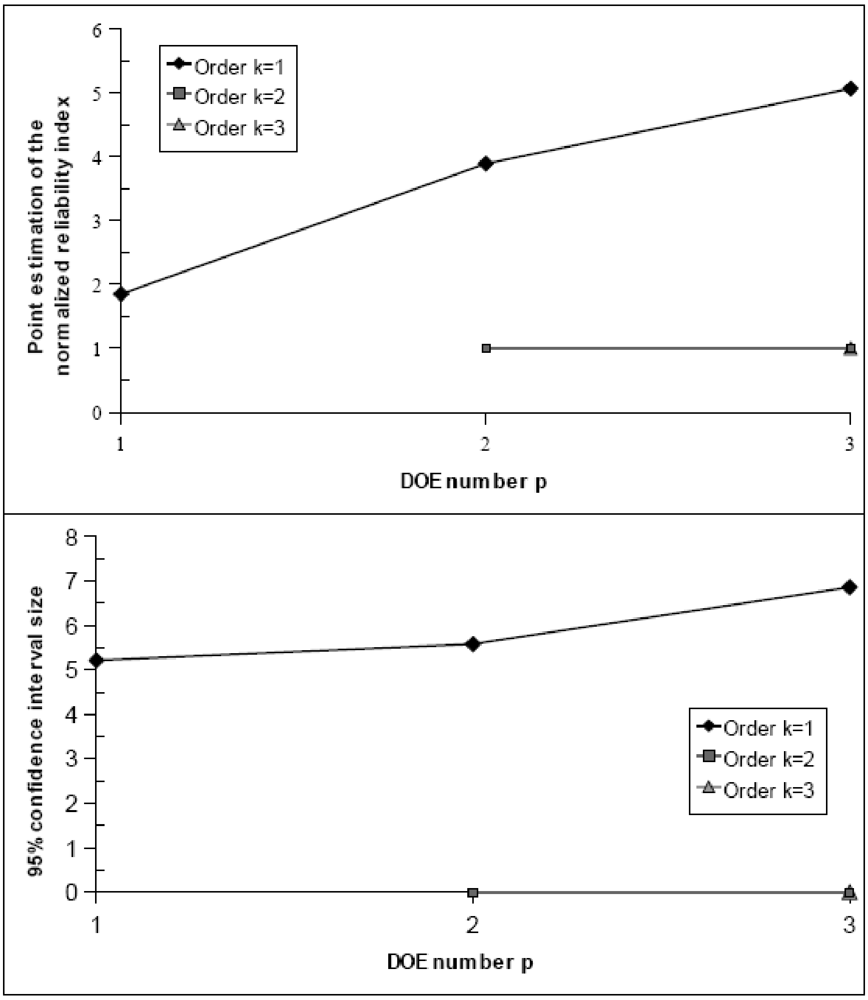

Figure 3. *Quadratic limit-state - Top: Plot of the convergence of the ratio* $\tilde{\beta}^{k,p}/\beta_{MC}$ *following the DOE number* p *- Bottom: Plot of the evolution of the convergence test* ϵ_β *following the DOE number* p

Table 1. *Quadratic limit-state - Results synthesis. Results are presented in terms of: lower bound* $\left(\tilde{\beta}^{k,p}_{min}/\beta_{MC}\right)$*, estimation* $\left(\tilde{\beta}^{k,p}/\beta_{MC}\right)$*, upper bound* $\left(\tilde{\beta}^{k,p}_{max}/\beta_{MC}\right)$

DOE p	$N^{(p)}$	Max chaos order p	**Order k=1**		
			$\tilde{\beta}^{k,p}_{\min}/\beta_{MC}$	$\tilde{\beta}^{k,p}/\beta_{MC}$	$\tilde{\beta}^{k,p}_{\max}/\beta_{MC}$
1	6	1	-0,757	1,849	4,455
2	12	2	1,099	3,889	6,679
3	20	3	1,636	5,066	8,496
DOE p	$N^{(p)}$	Max chaos order p	**Order k=2**		
			$\tilde{\beta}^{k,p}_{\min}/\beta_{MC}$	$\tilde{\beta}^{k,p}/\beta_{MC}$	$\tilde{\beta}^{k,p}_{\max}/\beta_{MC}$
1	6	1	NA	NA	NA
2	12	2	1,000	1,000	1,000
3	20	3	1,000	1,000	1,000
DOE p	$N^{(p)}$	Max chaos order p	**Order k=3**		
			$\tilde{\beta}^{k,p}_{\min}/\beta_{MC}$	$\tilde{\beta}^{k,p}/\beta_{MC}$	$\tilde{\beta}^{k,p}_{\max}/\beta_{MC}$
1	6	1	NA	NA	NA
2	12	2	NA	NA	NA
3	20	3	1,000	1,000	1,000

$N^{(1)} = 6$. As the reference value β_{MC} is known, we focus on the ratio $\tilde{\beta}^{k,p}/\beta_{MC}$. If $\tilde{\beta}^{k,p}/\beta_{MC} = 1$, then the model perfectly approximates the limit-state. The convergence of the reliability index and the convergence of the width of the confidence interval are plotted Figure 4. Numerical results are summed up Table 2. From a graphic analysis of Figure 4, one can conclude that a chaos of order 1 is unable to estimate the limit-state when chaos of order 2, 3 fit well the limit-state and chaos of order 4 gives the expected solution. This statement is confirmed by numerical results (Figure 4). A chaos of order 3 with a sample of size 30 gives a very good approximation ($\tilde{\beta}^{3,4}/\beta_{MC} = 0.997$ and $\epsilon_{beta} = 0.0057$). Whatever the size of the sample, the chaos of order 4 perfectly fits the limit-state and gives the reference value of β. Using a polynomial chaos of order 4, the size of the confidence interval is 0. It means that $\tilde{\beta}^{k,p} = \beta_{MC}$ and that the approximation gives the expected value of β. This way, the size of the confidence interval may be a good indicator of the quality of the approximation and a relevant stop criterion for the method. This example clearly shows the interest of the method in terms of decision-making according to the level of accuracy.

4.3. *Bending beam with uniform load*

This application deals with a bi-supported bending beam (length L between supports, rectangular cross section $b \times h$) with a uniform load q. Using the strength of

Table 2. *Exponential limit-state - Results synthesis. Results are presented in terms of: lower bound* $\left(\tilde{\beta}^{k,p}_{min}/\beta_{MC}\right)$*, estimation* $\left(\tilde{\beta}^{k,p}/\beta_{MC}\right)$*, upper bound* $\left(\tilde{\beta}^{k,p}_{max}/\beta_{MC}\right)$ *and convergence test* (ϵ_β)

DOE p	$N^{(p)}$	Max chaos order p	**Order k=1**			
			$\tilde{\beta}^{k,p}_{\mathtt{min}}/\beta_{MC}$	$\tilde{\beta}^{k,p}/\beta_{MC}$	$\tilde{\beta}^{k,p}_{\mathtt{max}}/\beta_{MC}$	ϵ_β
1	6	1	0,643	0,966	1,288	0,6677
2	12	2	0,894	0,944	0,994	0,1057
3	20	3	0,887	0,923	0,958	0,0772
4	30	4	0,896	0,925	0,954	0,0633
5	42	5	0,909	0,937	0,965	0,0603
6	56	6	0,905	0,938	0,971	0,0702
7	72	7	0,908	0,937	0,966	0,0619
DOE p	$N^{(p)}$	Max chaos order p	**Order k=2**			
			$\tilde{\beta}^{k,p}_{\mathtt{min}}/\beta_{MC}$	$\tilde{\beta}^{k,p}/\beta_{MC}$	$\tilde{\beta}^{k,p}_{\mathtt{max}}/\beta_{MC}$	ϵ_β
1	6	1	NA	NA	NA	NA
2	12	2	0,787	0,905	1,022	0,2600
3	20	3	0,984	1,005	1,025	0,0405
4	30	4	0,997	1,005	1,013	0,0157
5	42	5	0,997	1,005	1,012	0,0153
6	56	6	0,997	1,005	1,012	0,0150
7	72	7	1,000	1,006	1,011	0,0116
DOE p	$N^{(p)}$	Max chaos order p	**Order k=3**			
			$\tilde{\beta}^{k,p}_{\mathtt{min}}/\beta_{MC}$	$\tilde{\beta}^{k,p}/\beta_{MC}$	$\tilde{\beta}^{k,p}_{\mathtt{max}}/\beta_{MC}$	ϵ_β
1	6	1	NA	NA	NA	NA
2	12	2	NA	NA	NA	NA
3	20	3	0,882	0,941	1,000	0,1252
4	30	4	0,994	0,997	1,000	0,0057
5	42	5	0,997	0,998	1,000	0,0035
6	56	6	0,998	0,999	1,000	0,0023
7	72	7	0,998	0,999	1,000	0,0021
DOE p	$N^{(p)}$	Max chaos order p	**Order k=4**			
			$\tilde{\beta}^{k,p}_{\mathtt{min}}/\beta_{MC}$	$\tilde{\beta}^{k,p}/\beta_{MC}$	$\tilde{\beta}^{k,p}_{\mathtt{max}}/\beta_{MC}$	ϵ_β
1	6	1	NA	NA	NA	NA
2	12	2	NA	NA	NA	NA
3	20	3	NA	NA	NA	NA
4	30	4	1,000	1,000	1,000	0,0000
5	42	5	1,000	1,000	1,000	0,0000
6	56	6	1,000	1,000	1,000	0,0000
7	72	7	1,000	1,000	1,000	0,0000

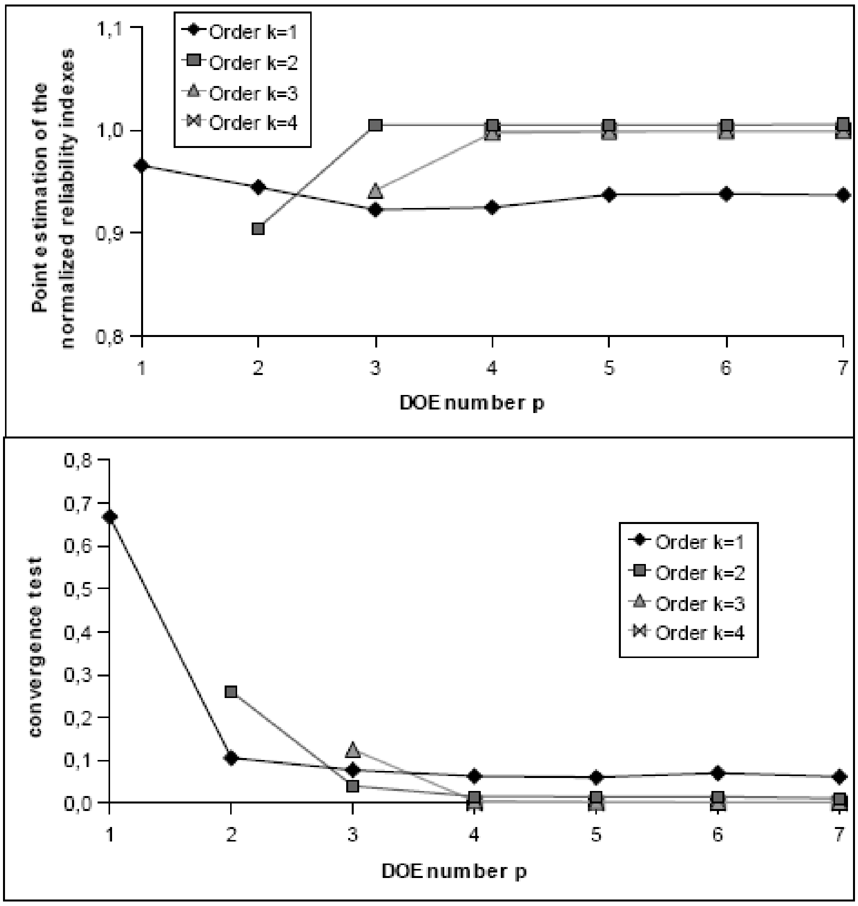

Figure 4. *Exponential limit-state - Top: Plot of the convergence of the ratio $\tilde{\beta}^{k,p}/\beta_{MC}$ following the DOE number p - Bottom: Plot of the evolution of the convergence test ϵ_β following the DOE number p*

material theory, the beam stress at the midpoint with a constant $b \times h$ rectangular cross section could be calculated:

$$\sigma = \frac{qhL^2}{16I} = \frac{3qL^2}{4bh^2} \qquad [19]$$

There are three random variables, the section height h, the load q and the beam width b. All these variables are modeled by a lognormal density (Table 3). Confidence level α is equal to 95% and the number B of resampling is set to 1000. According to a fatigue approach, the reliability index related to the probability that the stress σ exceeds the yield strength limit σ_D is computed:

$$\begin{aligned} P_f &= \text{Prob}\left[\sigma \geqslant \sigma_D\right] \\ \beta &= -\Phi^{-1}(P_f) \end{aligned}$$

The limit-state function reads:

$$G = \sigma_D - \sigma\left(q, h, b\right) \quad [20]$$

The reference reliability index is exact and calculated analytically because of the linearity of the performance function in the ξ-space. The reference value is $\beta_{exact} = 2.592$. We focus on the ratio $\tilde{\beta}^{k,p}/\beta_{exact}$. The convergence of the estimation of the reliability index and of the confidence interval width is shown Figure 5 and numerical results Table 4. As we can see, from 112 experiments chaos of order 3, 4 and 5 converge to β_{exact}. The polynomial chaos of order 2 is unable to correctly estimate the reliability index. However, the reference value is inside the confidence interval. The best choice is the polynomial chaos of order 5 with 112 experiments because the length of the confidence interval is zero and $\tilde{\beta}^{k,p}/\beta_{exact} = 1$: the expected value of β has been reached. However, the chaos of order 5 with 112 experiments is not necessarily the optimal choice. If we go into more details, the optimal choice is not obvious. Looking at Figure 5, the chaos of order 3 with 40 experiments is good enough to approximate the reliability index. Nevertheless, the analysis of statistical moments leads to a different conclusion. Concerning the estimation of the mean of the G function, from Figure 6 one can deduce that the chaos of order 3 is still a good choice but with 70 experiments. This statement is the same for the standard deviation (Figure 6). A closer look at the skewness and the kurtosis (Figure 6) shows that the polynomial chaos of order 4 with 70 experiments is an optimal choice. Finally, for this example, the optimal choice is the chaos of order 4 with 70 experiments. This way, the estimated value for the reliability index is $\tilde{\beta}^{k,p} = 2.5919$ and the 95% confidence interval is $[2.5909, 2.5928]$. Now, in the case where β_{exact} is unknown if we look at Figure 5 the optimal choice is the chaos of order 4 with 70 experiments.

REMARK. — In practice, the more the probability of failure is low, the more a fine approximation of the limit-state over a large domain is needed. This could involve an increase of the order of the PCE in the case of a low probability. From this statement of fact, we assume that the possibilities and limitations of the method are the ones inherent to the PCE. Besides, the analytical examples shows that the order of the PCE is correlated with the non-linearity of the limit-state and the level of probability.

Table 3. *Bending beam - Input variables*

Variable	PDF	Mean	c.o.v
σ_D	Deterministic	235 MPa	
L	Deterministic	6000 mm	
q	Lognormal	5 N/mm	0.2
b	Lognormal	102 mm	0.05
h	Lognormal	100 mm	0.05

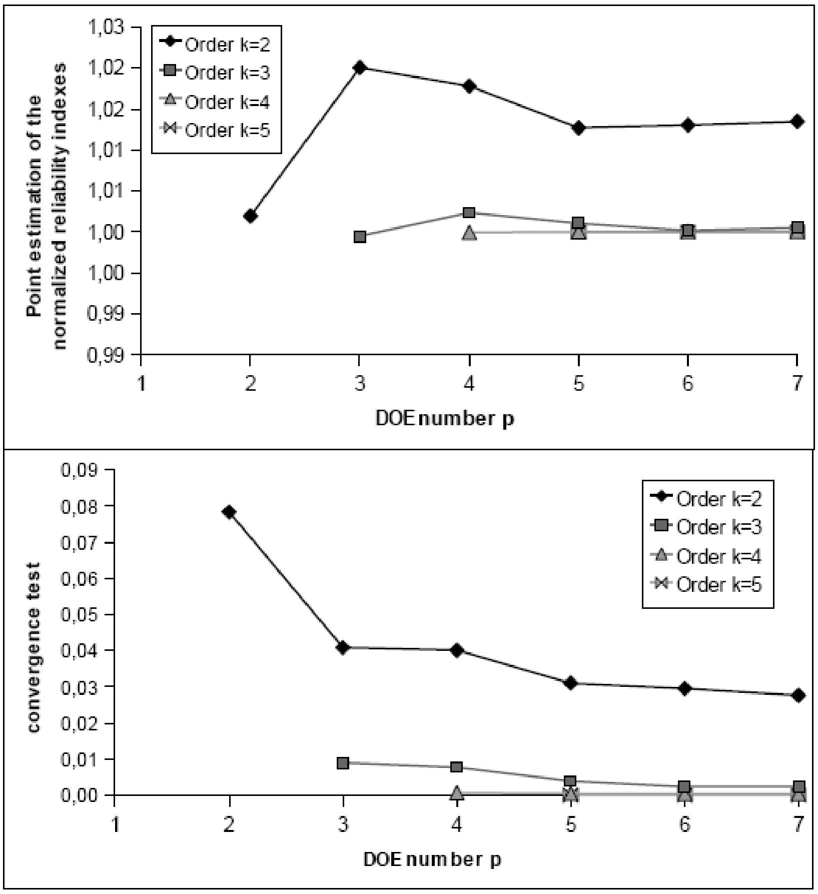

Figure 5. *Bending beam - Top: Plot of the convergence of the ratio $\tilde{\beta}^{k,p}/\beta_{exact}$ following the DOE number p - Bottom: Plot of the evolution of the convergence test ϵ_β following the DOE number p*

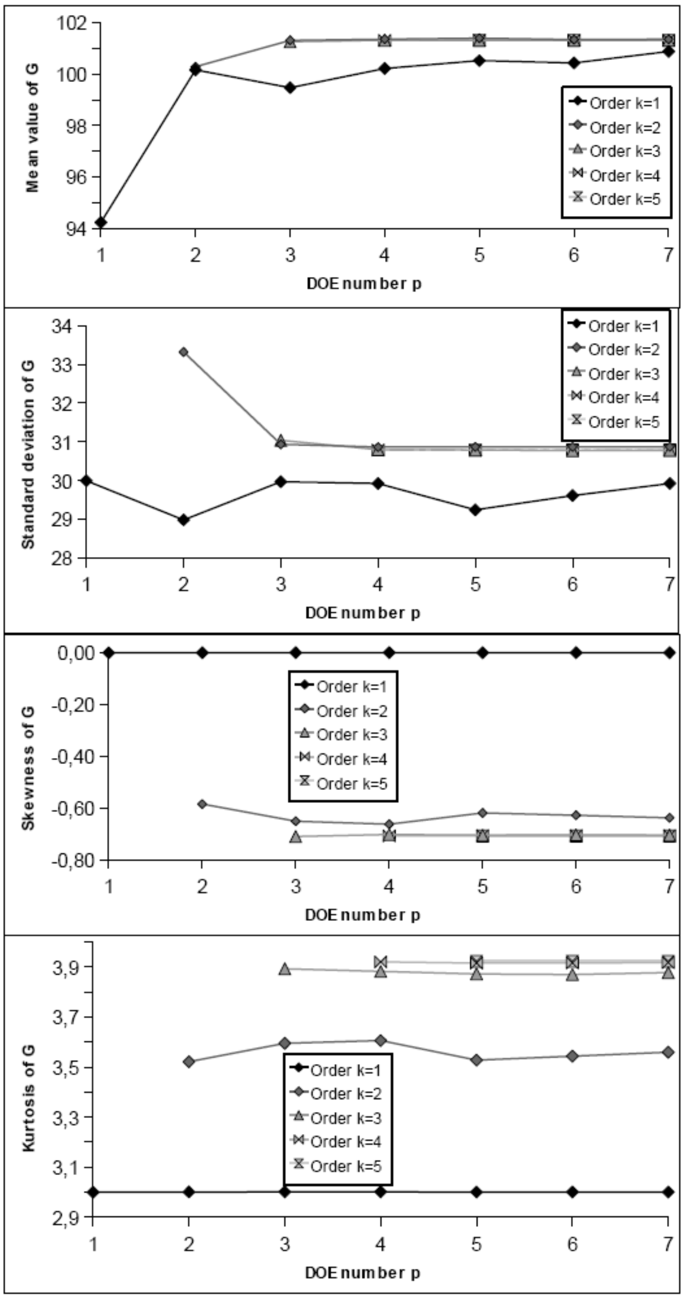

Figure 6. *Bending beam - Plot of the convergence of punctual estimation of the four first statistical moments on G*

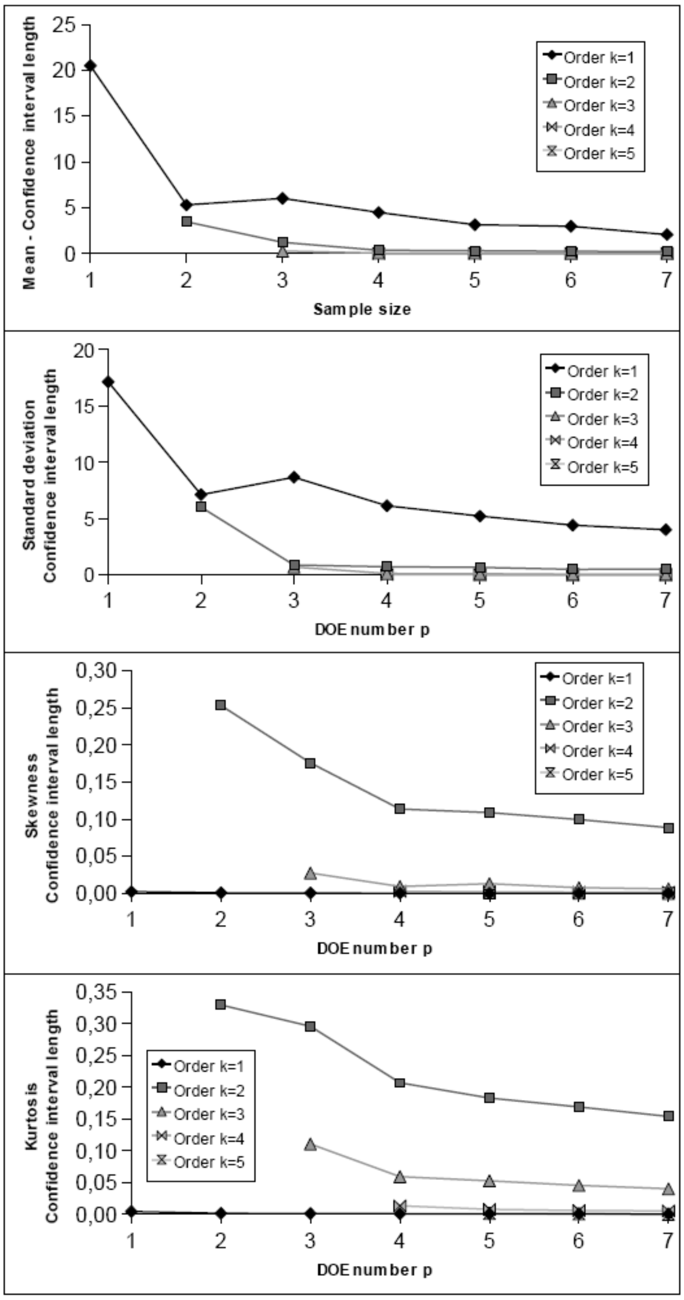

Figure 7. *Bending beam - Plot of the convergence of 95% confidence interval size on each moments of G*

Table 4. *Bending beam - Results synthesis. Results are presented in terms of: lower bound* $\left(\tilde{\beta}^{k,p}_{min}/\beta_{exact}\right)$*, estimation* $\left(\tilde{\beta}^{k,p}/\beta_{exact}\right)$*, upper bound* $\left(\tilde{\beta}^{k,p}_{max}/\beta_{exact}\right)$ *and convergence test* (ϵ_β)

DOE p	$N^{(p)}$	Max chaos order p	**Order k=2**			
			$\tilde{\beta}^{k,p}_{\mathtt{min}}/\beta_{exact}$	$\tilde{\beta}^{k,p}/\beta_{exact}$	$\tilde{\beta}^{k,p}_{\mathtt{max}}/\beta_{exact}$	ϵ_β
2	20	2	0,963	1,002	1,041	0,0782
3	40	3	1,000	1,020	1,040	0,0401
4	70	4	0,998	1,018	1,038	0,0394
5	112	5	0,997	1,013	1,028	0,0307
6	168	6	0,998	1,013	1,028	0,0292
7	240	7	1,000	1,013	1,027	0,0273
DOE p	$N^{(p)}$	Max chaos order p	**Order k=3**			
			$\tilde{\beta}^{k,p}_{\mathtt{min}}/\beta_{exact}$	$\tilde{\beta}^{k,p}/\beta_{exact}$	$\tilde{\beta}^{k,p}_{\mathtt{max}}/\beta_{exact}$	ϵ_β
2	20	2	NA	NA	NA	NA
3	40	3	0,995	0,999	1,004	0,0090
4	70	4	0,998	1,002	1,006	0,0078
5	112	5	0,999	1,001	1,003	0,0039
6	168	6	0,999	1,000	1,001	0,0023
7	240	7	0,999	1,001	1,002	0,0024
DOE p	$N^{(p)}$	Max chaos order p	**Order k=4**			
			$\tilde{\beta}^{k,p}_{\mathtt{min}}/\beta_{exact}$	$\tilde{\beta}^{k,p}/\beta_{exact}$	$\tilde{\beta}^{k,p}_{\mathtt{max}}/\beta_{exact}$	ϵ_β
2	20	2	NA	NA	NA	NA
3	40	3	NA	NA	NA	NA
4	70	4	1,000	1,000	1,000	0,0007
5	112	5	1,000	1,000	1,000	0,0005
6	168	6	1,000	1,000	1,000	0,0004
7	240	7	1,000	1,000	1,000	0,0004
DOE p	$N^{(p)}$	Max chaos order p	**Order k=5**			
			$\tilde{\beta}^{k,p}_{\mathtt{min}}/\beta_{exact}$	$\tilde{\beta}^{k,p}/\beta_{exact}$	$\tilde{\beta}^{k,p}_{\mathtt{max}}/\beta_{exact}$	ϵ_β
2	20	2	NA	NA	NA	NA
3	40	3	NA	NA	NA	NA
4	70	4	NA	NA	NA	NA
5	112	5	1,000	1,000	1,000	0,0000
6	168	6	1,000	1,000	1,000	0,0000
7	240	7	1,000	1,000	1,000	0,0000

4.4. *Academic examples: stop criterion definition*

The academic examples permit to assess the relevance of the criterion based on confidence interval length. In this example we can see that the precision on the reliability index is low if the confidence interval width on β is lower than 1% of the middle value. This value will be taken into account to stop the algorithm in the industrial example where no β reference value are available.

5. Application to finite elements models

5.1. *Fatigue analysis: the Dang Van criterion*

5.1.1. *Definition of the Dang Van criterion*

The Dang Van criterion (DangVan *et al.*, 1989) is a multiaxial fatigue criterion defined as follow (E_{DV} is the fatigue life time):

$$E_{DV} = \max_t \frac{\tau_{pr}(t) + \alpha P_H(t)}{\beta} \qquad [21]$$

with:

$$\tau_{pr}(t) = \frac{1}{2} \max \{|S_{1a}(t) - S_{2a}(t)|, |S_{2a}(t) - S_{3a}(t)|, |S_{3a}(t) - S_{1a}(t)|\} \qquad [22]$$

$S_{1a}(t), S_{2a}(t), S_{3a}(t)$ are the eigenvalues (or principal values) of the alternated deviatoric tensor $S_{ija}(t)$. α and β are material constants:

$$\alpha = 3\left(\frac{\tau_{-1}}{S_{-1}} - \frac{1}{2}\right), \beta = S_{-1} \qquad [23]$$

τ_{-1}, S_{-1}, are respectively the torsion endurance limit and the traction-compression endurance from uniaxial tests. The validity condition for the Dang Van criterion is:

$$\frac{\tau_{-1}}{S_{-1}} > \frac{1}{2} \qquad [24]$$

The fatigue criterion permits to position the multiaxial cycle against the material fatigue limit. If $E_{DV} < 1$ the criterion forecasts a crack nucleation beyond the fatigue lifetime of the material. On the contrary, if $E_{DV} > 1$ the crack initiates before the reach of the material fatigue limit.

5.1.2. *Use of the Dang Van criterion*

The use of the Dang Van model (Equation [21]) entails the computation of the coefficients α_{N_f} and β_{N_f}. These coefficients are computed using the torsion and bending Wöhler curves for the targeted life time (for computational purpose, Wöhler curves are approximated with a Basquin model). Once computed, the structure life time N_f is found numerically solving the following equation:

$$\tau_{pr} + \alpha_{N_f} P_H = \beta_{N_f} \qquad [25]$$

5.2. ***Academic model: fatigue failure of an angle bracket***

A finite element model that represents a steel angle bracket under fatigue loading is studied in this example. It was developed firstly in (Gayton *et al.*, 2009). The angle bracket, with its geometrical properties given Figure 8, is loaded by a scalar 10^6 cycles fatigue equivalent F_{eq} at its extremity. This parameter is obtained from a preliminary load sequence analysis. The damage indicator Γ is computed from an ANSYS finite element model. It is then compared to the value of 1.

The length d is set to 50 mm and material properties are supposed to be deterministic because we focus on the influence of the geometric uncertainties and the fatigue equivalent F_{eq}. The random variables are given Table 5.

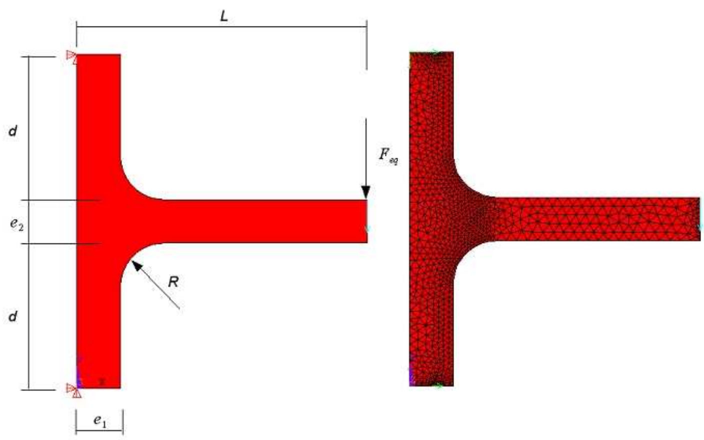

Figure 8. *Finite element model of the angle bracket under fatigue loading*

Reliability analysis Results are computed in terms of the generalized reliability index:

$$\beta = -\Phi\left(P_f\right)$$

Table 5. *Angle bracket - Input variables*

Variable	PDF	Mean	c.o.v
F_{eq} (N)	Normal	110	0.2
L (mm)	Normal	70	0.05
$e1$ (mm)	Normal	15	0.05
$e2$ (mm)	Normal	15	0.05
R (mm)	Normal	10	0.1
d (mm)	Deterministic	50	
E (GPa)	Deterministic	210	
τ_{-1} (MPa)	Deterministic	200	
S_{-1} (MPa)	Deterministic	300	

The probability of failure is defined as the probability that the damage indicator Γ is greater than 1:

$$P_f = \text{Prob}\left[\Gamma > 1\right] \quad [26]$$

and the limit-state function G:

$$G = 1 - \Gamma\left(F_{eq}, e1, e2, L, R\right) \quad [27]$$

For a given realization of the input parameters, a damage indicator Γ_i is computed using ANSYS then from Equation [27] the corresponding value G_i of the limit-state function is evaluated. Once all the mechanical computations ended, G is developed onto the polynomial chaos using full polynomial chaos whose coefficients are computed by regression (Equation [14]). The resampling procedure is applied on the polynomial chaos followed by sampling in order to compute the reliability index. The reliability index confidence interval is calculated using bootstrap BCa.

Simulation and results The parameters τ_{-1} and S_{-1} are set respectively to 200 MPa and 300 MPa. A Young's Modulus of 210 GPa and Poisson's ratio of 0.33 are selected for this application. The stop criterion is initialized to $\epsilon_\beta = 1\%$. Applying this criterion, the RPCM method stops at a polynomial chaos of order 5 with 924 experiments. Figure 9 shows the convergence of the reliability index $\tilde{\beta}^{k,p}$: the minimal size to obtain the convergence is 112. Combining the results of the two graphs, only the polynomial chaos of order 5 with 924 experiments satisfies the convergence test. Numerical results are summed up Table 6. Finally, using the RPCM method, the best representation for the performance function is a polynomial chaos of order 5 with 924 experiments. This way, the estimated value for the reliability index is $\beta = 3.3204$

and the confidence interval is $[3.3055, 3.3335]$. The reliability index $\beta = 3.3204$ corresponds to $P_f \approx 10^{-4}$ which requires roughly 10^6 MC runs. As a result, the reduction of the cost computation-time is obvious.

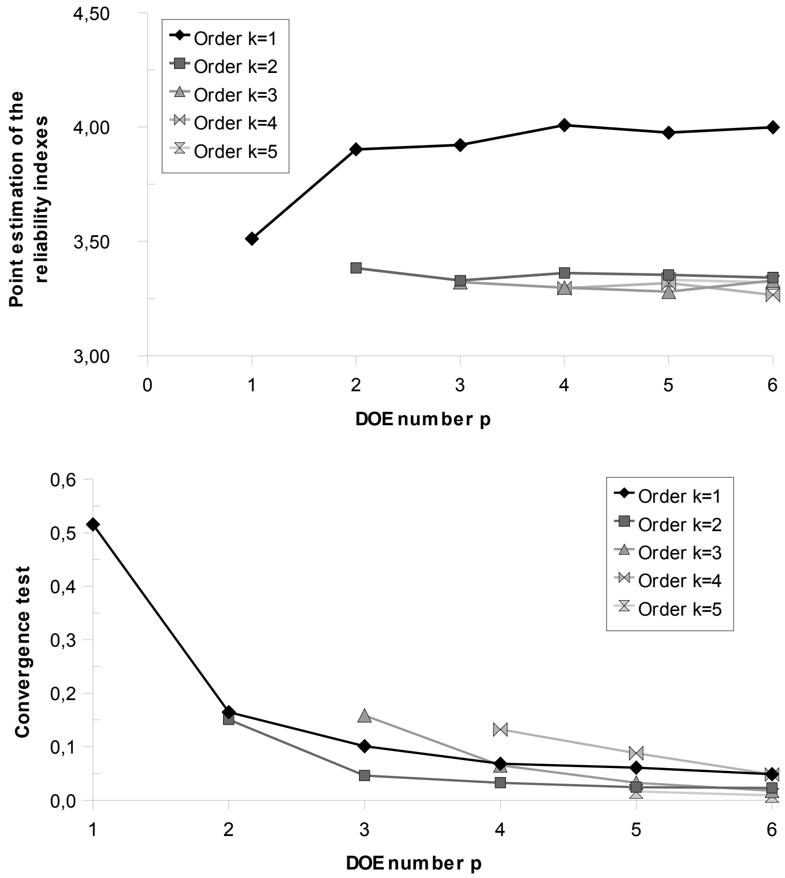

Figure 9. *Angle bracket - Top: Plot of the convergence of the ratio $\tilde{\beta}^{k,p}$ following the DOE number p - Bottom: Plot of the evolution of the convergence test ϵ_β following the DOE number p*

5.3. *Industrial model: Bogie support fixing*

The next considered problem is a train Bogie support fixing made of steel S235JR. The mesh (generated by ANSYS) is composed of 7515 10-Node Tetrahedral elements.

Table 6. *Angle bracket - Results synthesis. Results are presented in terms of: lower bound* $\left(\tilde{\beta}^{k,p}_{min}\right)$*, estimation* $\left(\tilde{\beta}^{k,p}\right)$*, upper bound* $\left(\tilde{\beta}^{k,p}_{max}\right)$ *and convergence test* (ϵ_β)

DOE p	$N^{(p)}$	Max chaos order p	**Order k=1**			
			$\tilde{\beta}^{k,p}_{\mathtt{min}}$	$\tilde{\beta}^{k,p}$	$\tilde{\beta}^{k,p}_{\mathtt{max}}$	ϵ_β
1	12	1	2,6066	3,5119	4,4172	0,5156
2	42	2	3,5813	3,9027	4,2240	0,1647
3	112	3	3,7231	3,9212	4,1193	0,1010
4	252	4	3,8719	4,0084	4,1449	0,0681
5	504	5	3,8544	3,9753	4,0962	0,0608
6	924	6	3,0920	3,9991	4,0962	0,0486
DOE p	$N^{(p)}$	Max chaos order p	**Order k=2**			
			$\tilde{\beta}^{k,p}_{\mathtt{min}}$	$\tilde{\beta}^{k,p}$	$\tilde{\beta}^{k,p}_{\mathtt{max}}$	ϵ_β
1	12	1	NA	NA	NA	NA
2	42	2	3,1285	3,3834	3,6382	0,1506
3	112	3	3,2520	3,3289	3,4058	0,0462
4	252	4	3,3070	3,3618	3,4166	0,0326
5	504	5	3,3127	3,3535	3,3943	0,0243
6	924	6	3,3033	3,3418	3,3803	0,0230
DOE p	$N^{(p)}$	Max chaos order p	**Order k=3**			
			$\tilde{\beta}^{k,p}_{\mathtt{min}}$	$\tilde{\beta}^{k,p}$	$\tilde{\beta}^{k,p}_{\mathtt{max}}$	ϵ_β
1	12	1	NA	NA	NA	NA
2	42	2	NA	NA	NA	NA
3	112	3	3,0586	3,3222	3,5857	0,1587
4	252	4	3,1904	3,2978	3,4052	0,0651
5	504	5	3,2261	3,2798	3,3334	0,0327
6	924	6	3,2997	3,3289	3,3580	0,0175
DOE p	$N^{(p)}$	Max chaos order p	**Order k=4**			
			$\tilde{\beta}^{k,p}_{\mathtt{min}}$	$\tilde{\beta}^{k,p}$	$\tilde{\beta}^{k,p}_{\mathtt{max}}$	ϵ_β
1	12	1	NA	NA	NA	NA
2	42	2	NA	NA	NA	NA
3	112	3	NA	NA	NA	NA
4	252	4	3,0776	3,2953	3,5130	0,1321
5	504	5	3,1720	3,3177	3,4633	0,0878
6	924	6	3,1878	3,2661	3,3444	0,0480
DOE p	$N^{(p)}$	Max chaos order p	**Order k=5**			
			$\tilde{\beta}^{k,p}_{\mathtt{min}}$	$\tilde{\beta}^{k,p}$	$\tilde{\beta}^{k,p}_{\mathtt{max}}$	ϵ_β
1	12	1	NA	NA	NA	NA
2	42	2	NA	NA	NA	NA
3	112	3	NA	NA	NA	NA
4	252	4	NA	NA	NA	NA
5	504	5	3,3053	3,3323	3,3593	0,0162
6	924	6	3,3055	3,3204	3,3352	0,0089

Reliability analysis Results are computed in terms of the generalized reliability index:

$$\beta = -\Phi\left(P_f\right)$$

The probability of failure is defined as the probability that the fatigue lifetime is lower than the 10^6 target value:

$$P_f = \text{Prob}\left[N_f < 10^6\right] \qquad [28]$$

and the limit-state function G:

$$G = N_f - N\left(P, \tau_{-1}, \sigma_{-1}\right) \qquad [29]$$

For a given realization of the input parameters, the fatigue lifetime N_i is computed using the Dang Van model (Equation [21]) then from equation 29 the corresponding value G_i of the limit-state function is evaluated. Once all the mechanical computations ended, G is developed onto the polynomial chaos using full polynomial chaos whose coefficients are computed by regression (Equation [14]). The resampling procedure is applied on the polynomial chaos followed by sampling in order to compute the reliability index. Reliability index confidence interval is calculated using bootstrap BCa.

Simulation and results Due to data privacy, results are normalized using a reference value β_{ref} which will be considered as unknown. Boundary conditions are shown Figure 10. There are three boundary conditions:

1) The gravity: $g = 9806.6 mm/s^2$;

2) A pressure $P = 0.2 MPa$ is applied to the surface in contact with the transverse part of the Bogie;

3) Displacements are blocked in all directions for each hole.

There are three random variables: the pressure P, the shear fatigue limit τ_{-1} and the tensile fatigue limit σ_{-1}. τ_{-1} and σ_{-1} are modeled by a lognormal density and the pressure P by a truncated lognormal density in the [0.18 0.3] interval (Table 7). The Young's modulus is deterministic because its variability does not affect fatigue results when using the Dang Van criterion (Equation [21]). It is supposed to be constant and equal to 200 GPa. Load ratio is -1 (symmetric alternated). The confidence level is set to 95% with $B = 1000$ resamples. The fatigue limit objective is $N_f = 10^6$ cycles. In this example, the stop criterion is setup to $\epsilon_\beta = 1\%$.

Numerical results are summed up Table 8. Using this level of accuracy, the RPCM method stops on a polynomial chaos of order 3 with 40 experiments. However, in order to investigate the behavior of the RPCM method, we did more experiments increasing the chaos order. The convergence graph of $\widetilde{\beta}^{k,p}/\beta_{ref}$ is plotted Figure 11: the polynomial chaos of order 1 never converge when from a DOE of size 40 polynomial chaos of order 2, 3 and 4 converge. Besides, Figure 11 demonstrates that only chaoses of order 3 and 4 satisfy the convergence test whatever the size of the DOE. Moreover, Figure 11 shows that an increase of the chaos order does not change the accuracy and the performance of the RPCM method. To conclude, according to the RPCM method, with an error of 1%, the best meta-model is a polynomial chaos of order 3 with 40 experiments.

Table 7. *Bogie - Parameters definition*

Variable	PDF	Mean	c.o.v	Truncation
P (MPa)	Lognormal	0.2	0.2	$[0.18 - 0.3]$
σ_{-1} (MPa)	Lognormal	213	0.1	
τ_{-1} (MPa)	Lognormal	127.8	0.1	
E (GPa)	Deterministic	200		

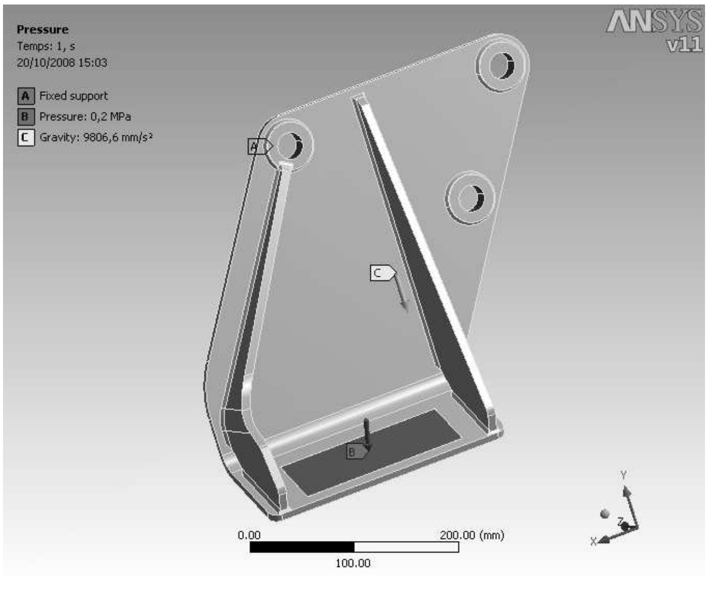

Figure 10. *Bogie - Boundaries conditions*

Table 8. *Bogie - Results synthesis. Results are presented in terms of: lower bound* $\left(\tilde{\beta}^{k,p}_{min}\right)$*, estimation* $\left(\tilde{\beta}^{k,p}\right)$*, upper bound* $\left(\tilde{\beta}^{k,p}_{max}\right)$ *and convergence test* (ϵ_β)

DOE p	$N^{(p)}$	Max chaos order p	**Order k=1**			
			$\tilde{\beta}^{k,p}_{\mathtt{min}}$	$\tilde{\beta}^{k,p}$	$\tilde{\beta}^{k,p}_{\mathtt{max}}$	ϵ_β
2	20	2	0,9083	1,0844	1,2604	0,3248
3	40	3	0,9041	1,0449	1,1858	0,2696
4	70	4	0,9110	1,0215	1,1319	0,2163
5	112	5	0,9654	1,0278	1,0901	0,1213
6	168	6	0,9163	0,9942	1,0721	0,1567
7	240	7	0,9135	0,9792	1,0449	0,1342
DOE p	$N^{(p)}$	Max chaos order p	**Order k=2**			
			$\tilde{\beta}^{k,p}_{\mathtt{min}}$	$\tilde{\beta}^{k,p}$	$\tilde{\beta}^{k,p}_{\mathtt{max}}$	ϵ_β
2	20	2	0,8338	0,9197	1,0055	0,1867
3	40	3	0,9727	0,9895	1,0063	0,0339
4	70	4	0,9785	0,9913	1,0041	0,0259
5	112	5	0,9811	0,9923	1,0035	0,0226
6	168	6	0,9808	0,9912	1,0015	0,0208
7	240	7	0,9812	0,9914	1,0016	0,0206
DOE p	$N^{(p)}$	Max chaos order p	**Order k=3**			
			$\tilde{\beta}^{k,p}_{\mathtt{min}}$	$\tilde{\beta}^{k,p}$	$\tilde{\beta}^{k,p}_{\mathtt{max}}$	ϵ_β
2	20	2	NA	NA	NA	NA
3	40	3	0,9953	0,9978	1,0004	0,0051
4	70	4	0,9982	0,9992	1,0002	0,0019
5	112	5	0,9987	0,9994	1,0002	0,0015
6	168	6	0,9989	0,9996	1,0002	0,0013
7	240	7	0,9991	0,9996	1,0002	0,0011
DOE p	$N^{(p)}$	Max chaos order p	**Order k=4**			
			$\tilde{\beta}^{k,p}_{\mathtt{min}}$	$\tilde{\beta}^{k,p}$	$\tilde{\beta}^{k,p}_{\mathtt{max}}$	ϵ_β
2	20	2	NA	NA	NA	NA
3	40	3	NA	NA	NA	NA
4	70	4	0,9999	1,0000	1,0002	0,0003
5	112	5	0,9999	1,0000	1,0000	0,0001
6	168	6	1,0000	1,0000	1,0000	0,0000
7	240	7	1,0000	1,0000	1,0000	0,0000

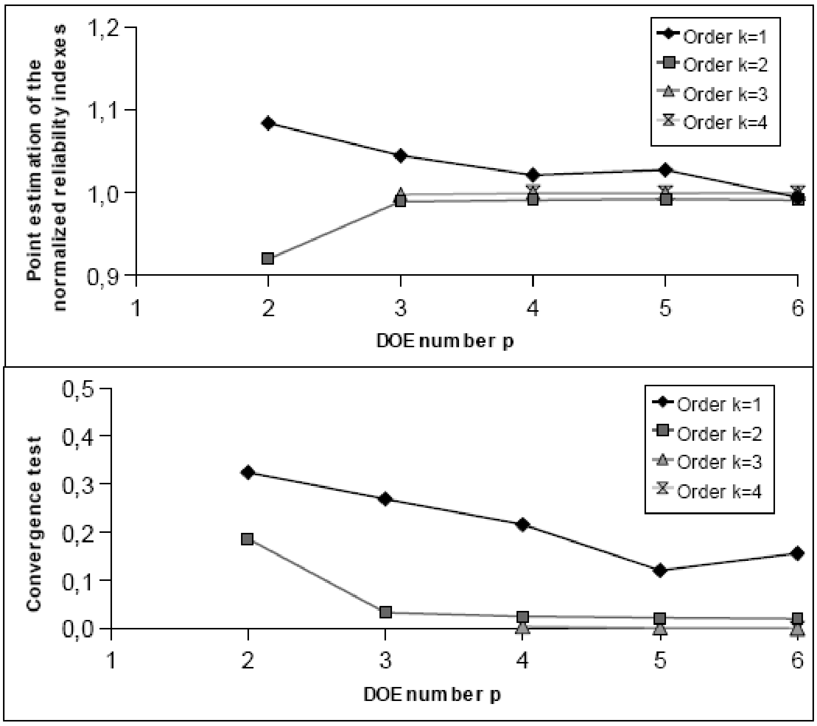

Figure 11. *Bogie - Top: Plot of the convergence of the ratio $\tilde{\beta}^{k,p}/\beta_{ref}$ following the DOE number p - Bottom: Plot of the evolution of the convergence test ϵ_β following the DOE number p*

6. Conclusion

Recently, (Blatman *et al.*, 2008) introduced the so-called sparse polynomial chaos expansion. The goal is to reduce the dimensionality of the PC basis. The strategies for truncating the PC expansion are inspired by the *sparsity-of-effects* principle, which states that most models are principally governed by main effects and low-order interactions. In (Blatman *et al.*, 2010), a new truncation scheme (referred to as *hyperbolic*) is proposed which retains in priority the basis terms associated with low-order interactions. The sparse PC expansion consists in an iterative procedure based on stepwise regression. To assess the meta-model, a leave-one-out error estimator is used. It is based on cross-validation technique. As the RPCM method is like a wrapper, from our point of view there are no difficulties to integrate sparse PCE which is a perspective for future work. Even if the polynomial chaos are very useful for sensitivity analysis, competing approaches are developed for reliability analysis: Support Vector Machine

(Deheeger *et al.*, 2007), Kriging (Echard *et al.*, 2009). It will be interesting to quantify the influence of such techniques onto the confidence intervals on the reliability index.

The proposed approach takes advantage of the use of polynomial chaos expansion for reliability analysis. It is applied selecting full successive polynomial chaos order and takes advantage of two efficient methods to perform reliability analysis: the polynomial chaos to build the response and resampling techniques for validation. The Bootstrap is used to compute confidence intervals which indicate the reliability of an estimate according to a confidence level. This way, confidence intervals are a natural complement to the reliability analysis. The originality comes from combining such methods in an industrial context to improve design taking advantages of existing DOE. Providing a learning step and an enrichment step, the RPCM method finds the best chaos order according to the reliability and accuracy goal. The method allows to explore the database too. The truncation order of the polynomial expansion is linked to the accuracy on estimating the reliability index. This way, RPCM is a tool to help to build meta-models taking the available data into account. In the case where the DOE is known *a priori* and cannot be modified, the best polynomial chaos degree will be the one which gives the smallest confidence interval. This way, one can decide if the DOE contains enough information to give an accurate result. The information provided by the confidence interval is very important because the knowledge of the reliability index range gives confidence or not in the result. Such an information helps to make a decision providing important information on the accuracy of the failure probability predictions knowing the input randomness. Furthermore, it can help to assess product life extension or improve the fixing of inspection times.

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