Polarization of contact forces in multi-contact systems

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ABSTRACT. *The aim of this study is to identify the homogenized laws modeling the overall behaviour of multi-contact systems. At the moment, these systems are generally analyzed either by continuum mechanics or micro-mechanics and a multi-scale approach. These approaches differ from the phenomenological approach traditionally used for modeling the behavior of solid materials which is based on mathematical formulations developed in the framework of thermodynamics, whose constants are determined from results of laboratory tests. The lack of basic physics in these formulations leads to mathematical models that are often complex and difficult to identify. The multi-scale approach appears well suited to address these difficulties. This study aims at quantification using the Discrete Element method (Jean, 1999; Fortin et al., 2005) polarization phenomena of contact forces.*

RÉSUMÉ. *Ce travail consiste à identifier des lois de comportement homogénéisées rendant compte du comportement global des systèmes multicontacts. A l'heure actuelle, ces systèmes sont généralement analysés avec la mécanique des milieux continus et la micromécanique ou approche multi-échelle. Cette étude a pour but de quantifier, à l'aide de la méthode des eléments discrets, (Jean, 1999 ; Fortin et al., 2005) les phénomènes de polarisation des efforts de contact.*

KEYWORDS: *discrete element method, multi-scale method, friction.*

MOTS-CLÉS : *méthode des éléments discrets, méthode multi-échelle, frottement.*

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1. Context

The aim of this study is to identify the homogenized behavior laws modeling the overall behavior of multi-contact systems. At the moment, these systems are generally analyzed either with continuum mechanics or micro-mechanics and multi-scale approach. These approaches differ from the phenomenological approach traditionally used for modeling the behavior of solid materials, which is based on mathematical formulations developed in the framework of thermodynamics, whose constants are determined from results of laboratory tests. The lack of basic physics in these formulations leads to mathematical models that are often complex and difficult to identify. The multi-scale approach appears well suited to address these difficulties. This study aims to quantify polarization phenomena of contact forces using the Discrete Element method (Jean, 1999; Fortin *et al.*, 2005). Polarization means that the main part of the contact forces in a relevant zone, for example on the walls of a silo, are on the boundary of the Coulomb cone.

2. Continuous approaches

Classically, to numerically study the behavior of a deformable body submitted to various solicitations, we use the Finite Element Method (FEM) which is an important tool for the analysis of structures. It is based on the Mechanics of Continuous Fields. Even though the bodies that are considered are not continuous, the hypothesis of continuity brings a simplification which makes possible the resolution of the problems of classic mechanics.

2.1. *Stability of multi-contacts systems*

Contrary to a liquid, a granular material can offer an oblique free surface. For a dry and non cohesive material, it is, however, impossible to tilt vertically : a pile of sand or earth cannot form a bank possessing a bigger slope than a certain critical angle according to the material and the geometry of the sand pile. Careful observation of the formation of a sand pile shows in fact that there are two critical angles : the angle of starting up θ_{start}, and the angle of stopping θ_{stop}. As soon as the angle of inclination exceeds θ_{start} an avalanche occurs : the material slides on the surface, which reduces the slope of the sand pile. This flow stops as soon as the angle of inclination becomes lower than θ_{stop}.

The most direct interpretation would be to associate this coefficient of macroscopic friction with the microscopic friction between grains. Nevertheless, a pile of perfectly smooth grains, as soon as the first layer is maintained fixed, can also lead to an oblique free surface. The grain/grain friction is thus not necessary for the existence of a critical angle. Where then does this stability come from ? A first element comes from the rigidity of the grains. To start moving, the pile has to dilate.

For dense flows on a free surface, the integrated approach of Saint-Venant is the most widely used. It consists in integrating the equations of preservation on the thickness of the layer into movement.

$$\frac{\partial h}{\partial t}(x,t) = -\nabla \cdot q(x,t) + f(x,t) \qquad [1]$$

This approach allows freedom the internal rheology of the layer in movement by considering only its averaged effects on the thickness of the layer in movement. The missing relations are generally chosen so as to report the studied phenomenon. We then obtain a complete set of equations which describes the dynamics of the granular material through the evolution of overall heights as the thickness of the layer in movement or the average flow. The flow of the granular material is connected in a thin boundary layer moving down slopes of the growing sand pile. There exists a function $m(x,t) \geq 0$ such that :

$$q(x,t) = -m(x,t)\nabla h(x,t). \qquad [2]$$

For numerical simulation, this equation has been discretized in time with an implicit Euler scheme, and a Finite Element Method is used for the space discretization of the domain Omega, Figure 1 (see (Dumont *et al.*, 2009) for more details).

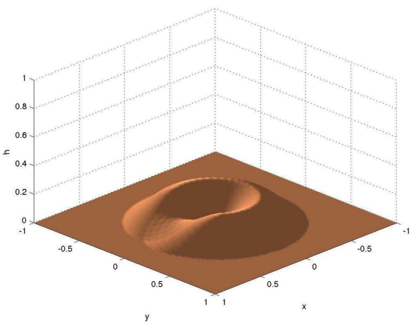


Figure 1. *The sand source is localized upon the center of the domain*

However, this type of approach is purely descriptive and does not allow an understanding of the microscopic mechanisms responsible for continuous behaviors. On the scale of the grain, the two fundamental questions on the statics of granular assemblies concern the distribution of strengths when the assembly is at equilibrium and the stability conditions of this equilibrium.

2.2. *Rheology of the ensiled materials*

One of the important problems in the design of silos is the prediction of the vertical wall stress, induced by the stored granular material. This stress distribution depends of course on the granular material parameters and on the flow during the process of discharge. By modeling the ensiled granular medium as a continuous medium we can obtain relatively simple models for the calculation of the vertical wall stress. In 1885, Janssen (Janssen, 1895) offered a first model that provided a qualitative understanding of the saturation effect in granular silos. The idea of the Janssen model is that the vertical wall stress does not vary linearly with depth. The simplicity of the Janssen model (Janssen, 1895) comes because the granular media is considered as a continuous environment in a quasi-static state, that is that movements inside the silo are small enough for the global movement of the material to be considered as equal to zero. The main directions are in a perpendicular vertical plane on both walls of the silo (Figure (2)).

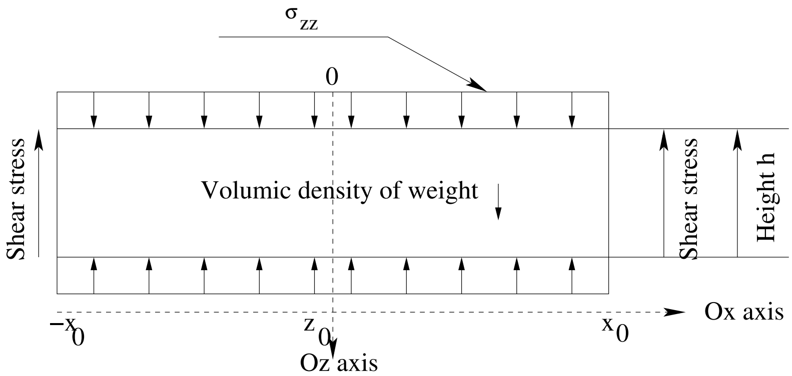


Figure 2. *Representation of a slice of ensiled matter*

The Janssen model also supposes the existence of a constant ratio λ between the horizontal component of the stress tensor and its vertical component. This implies that $\lambda = \frac{\sigma_{xx}}{\sigma_{zz}}$ is constant in all the silo. On the other hand, the ensiled material is supposed to be in a state of break described by the Mohr-Coulomb criterion

$$\tau_{xz} = \mu\sigma_{xx} \qquad \text{with} \qquad \mu = \tan\phi, \qquad [3]$$

where σ_{xx}, σ_{zz} indicate the main stress and τ_{xz} the shear stress. Finally, the theory of Janssen is limited to the non cohesive, isotropic and homogeneous materials. Thus, Janssen was interested in the determination of the stress tensor assumed to depend only on z. The origin of the axis z in the silo corresponds to the free surface of the material. On the other hand, Janssen considers that the upper surface of the ensiled material is without stress ($\sigma_{zz} = 0$ en $z = 0$). We finally obtain the stress expression

$$\begin{aligned} \sigma_{zz} &= \frac{\gamma R_h}{\mu\lambda}\left(1 - \exp\left(-\frac{\mu\lambda}{R_h}z\right)\right) \\ \sigma_{xx} &= \lambda\sigma_{zz} \\ \tau_{xz} &= \mu\lambda\sigma_{zz} \end{aligned} \quad [4]$$

where γ represents the volume weight, and R_h indicates the hydraulic beam. However, the Janssen theory rests on certain contradictory hypotheses and contains some limitations ; the Janssen theory assumes that σ_{xx} et σ_{zz} are the main stress connected by a constant coefficient in all the silo. These limitations of the Janssen theory explain the significant gaps existing between the stresses measured experimentally and those predicted by the Janssen theory. We notice that the hypothesis of continuity seems difficult to admit for systems which consist of several stiff or deformable parts, which are inter-connected. We then speak of multi-contact systems (Rahmoun, 2006).

3. Discrete approach

In fields where a collection of bodies between which the one-sided connections, usually affected by Coulomb friction, may become established or broken, these equations of constraints lead to a problem of non linear complementarity which cannot be solved by a linear programming method. The separation of surfaces in the case of sliding results from non compliance with the hypothesis of normality which implies a speed corresponding to the dilation of the interface.

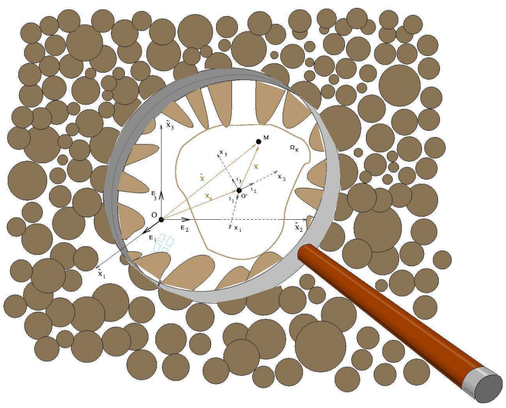


Figure 3. *Multi-contact systems*

To each couple of particles Ω_i and Ω_j which may enter in contact, is associated a local reference whose axes are oriented according to the two unit vectors $\mathbf{n}$ and $\mathbf{t}$, respectively, normal and tangential vectors in the contact plane. The normal $\mathbf{n}$ is directed from Ω_j to Ω_i. The dual variables are $\dot{\mathbf{u}}$, the local relative velocity between Ω_i and Ω_j, and the contact reaction $\mathbf{r}$ of Ω_j on Ω_i. In the local base, they are written by

$$\dot{\mathbf{u}} = \dot{\mathbf{u}}_t + \dot{u}_n.\mathbf{n}\ , \qquad \mathbf{r} = \mathbf{r_t} + r_n.\mathbf{n}\ , \qquad [5]$$

where $\dot{u}_n$ is the normal velocity, $\dot{\mathbf{u}}_\mathbf{t}$ the sliding velocity, r_n the contact pressure and $\mathbf{r_t}$ the friction force. The introduction of Coulomb friction K_μ defined by

$$K_\mu = \{(r_n, \mathbf{r}_t) \text{ such that } f(r_n, \mathbf{r}_t) = ||\mathbf{r}_t|| - \mu r_n \leq 0\} \qquad [6]$$

leads to a non-linear problem which cannot be solved by a linear programming method. It is possible to prove that the Coulomb unilateral contact law with dry friction does not satisfy the cyclic monotonic condition of Rockaffelar. Thus there does not exist an associated formulation in terms of sub-differential using a pseudo-potential. However one can associate a bi-potential as follows :

$$b_c(-\dot{\mathbf{u}}, \mathbf{r}) = \Psi_{\mathbb{R}^-}(-\dot{u}_n) + \Psi_{K_\mu}(\mathbf{r}) + \mu r_n|| - \dot{\mathbf{u}}_t||. \qquad [7]$$

The condition of non inter-penetrability $\dot{u}_n \geq 0$ is represented by the indicatory function of $\mathbb{R}^-$, noted $\Psi_{\mathbb{R}^-}(-\dot{u}_n)$, which is equal to zero when $-\dot{u}_n \leq 0$ and to $+\infty$ otherwise. The contact bi-potential also takes infinite values if the condition $\mathbf{r} \in K_\mu$ is not satisfied. This bi-potential of contact is bi-convex (convex with respect to each of the variables) and satisfies :

$$\forall - \dot{\mathbf{u}}, \mathbf{r} \in \mathbb{R}^3, \qquad b_c(-\dot{\mathbf{u}}, \mathbf{r}) \geq -\dot{\mathbf{u}}.\mathbf{r}. \qquad [8]$$

Moreover, the couples for whom equality is reached in the previous relation, are called extremal couples :

$$b_c(-\dot{\mathbf{u}}, \mathbf{r}) = -\dot{\mathbf{u}}.\mathbf{r} \Leftrightarrow \mu r_n|| - \dot{\mathbf{u}}_t|| = -(\dot{\mathbf{u}}_t.\mathbf{r}_t + \dot{u}_n r_n). \qquad [9]$$

These couples verify the Coulomb unilateral contact law with dry friction and the inverse law, which can be written implicitly :

$$-\dot{\mathbf{u}} \in \partial_r b_c(-\dot{\mathbf{u}}, \mathbf{r}), \qquad \mathbf{r} \in \partial_{-\dot{\mathbf{u}}} b_c(-\dot{\mathbf{u}}, \mathbf{r}) \qquad [10]$$

where $\partial_x b_c$ denotes the sub-differential of b_c with respect to the variable x. Classically the resolution of the Coulomb unilateral contact law with dry friction needs two principles of minimization : the first one for unilateral contact, and the second one for friction. The use of the bi-potential only needs one variational principle where the contact and the friction are coupled. For the resolution of the unilateral contact law with bi-potential formalism, we use the augmented Lagrangian method. First, let us write a relation as follows :

$$\forall \mathbf{r}' \in K_\mu, \quad b_c(-\dot{\mathbf{u}}, \mathbf{r}') - b_c(-\dot{\mathbf{u}}, \mathbf{r}) \geq -\dot{\mathbf{u}}(\mathbf{r}' - \mathbf{r}). \qquad [11]$$

Now let us choose a positive arbitrary coefficient ρ, whose value will be fixed later to ensure the numerical convergence of the algorithm. Then inequality (11) can be written :

$$\forall \mathbf{r}' \in K_\mu, \quad \rho b_c(-\dot{\mathbf{u}}, \mathbf{r}') - \rho b_c(-\dot{\mathbf{u}}, \mathbf{r}) + [\mathbf{r} - (\mathbf{r} + \rho(-\dot{\mathbf{u}}))].(\mathbf{r}' - \mathbf{r}) \geq 0. \qquad [12]$$

Using now the definition (7) of the contact bi-potential, relation (12) becomes with $\dot{u}_n \geq 0$ and $\mathbf{r} \in K_\mu$:

$$\forall \mathbf{r}' \in K_\mu, \qquad (\mathbf{r} - \tau).(\mathbf{r}' - \mathbf{r}) \geq 0 \qquad [13]$$

where

$$\tau = \mathbf{r} - \rho[\dot{\mathbf{u}}_t + (\dot{u}_n + \mu|| - \dot{\mathbf{u}}_t||).\mathbf{n}]$$

denotes the augmented reaction. Relation (13) implies that $\mathbf{r}$ is the projection of τ onto Coulomb cone :

$$\mathbf{r} = proj(\tau, K_\mu) \qquad [14]$$

and can be solved with a Usawa-like algorithm ; let $(-\dot{\mathbf{u}}^i, \mathbf{r}^i)$ be an approximation of $(-\dot{\mathbf{u}}, \mathbf{r})$ at the iteration i. Then the calculus of $\mathbf{r}^{i+1}$ is split into two steps :

$$\begin{array}{ll} \text{prediction :} & \tau^{i+1} = \mathbf{r}^i - \rho[\dot{\mathbf{u}}_t^i + (\dot{\mathbf{u}}_n^i + \mu|| - \dot{\mathbf{u}}_t^i||).\mathbf{n}] \,, \\ \text{correction :} & \mathbf{r}^{i+1} = proj(\tau^{i+1}, K_\mu). \end{array} \qquad [15]$$

where the projection of Coulomb's cone leads, according to the value of τ, to one of these states : no contact, contact with adherence or sliding contact, (Figure 4). Conventionally, at each time step, the contact forces in the system are determined repeatedly by the method of successive balances based on a Gauss-Seidel algorithm for the 2D version. Each contact force is calculated by adopting temporary values over the other contacts. Convergence is obtained when the force confirms the unilateral contact law with dry friction.

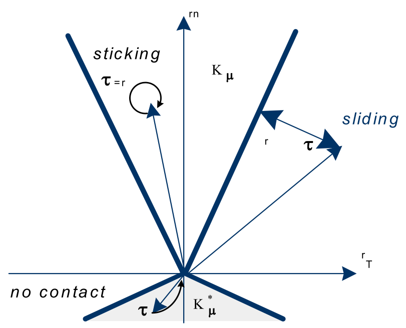


Figure 4. *Projection of the augmented reaction onto the friction cone*

4. Homogenized laws of behavior

While the stress tensor has for a long time been defined for a continuous environment, the definition of an average stress tensor from the strengths being applied on the scale of the grain itself is much less clear for granular media. On one hand, there are several different definitions which do not generally lead to a symmetric average stress tensor. We can find a comparison between the various expressions obtained as well as an analysis of an order of height of the antisymmetric part of the average stress tensors defined. Eventually, it is important to note that all these works are based on a static analysis of the problem, and are valid only for a quasi-static arrangement of granular media.

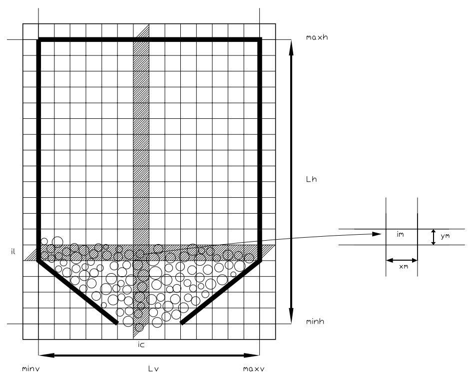


Figure 5. *Homogenization procedures : to define a Representation Elementary Volume*

In the dynamic state, only J.J. Moreau proposes a generalization of the previous definitions based on the notion of internal moment. Hence, two problems still remain open ; the first concerns the choice of a substantial definition of the average stress tensor in a granular medium in quasi-static conditions. The second concerns an extension of this definition in dynamics. To try to clarify the problem, we have proposed a general definition of the average stress tensor for a granular medium, valid in statics as well as in dynamics. This definition, which follows (de Saxcé *et al.*, 2004) takes into account not only the strengths of contact, but also the strengths of volume (gravity and slowness) being applied on the scale of grains. The average stress tensor so defined possesses the properties of a Cauchy stress tensor : it is symmetric and invariant by translation.

$$\Sigma_p = \frac{1}{V_b}\left(\int_{V_b} \vec{x} \otimes \left[\rho(\vec{g} - \ddot{\vec{x}})\right] dV + \vec{x}_c \otimes \vec{r}\right) = \Sigma_r + \Sigma_g + \Sigma_\gamma \qquad [16]$$

where Σ_r, Σ_g and Σ_γ denote the respective contributions of the reaction force, of the gravity force and of the acceleration forces. We obtain, after computations, (de Saxcé *et al.*, 2004).

$$\Sigma_p = \rho g a \left\{ \begin{pmatrix} 0 & \frac{sin\varphi}{6} \\ \frac{sin\varphi}{6} & -cos\varphi \end{pmatrix} + \frac{\mathcal{L}}{a} \begin{pmatrix} \frac{sin\varphi}{3} & 0 \\ 0 & \frac{sin\varphi}{3} \end{pmatrix} \right\} \qquad [17]$$

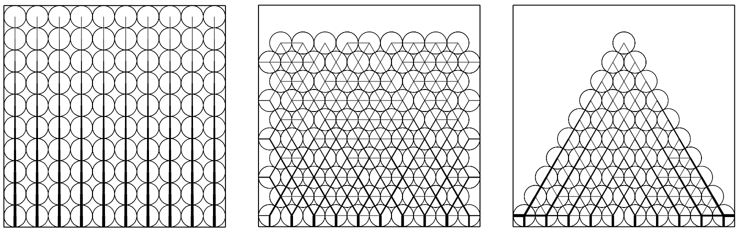
where $\mathcal{L}$ can be interpreted as the mean free path of the bead between two collisions.

5. Application

5.1. *Test case*

Quasi-static examples, presented in Fig 6, are considered. A regular array of equal-size disks confined in a box is considered. The box is composed of two vertical and two horizontal flat walls. The particles as well as the walls are assumed to be perfectly rigid. Each particle is subjected to the gravitation force g and to contact force resulting from neighborhood particles and walls. For three different geometries, the normal contact forces for each particle are computed. The assumptions and results are displayed in Figure 6.

The widths of inter-center segments are proportional to the corresponding normal contact force intensity. For the rectangular arrangement, for obvious physical reasons, the normal force increases when the height of the disk center with respect to the base of



(a) (b) (c)

Figure 6. *Normal contact forces(a) rectangular - (b) triangular - (c) cannon ball*

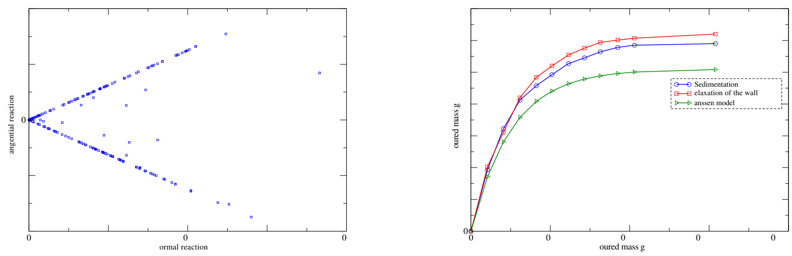
the box decreases. For the triangular arrangement, we observe that contacts between the particles and the vertical wall exist. This situation is typical regarding the complex behavior of granular materials. By comparison with the analytical computation, we obtain the same values concerning the normal contact forces. For the cannon-ball arrangement, friction, which is not displayed here, is essential to the stability of the system.

5.2. *Polarization of contact forces*

First, we consider a silo where the properties of the contact stresses on the lateral walls allow an explanation of the macroscopic properties, such as the stopping of the flow at the bottom of the silo, or the modeling of the stresses in the silo with the Janssen model. In the experiment presented below, we have tried to determine if the contact stresses on the vertical walls are in the so-called Coulomb conditions, i.e. the normal and the tangential reaction are on the slip surface of the Coulomb cone. We have considered here a sample of 350 particles, with a radius equal to 1.5mm ($\pm 10\%$) in a silo. The static friction coefficient is equal to 0.25. The first part of the experiment was to settle the sample under the influence of gravity, and then to relax the system by lowering the bottom wall in order to put the system in the Coulomb condition (Figure 7a). At the end of the relaxation, we compute the average constraint on the lateral walls, and compare the results with those given by the Janssen approach (Figure 7b).

In a second step, we have considered a shear flow, Figure 8, where the quantification of the polarization will permit us to propose a model of boundary layer for this type of flow, like what has been achieved in the case of masonry (Lebon *et al.*, 2008). This type of study has already been done with the DEM in the case of non-sheared media, as for example in a rotating drum (Renouf *et al.*, 2005).

In the first term of (17), the σ_{yy} component is negative, which corresponds, with our sign convention, to a compression stress. It can be interpreted as due to the contact reaction onto the bead (induced by the action of gravity) which prevents it from penetrating through the plane. The σ_{xy} component of the first term represents the friction effect. In the second term of (17), the resulting hydrostatic stress is positive, which



(a) (b)

Figure 7. *(a) Polarization of contact forces - (b) Comparizon Janssen - DEM*

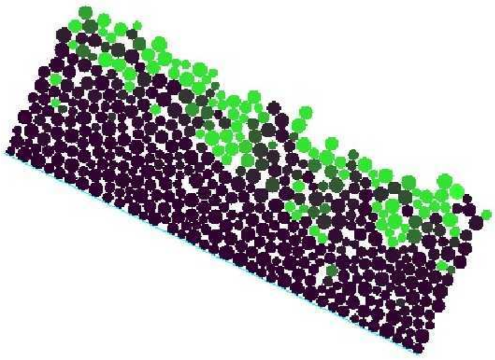


Figure 8. *Shear flow*

corresponds to a traction state due to the centrifugal effects. We call it the centrifugal stress tensor. However, in a dense granular medium, because of frequent collisions, the mean free path of particles is small with respect to their size. Hence, the ratio $\frac{\mathcal{L}}{a}$ remains small with respect to 1 and the second term in (17) can be neglected. It is worthwhile noting that both contact reaction and gravity produce non-symmetrical tensors Σ_r, Σ_g. This shows that it is important not to neglect the inertia tensor Σ_r in the calculation of the mean stress tensor. As the inertia forces have to balance the other forces, contact reactions and gravity, they have the same order of magnitude.

6. Conclusion

In this study we have shown both for a quasi-static example as in the silo and for a dynamic example as in a shear flow over a sandpile that the position of the contact forces on the Coulomb cone enables to explain the overall behavior of the granular material.

Aknowledgements

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