A quasi-symmetric formulation for contact between deformable bodies

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ABSTRACT. *In the frame of contact issues between 3D deformable bodies with non-matching finite element discretizations of possibly very different mesh sizes, a quasi-symmetric formulation is proposed to obtain satisfactory results whichever body is selected as* master *or* slave. *This approach is not based on usual* mortar elements *in order to avoid the creation of an additional integration surface. It draws its inspiration from a symmetric treatment of the contact conditions, where the formulation is made compatible by replacing the Lagrange multiplier of the* master *body by the projection of the* slave *one. Numerical results are obtained within the FORGE3® finite element software. Numerous 3D test cases numerically show that this approach actually solves the main issues of contact between deformable bodies, in a rather simple way.*

RÉSUMÉ. *Dans le cadre des problèmes de contact entre corps déformables 3D ayant des discrétisations éléments finis incompatibles avec éventuellement des tailles de mailles fort différentes, nous proposons une formulation quasi symétrique permettant d'obtenir d'excellents résultats quel que soit le choix du corps* esclave *et du corps* maître. *Cette approche ne repose pas sur l'utilisation d'*éléments de mortier *habituels afin d'éviter la création d'une nouvelle surface d'intégration. Elle s'inspire plutôt d'un traitement symétrique des conditions de contact, où la formulation est rendue compatible en remplaçant le multiplicateur de Lagrange du corps* maître *par la projection de celui de l'*esclave. *Les résultats numériques sont obtenus au sein du logiciel FORGE3® sur de nombreux cas tests 3D. Ils montrent que cette approche résout de manière simple les principales difficultés du contact entre corps déformables.*

KEYWORDS: *contact, deformable bodies, non-matching meshes, contact elements, mortar elements.*

MOTS-CLÉS : *contact, corps déformables, maillages non compatibles, éléments de contact, éléments de mortier.*

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1. Introduction

Contact between deformable bodies gives rise to issues that are both simple, in terms of principle, and complex, in terms of practical solutions. They arise after the finite element discretization of antagonistic bodies. A quick and basic analysis shows that it is not possible to symmetrically enforce the non-penetration conditions in the general case, that is when the interface meshes are not matching. In fact, a perfectly symmetric formulation turns to be over-constrained: the interface becomes artificially stiff and locked (Chenot *et al.* 2002; El-Abbasi *et al.* 2001b; Puso *et al.* 2004a). All recent studies converge toward a non-symmetric treatment of contact, where one body is regarded as the *master*, which imposes the contact condition, while the other body is regarded as the *slave*, which undergoes it. In order to properly discretize the contact conditions, it is necessary that the *slave* body be much finely distretized than the *master* body, which is not possible to fulfil, for instance with an assembly of numerous contacting bodies or when the contact interfaces continuously evolve during non-steady problems. In the extreme case, that is when the *master* body is much finely discretized than the *slave* one, the finite element formulation diverges (Chenot *et al.* 2002; El-Abbasi *et al.* 2001b; Hild 1998): the smaller the finite element size of the *master* interface, the worse the finite element solution.

To solve this issue, most recent studies have utilised the frame of *mortar elements*. It allows properly dealing with incompatible contact interfaces and taking into account possible relative over-refinements of the *master* body by enriching the contact interface (Baillet *et al.* 2003; Ben Dhia *et al.* 2002; El-Abbasi *et al.* 2001a; Hild 2000; Puso *et al.* 2004a; Puso *et al.* 2004b). Most proposed formulations can easily be integrated into 2D software, however at the price of a significant increase of the computational cost. In 3D, the definition of the mortar elements turns to be much trickier (Fernandez 2004; Puso *et al.* 2004a; Puso *et al.* 2004b), particularly in the most general cases where the contact interface is broken up into several non-closely related pieces that are continuously evolving with time.

In this paper, a significant variant of the over-mentioned formulations is presented. It was first envisaged in (Ben Dia *et al.* 2001) in the Arlequin framework, and applied to metal forming problems in (Fourment *et al.* 2003). It can alternatively be regarded as an extension of the *symmetric* contact formulation that has been developed by several authors (see for instance (Habraken *et al.* 1998)) but that is only suitable when the ratio between the antagonistic bodies discretizations is very large. In this new formulation, the over-constraint issue of the *symmetric* formulation is avoided by refraining from defining an undesired Lagrange multiplier on the *master* body, as it is done with *usual master/slave* or *mortar* approaches. This multiplier is then replaced by a projection of the *slave* body multiplier. This formulation turns to be easy to implement into 2D as well as 3D finite element codes (Section 3 of this paper). Theoretically developed for an *integrated (facet-to-facet)* contact formulation, it can easily be extended to *nodal (node-to-facet)* formulation that is utilized in some commercial packages (Section 4 of this paper). It allows obtaining nice numerical results for reference test problems proposed in literature (El-Abbasi *et al.* 2001b; Hild 1998) or for more qualitative ones (Section 5 of this paper).

2. Problem equations

Two elastic bodies a et b are contacting (see Figure 1). For $i = a,b$, u_i is the displacement field, $\varepsilon(u_i)$ the linearized strain tensor [1], $\sigma(u_i)$ the stress tensor with $\boldsymbol{K}_i$ the fourth order tensor of the Hooke's law [1].

$$\varepsilon(u_i) = \frac{1}{2}\left(\nabla u_i + \nabla^t u_i\right) \quad \text{and} \quad \sigma(u_i) = \boldsymbol{K}_i\, \varepsilon(u_i) \qquad [1]$$

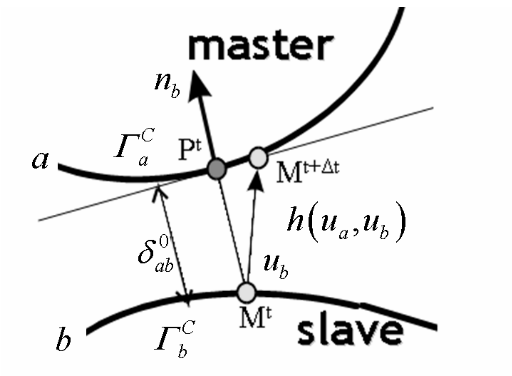

Figure 1. *Contacting bodies*

Volume forces f_i are applied on Ω_i. Its surface is divided into Γ_i^T, on which surface forces T_i are applied, Γ^C the contact surface without friction (for sake of simplicity only). The balance equations are given in [2] and the boundary conditions on the contact surface in [3] :

$$\begin{cases} div\left(\sigma(u_i)\right) + f_i = 0, & \text{on } \Omega_\mathrm{i} \\ \sigma(u_i)\, n_i = T_i, & \text{on } \Gamma_\mathrm{i}^\mathrm{T} \end{cases} \qquad [2]$$

$$0 \le \delta(x) \quad \text{with} \quad u_i^t = u_i - (u_i.n_i)\, n_i = 0 \quad \text{on } \Gamma^\mathrm{C} \qquad [3]$$

where n_i is the outward normal to Ω_i and $\delta(x)$ is the distance function between *a* and *b*, which is negative when there is penetration. $\delta(x)$ is linearized into $h(u_a,u_b)$ by using the initial value δ_{ab}^0 of the distance function [4], and the weak forms of the mixed balance and contact equations are then written in [5].

$$h(u_a,u_b) = (u_a.n_a + u_b.n_b) - \delta_{ab}^0 \leq 0, \quad \text{on } \Gamma^C \qquad [4]$$

$$\begin{cases} \forall u_j^*, \displaystyle\sum_{i=a,b}\left(\int_{\Omega_i}\sigma(u_i):\varepsilon(u_j^*)\,dw_i - \int_{\Omega_i} f_i.u_j^* dw_i - \int_{\Gamma_i^T} T_i.u_j^* ds_i\right) - \int_{\Gamma^C}\lambda \underbrace{h(u_j^*)}_{\substack{h(u_a^*)=h(u_a^*,u_b)\\ h(u_b^*)=h(u_a,u_b^*)}} ds = 0 \\ \forall \lambda^* \leq 0, \displaystyle\int_{\Gamma^C}(\lambda-\lambda^*)\,h(u_a,u_b)\,ds \leq 0 \end{cases} \qquad [5]$$

where λ is the Lagrange multiplier of the mixed problem, which is identified to the contact normal stress, $\sigma_n = (\sigma n).n$. The inequality of [5] provides the weak form of the unilateral contact equation:

$$\forall \lambda^* \leq 0, \quad -\int_{\Gamma^C}\lambda^* h(u_a,u_b)\,ds \leq 0 \qquad [6]$$

The key point of contact treatment is based on an accurate integration of the discretized Lagrangian term of the mixed problem (El-Abbasi *et al.* 2001a):

$$b(\lambda,u) = \int_{\Gamma^C}\lambda\; h(u_a,u_b)\,ds \qquad [7]$$

The problem [5] is then discretized with iso-parametric (or quasi-iso parametric) finite elements $(x_i^h, u_i^h, \lambda^h)$ using shape functions N_i^k, for $i = a,b$:

$$\forall x \in \Omega_i, \quad x_i^h(x) = \sum_k x_i^k N_i^k(x) \quad ; \quad u_i^h(x) = \sum_k u_i^k N_i^k(x) \qquad [8]$$

$$\forall x \in \Gamma^C, \quad \lambda^h(x) = \sum_k \lambda^k N^k(x) \qquad [9]$$

3. Quasi-symmetric formulation

After finite element discretization, the two surfaces Γ_a^C and Γ_b^C are not similar, so that it is necessary to define on which surface $b\left(\lambda^h, u^h\right)$ should be integrated. Contrary to most of usual *mortar elements* approaches (Baillet *et al.* 2003; Ben Dhia *et al.* 2002; El-Abbasi *et al.* 2001a; Hild 2000; Puso *et al.* 2004a; Puso *et al.* 2004b), we wish to avoid introducing a new integration surface, which can be very tricky to build in 3D (Fernandez 2004; Puso *et al.* 2004a; Puso *et al.* 2004b) and to only use the available discretized surfaces Γ_a^C and Γ_b^C.

3.1. *Usual master/slave formulation*

In the *usual master/slave* formulation, $b\left(\lambda^h, u^h\right)$ is integrated on a single surface, Γ_a^C, the *slave* body surface, while Γ_b^C is the surface of the *master* body which imposes its geometry, so λ^h is only interpolated on Γ_a^C [11]:

$$b_1\left(\lambda_a^h, u^h\right) = \int_{\Gamma_a^C} \lambda_a^h \; h\left(u_a^h, u_b^h\right) ds \qquad [10]$$

$$\forall x \in \Gamma_a^C, \quad \lambda_a^h\left(x\right) = \sum_k \lambda_a^k N_a^k\left(x\right) \qquad [11]$$

In practice, it is tried hard to select as *slave* body the one which is more finely discretized, in other words the body which will provide the more accurate integration of $b_1\left(\lambda_a^h, u_h\right)$. When the two bodies are similarly discretized, it is equivalent to integrate on Γ_a^C or Γ_b^C, and both integrations produce similar results.

3.2. *Symmetric formulation*

If the *slave* body is discretized in a much coarser way than the *master*, which is sometimes inevitable, then the integration of $b_1\left(\lambda_a^h, u_h\right)$ is not satisfactory. Consequently, the contact condition is very poorly taken into account. In this case, it can be judicious to integrate $b\left(\lambda^h, u^h\right)$ symmetrically on both contact surfaces (Habraken *et al.* 1998), which requires interpolating λ^h on both Γ_a^C [11] and Γ_b^C:

$$b_2\left(\lambda^h,u^h\right)=\frac{1}{2}\left(\int_{\Gamma_a^C}\lambda_a^h\ h\left(u_a^h,u_b^h\right)ds+\int_{\Gamma_b^C}\lambda_b^h\ h\left(u_b^h,u_a^h\right)ds\right) \qquad [12]$$

$$\forall x\in\Gamma_b^C,\quad \lambda_b^h\left(x\right)=\sum_k\lambda_b^k N_b^k\left(x\right) \qquad [13]$$

$b_2\left(\lambda^h,u^h\right)$ allows integrating $b\left(\lambda^h,u^h\right)$ with accuracy in each zone of the contact interface where one of the bodies is more finely discretized than the other. In fact, by considering that the contribution of the less discretized body is negligible, it results into using everywhere the best discretization. However, if the discretizations of Γ_a^C and Γ_b^C are similar then $b_2\left(\lambda^h,u^h\right)$ introduces too many contact conditions, and the formulation proves to be over-constrained (Chenot *et al.* 2002; El-Abbasi *et al.* 2001b), resulting into artificially stiffening of the contact interface.

3.3. *Quasi-symmetric formulation*

In order to adapt to all kind of situations, these two approaches have to be combined. A *symmetric master/slave* formulation has to be built. The *master/slave* formulation is characterized by the interpolation of λ^h on the *slave* body only [11], and the *symmetric* formulation is characterized by the integration of $b\left(\lambda^h,u^h\right)$ on the two surfaces Γ_a^C and Γ_b^C. Their combination results into replacing, in [12], λ_b^h by an approximation $\tilde{\lambda}_b^h$ that is calculated from λ_a^h only. The simplest approximation that comes to mind is the orthogonal projection $\pi_b^h\left(\lambda_a^h\right)$ of λ_a^h on Γ_b^C. $b\left(\lambda^h,u^h\right)$ is then calculated as follows:

$$b_3\left(\lambda_a^h,u^h\right)=\frac{1}{2}\left(\int_{\Gamma_a^C}\lambda_a^h\ h\left(u_a^h,u_b^h\right)ds+\int_{\Gamma_b^C}\underbrace{\pi_b^h\left(\lambda_a^h\right)}_{\tilde{\lambda}_b^h}\ h\left(u_b^h,u_a^h\right)ds\right) \qquad [14]$$

The utilized finite element discretizations suggest to interpolate $\tilde{\lambda}_b^h=\pi_b^h\left(\lambda_a^h\right)$ with the interpolation functions N_b of λ_b^h, and to use the nodal values of $\pi_b^h\left(\lambda_a^h\right)$ for this interpolation:

$$\forall x\in\Gamma_b^C,\quad \tilde{\lambda}_b^h\left(x\right)=\pi_b^h\left(\lambda_a^h\right)\left(x\right)=\sum_k\left(\pi_b^h\left(\lambda_a^h\right)\left(x_b^k\right)\right)N_b^k\left(x\right) \qquad [15]$$

These $\pi_b^h\left(\lambda_a^h\right)\left(x_b^k\right)$ values are calculated in the simplest way, from the values of λ_a^h at $\pi_a^h\left(x_b^k\right)$, the orthogonal projection of the nodes x_b^k of Γ_b^C on Γ_a^C :

$$\forall x \in \Gamma_b^C, \tilde{\lambda}_b^h(x) = \sum_k \lambda_a^h\left(\pi_a^h\left(x_b^h\right)\right) N_b^k(x) \quad \text{with} \quad \pi_a^h\left(x_b^h\right) = \sum_{l \in f_a^k} x_a^l N_a^l\left(\xi_a^k\right) \quad [16]$$

where f_a^k is the facet of Γ_a^C containing node x_b^k, and ξ_a^k is the barycentric coordinate of x_b^k on f_a^k, which is given by the inverse interpolation of [8]. So,

$$\forall x \in \Gamma_b^C, \quad \tilde{\lambda}_b^h(x) = \sum_k \left(\sum_{l \in f_a^k} \lambda_a^l N_a^l\left(\xi_a^k\right) \right) N_b^k(x) \quad [17]$$

3.4. *Discretized Quasi-symmetric contact equation*

The discretized form of the *quasi-symmetric* contact equation results from the discretization of the weak form [6], or else from the differentiation of $b_3\left(\lambda_a^h, u^h\right)$ with respect to λ_a^k :

$$\forall k' \in \Gamma_a^C, \quad \int_{\Gamma_a^C} N_a^{k'}(x) h\left(u_a^h, u_b^h\right) ds + \sum_{l' \text{ t.q. } f_a^{l'} \ni k'} \int_{\Gamma_b^C} N_a^{k'}\left(\xi_a^{l'}\right) N_b^{l'}(x) h\left(u_b^h, u_a^h\right) ds \leq 0 \quad [18]$$

It can be noticed that the first part of equation [18] provides the usual *master/slave* contribution, while the second part enriches the formulation with an additional term that makes it possible to take into account the contact analysis of *b* with *a*. This new contribution can be dominant if the *master* body is more finely discretized than the *slave*. From a practical standpoint, it is remarked that this formulation requires two contact analyses, in the same way as the *symmetric* formulation and the usual *mortar elements* formulations. It also results into an increase of the bandwidth of the discretized contact equations, proportionally to the addition of contributions from the *master* body contact analysis.

4. Nodal (node-to-facet) contact formulation

When a node-to-facet formulation is used, as it is the case in the frame of the present numerical implementations within the FORGE3® software, the Lagrange multiplier λ_a^h is not continuously interpolated as in [9], in order to exactly impose the contact condition in each node of the mesh. In this *nodal* formulation, the interpolation functions of λ_a^h could be regarded as Dirac functions that are defined at the nodes only (or by piece-wise functions on the dual mesh). The *usual master/slave* [10], *symmetric* [12] and *quasi-symmetric* [14] formulations presented above are then written as follows:

$$\textit{Usual master/slave}: \quad b_1\left(\lambda_a^h, u^h\right) = \sum_{k \in \Gamma_a^C} \lambda_a^k \; h\left(u_a^k, u_b^k\right) \qquad [19]$$

$$\textit{Symmetric}: \quad b_2\left(\lambda^h, u^h\right) = \tfrac{1}{2}\left(\sum_{k \in \Gamma_a^C} \lambda_a^k \; h\left(u_a^k, u_b^k\right) + \sum_{l \in \Gamma_b^C} \lambda_b^l \; h\left(u_b^l, u_a^l\right) \right) \qquad [20]$$

$$\textit{Quasi-symmetric}: \quad b_3\left(\lambda_a^h, u^h\right) = \tfrac{1}{2}\left(\begin{array}{l} \displaystyle\sum_{k \in \Gamma_a^C} \lambda_a^k \; h\left(u_a^k, u_b^k\right) + \\ \displaystyle\sum_{l \in \Gamma_b^C} \left(\pi_b^h\left(\lambda_a^h\right)\left(x_b^l\right)\right) h\left(u_b^l, u_a^l\right) \end{array} \right) \qquad [21]$$

$b_3\left(\lambda_a^h, u^h\right)$ requires calculating $\pi_b^h\left(\lambda_a^h\right)\left(x_b^l\right)$, the value of $\pi_b^h\left(\lambda_a^h\right)$ at node x_b^l, which is only possible by continuously extending the definition of λ_a^h. This is done using the simplest and more natural extension, the finite element interpolation utilized in previous sections [11], which provides, using previous notations of [16]:

$$\forall l \in \Gamma_b^C, \quad \pi_b^h\left(\lambda_a^h\right)\left(x_b^l\right) = \sum_{k \in f_a^l} \lambda_a^k N_a^k\left(\xi_a^l\right) \qquad [22]$$

and then: $$b_3\left(\lambda_a^h, u^h\right) = \tfrac{1}{2}\left(\sum_{k \in \Gamma_a^C} \lambda_a^k \; h\left(u_a^k, u_b^k\right) + \sum_{l \in \Gamma_b^C} \left(\sum_{k \in f_a^l} \lambda_a^k N_a^k\left(\xi_a^l\right) \right) h\left(u_b^l, u_a^l\right) \right) \qquad [23]$$

The discretized *quasi-symmetric* nodal unilateral contact condition is written as:

$$\forall k \in \Gamma_a^C, \quad h\left(u_a^k, u_b^k\right) + \sum_{l \in \Gamma_b^C \text{ t.q. } f_a^l \ni k} N_a^k\left(\xi_a^l\right) h\left(u_b^l, u_a^l\right) \leq 0 \qquad [24]$$

As in the integrated formulation, and even more clearly, it is checked that the first term of [24] is the usual contact condition at the nodes of the *slave* body, while the second term is a contribution of the nodes of the *master* body, weighted by the interpolation functions of the *slave* body.

5. Numerical results

5.1. *Convergence analysis*

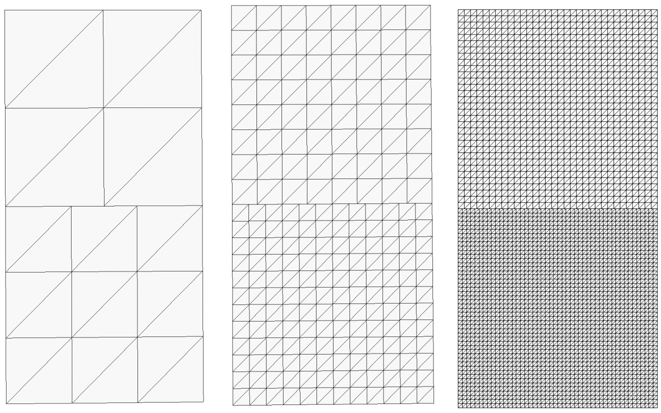

Figure 2. *Examples of non-matching meshes utilized for the squeezed elastic bars*

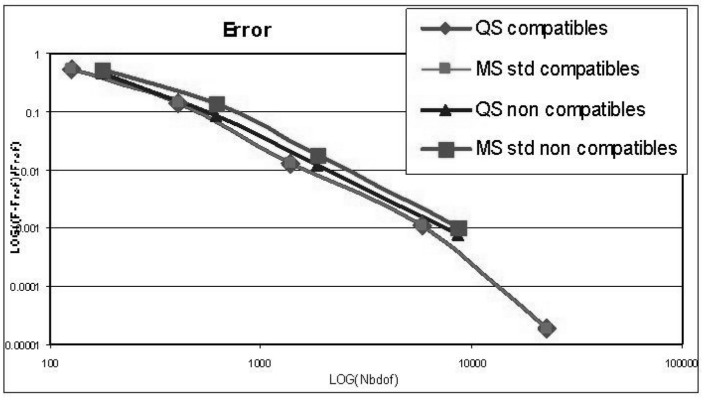

Figure 3. *Squeezing error versus the number of degrees of freedom for matching (compatible) and non-matching (incompatible) meshes, with the usual (MS std) and quasi-symmetric (QS) formulations*

Two identical elastic bars are squeezed between two rigid dies under plane deformation conditions (Hild 1998). The contact between the bodies is perfectly sticking. The problem solution is calculated on a reference mesh, which is four times finer than the finest studied mesh. The error on the squeezing force (see Figure 3) is calculated for several matching and non-matching meshes (see in Figure 2 the most extreme cases) of increasing number of degrees of freedom, with the *usual* (*master/slave*) and *quasi-symmetric* formulations. With matching meshes, both formulations provide exactly the same results, while with non-matching meshes, the *quasi-symmetric* formulation systematically provides better results. However, for this not very severe case where the incompatibility between the meshes is low, all results are almost similar.

5.2. *Patch test*

Figure 4 shows another squeezing test proposed in (El-Abbasi *et al.* 2001b) with two specific meshes of piled up cubes under uniform boundary conditions that are imposed on the upper cube. The diagram of Figure 5 summarizes the obtained results in the elastic case, with an imposed pressure (case n°1) or an imposed displacement (case n°2), and in the rigid plastic Newtonian case, with an imposed pressure (case n°3) or an imposed displacement (case n°4), with the *usual* and *quasi-symmetric* formulations. With the *quasi-symmetric* formulation, the error is systematically two times smaller, and the quality of the interface fields much higher.

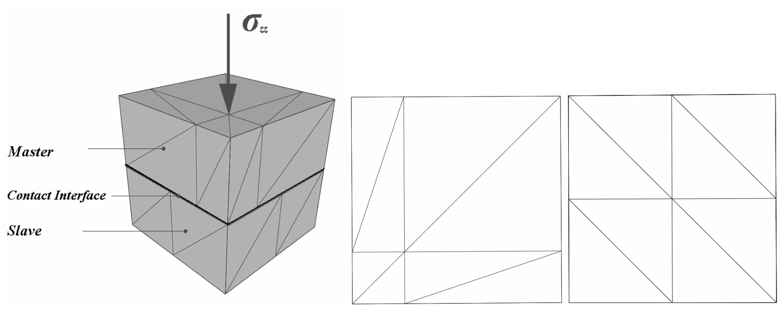

Figure 4. *Contact patch test from (El-Abbasi et al. 2001b) with interface meshes*

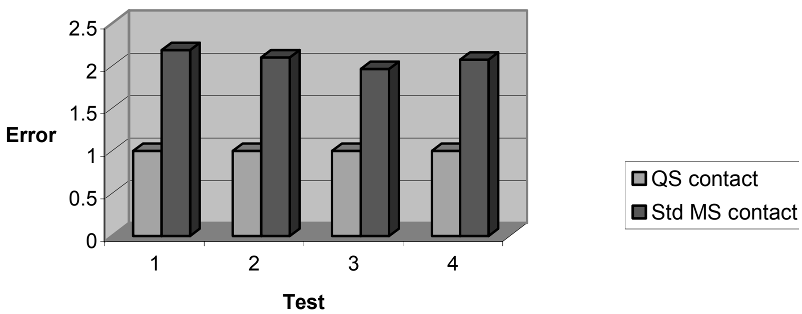

Figure 5. *Relative error on the normal stress field transmission for the contact patch test , with the usual (MS std) and quasi-symmetric (QS) formulation*

5.3. *Contact between deformable cubes*

In the example of Figure 6, the squeezed cubes are meshed with either identical or strongly dissimilar meshes. With the *usual* formulation, the vertical velocity field exhibits a satisfactory continuity at the interface with matching meshes, and when the *slave* cube is more finely meshed. On the other hand, when the *master* cube is more finely meshed, the continuity condition is only locally satisfied, and the resulting velocity field is globally quite erroneous. With the *quasi-symmetric* formulation, the results are almost identical whichever cube is more finely meshed, which so perfectly justifies the *quasi-symmetric* name of the formulation.

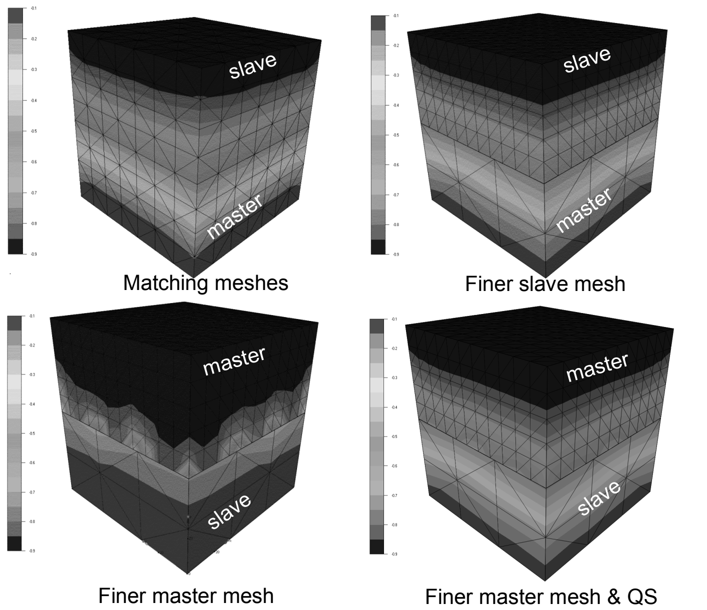

Figure 6. *Isovalues of the vertical velocity field. Up-left: matching meshes. Up-right: finer* slave *mesh. Down: finer* master *mesh with the* usual *(left) and* quasi-symmetric *(right) formulations*

6. Conclusions

This paper suggests an alternative to *mortar elements* for the treatment of contact between deformable bodies with non-matching meshes. Rather than improving the interpolation spaces of the displacement fields on the contact interface, as it is done with the *mortar* approach, it is preferred to adjust the interpolation fields of the contact multipliers in the frame of a *symmetric* formulation. From a practical standpoint, this new formulation turns to be simpler to implement into finite element software. On the other hand, the whole of performed numerical applications shows that the obtained results have the same quality as with *mortar elements*, and that the qualifier of *quasi-symmetric* is well justified. Applications to more complex 3D metal forming have been carried out and will be presented in a forthcoming paper.

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