Non uniform warping for beams

Theory and numerical applications

Rached El Fatmi

Laboratoire de Génie Civil, Ecole Nationale d'Ingénieurs de Tunis
BP 37, 1002 Le Belvédère
Tunis, Tunisie

rached.elfatmi@enit.rnu.tn

ABSTRACT. *A non-uniform warping beam theory including the effects of torsion and shear forces is presented. Based on a displacement model using three warping parameters associated to the three St Venant warping functions corresponding to torsion and shear forces, this theory is free from the classical assumptions on the warpings or on the shears, and valid for any kind of homogeneous elastic and isotropic cross-section. This general theory is applied to analyze, for a representative set of cross-sections, the elastic behavior of cantilever beams subjected to torsion or shear-bending. Numerical results are given for the one-dimensional structural behavior and the three-dimensional stresses distributions; for the stresses in the critical region of the built-in section, comparisons with three-dimensional finite elements computations are presented. The study clearly shows when the effect of the restrained warping is localized or not.*

RÉSUMÉ. *Une théorie de gauchissement non uniforme prenant en compte les effets de la torsion et des efforts tranchants est présentée. Basée sur un modéle cinématique utilisant trois paramètres de gauchissement associés aux trois fonctions de gauchissement de torsion et d'efforts tranchants de St Venant, la théorie, qui s'affranchit naturellement des hypothèses classiques sur les gauchissements ou les cisaillements, est valable pour toute section homogène élastique et isotrope. Cette théorie générale est appliquée pour analyser le comportement élastique de la torsion ou de la flexion-simple de poutres consoles et pour différents types de section. Les résultats numériques portent sur le comportement unidimensionnel de la poutre et sur la distribution tridimensionnelle des contraintes ; les contraintes au droit de la zone critique de l'encastrement sont comparées à celles résultant de calculs par éléments finis tridimensionnels. L'étude permet de préciser quand l'effet du gauchissement empêché est localisé ou pas.*

KEYWORDS: *warping, torsion, shear force, St Venant, coupling, axial stress, shear.*

MOTS-CLÈS : *gauchissement, torsion, effort tranchant, St Venant, couplage, contrainte axiale, cisaillement.*

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1. Introduction

In general case of loading and boundary condition, the warping is non-uniform along the axis of a beam. This leads to a beam mechanical behavior that may be enough different from that predicted by St Venant beam theory (SV-BT) (Ladevèze *et al.*, 1998) or other theories which are restricted to uniform warping. Further, concerning St Venant's principle, the effect of the critical region where the warping is restrained, is not always localized to that region. To better describe the warping effects, *high order* beam theories have been proposed; they have generally been performed to study, separately, the effects of torsional warping (Vlasov, 1961; Kim *et al.*, 2005) and shear force warping (Wang *et al.*, 2000; Dufort *et al.*, 2001). These theories are based on displacement models ($\boldsymbol{\xi}$) including a warping (w) of the following shape:

$$\boldsymbol{\xi}(x,\boldsymbol{X}) = \boldsymbol{v}(x) + \boldsymbol{\theta}(x) \wedge \boldsymbol{X} + w(x,\boldsymbol{X})\ \boldsymbol{x} \quad \text{with} \quad w(x,\boldsymbol{X}) = \eta(x)\ \psi(\boldsymbol{X})$$

where, $\boldsymbol{x}$ is the unit vector of the beam axis, $\boldsymbol{X}$ the in-section vector position, $(\boldsymbol{v},\boldsymbol{\theta})$ the cross-sectional displacements, η the warping parameter and ψ a warping mode associated to torsion or to one of the shear forces; (vectors are highlighted in boldface characters). In each case, ψ is supposed to represent the corresponding SV-warping-function, which is considered as the reference to describe the natural warping of a cross-section (CS). Further, η may be independent or linked to the cross-sectional strains, which can reduce the number of degrees of freedom (for the torsion, η is taken as the twisting rate (e.g.(Vlasov, 1961)), and for the shear-bending η is taken as the cross-sectional shear strain (e.g. (Dufort *et al.*, 2001)).

Non-uniform warping theories agree for the structural behavior of the beam in the case of bi-symmetrical-CS. There is also an agreement about the expression of the additional axial stresses due to warping. However, the situation is not so clear for the shear stresses because these ones are intimately associated to the choices of the warping mode and the warping parameter (independent or not). Also, there is not an agreement concerning the effect of the non symmetry of the CS. In most of the works, for non-symmetrical-CS, if the bending moments refer to the centroid while the torsional moment refers to the shear center, torsional and bending effects remain uncoupled for non-uniform warping theories as they were in classical beam theories. However, on an other side, (Kim *et al.*, 2005) has shown that a (new) flexural-torsional coupling is induced by the non-uniformity of the warping.

In order to obtain a beam theory valid for any CS and able to highlight all these aspects, we propose a beam theory based on the following warping model

$$w(x,\boldsymbol{X}) = \eta_x(x)\ \psi^x(\boldsymbol{X}) + \eta_y(x)\ \psi^y(\boldsymbol{X}) + \eta_z(x)\ \psi^z(\boldsymbol{X})$$

using three independent warping parameters (η_x, η_y, η_z) associated to three warping functions (ψ^x, ψ^y, ψ^z) which are "exactly" the SV-warping-functions corresponding to torsion and shear forces. This model, that could be considered as the most general one, leads to a non-uniform beam theory (denoted herein by NUW-BT) free from the classical assumptions on the warping functions or on the shear distributions (e.g. Vlasov assumptions for thin-walled profiles). Starting from of the model displacement, the theory is derived using, as classical, the virtual work principle. In the present paper, the description of the NUW-BT will be restrained to its key points; however, one can find the theoretical developments in (El Fatmi, 2007c; El Fatmi, 2007b). It should be noted that the numerical applications of the NUW-BT need, for a given cross-section, to first compute all its characteristics: the cross-sectional constants and the SV-warping-functions. This is achieved in this study by using the software tool designated by *SECOPE* which is available within the finite elements code *CASTEM*. *SECOPE* has been developed conforming to the numerical method proposed by (El Fatmi *et al.*, 2004) to compute the complete three-dimensional (3D) SV-solution, within the frame of the *exact* beam theory established by (Ladevèze *et al.*, 1998). To illustrate the NUW-BT, the applications are devoted to the analyzes of cantilever beams subjected to torsion or shear-bending, and for different kinds of sections.

2. Non-uniform warping theory

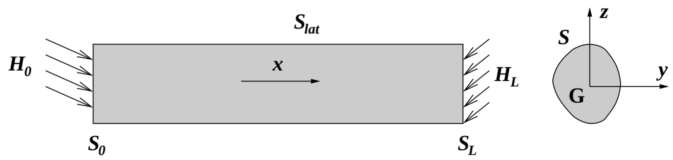

Figure 1. *The reference problem*

The reference problem shown in Figure 1 is a 3D equilibrium beam problem. The beam is of section S and length L. S_{lat} is the lateral surface and S_0 and S_L are the extremity sections. $\boldsymbol{y}$ and $\boldsymbol{z}$ are the inertia unit vectors of the CS. The material constituting the beam is characterized by the Young's modulus E and the shear modulus G (or the Poisson's ratio ν). The beam is in equilibrium under[1] surface force densities $\boldsymbol{H}_0$ and $\boldsymbol{H}_L$ acting on S_0 and S_L, respectively.

Let $\boldsymbol{X}$=$\boldsymbol{GM}$ and $\overline{\boldsymbol{X}}$=$\boldsymbol{CM}$ the in-section vectors that refer to the centroïde G and to the shear center C of the section, respectively. The displacement model is the following:

$$\boldsymbol{\xi}(\boldsymbol{v},\boldsymbol{\theta},\boldsymbol{\eta}) = \boldsymbol{v}(x) + \theta_x(x)\boldsymbol{x} \wedge \overline{\boldsymbol{X}} + (\theta_y(x)\boldsymbol{y} + \theta_z(x)\boldsymbol{z}) \wedge \boldsymbol{X} + [\eta_i(x) \cdot \psi^i(\boldsymbol{X})]\boldsymbol{x} \quad [1]$$

where $\eta_i\psi^i$ is a sum using the repeated indices convention with $i \in \{x, y, z\}$, $(\boldsymbol{v}, \boldsymbol{\theta})$ are the cross-sectional displacements, η_i ($\boldsymbol{\eta}$=(η_x, η_y, η_z)) the warping parameters, and ψ^i the SV-warping-functions corresponding to torsion and shear forces. The

1. This is conform to the original SV-problem, and for the sake of simplicity we consider the same conditions; for other loadings and boundary conditions, see (El Fatmi, 2007a; El Fatmi, 2007b).

beam theory associated with this displacement, parametrized by $(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{\eta})$, is derived by the virtual work principle. Let us introduce first $\widehat{\boldsymbol{\xi}} = \boldsymbol{\xi}(\widehat{\boldsymbol{v}}, \widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{\eta}})$ denoting a virtual displacement and $\widehat{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}(\widehat{\boldsymbol{\xi}})$ the corresponding strain tensor. With $\widehat{\boldsymbol{\gamma}}$=$\widehat{\boldsymbol{v}}'$+$\boldsymbol{x} \wedge \widehat{\boldsymbol{\theta}}$ and $\widehat{\boldsymbol{\chi}}$=$\widehat{\boldsymbol{\theta}}'$ ($(.)'$ expresses the derivative with respect to x), the non zero components of $\widehat{\boldsymbol{\varepsilon}}$ can be written:

$$\widehat{\varepsilon}_{xx}=\widehat{\gamma}_x+z\widehat{\chi}_y-y\widehat{\chi}_z+\widehat{\eta}_i'\psi^i \ ; \ \begin{bmatrix} 2\widehat{\varepsilon}_{xy} \\ 2\widehat{\varepsilon}_{xz} \end{bmatrix}=\widehat{\gamma}_y\boldsymbol{y}+\widehat{\gamma}_z\boldsymbol{z}+\widehat{\chi}_x(\boldsymbol{x}\wedge\overline{\boldsymbol{X}})+\widehat{\boldsymbol{\eta}}_i\boldsymbol{\nabla}\psi^i \quad [2]$$

$\boldsymbol{\sigma}$ being the stress tensor, the internal virtual work is W_i= $-\int_L \langle \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\widehat{\boldsymbol{\xi}}) \rangle dx$, where the compact notation $\langle (\cdot) \rangle$ denotes $\int_S (\cdot)\ dS$. Using the expression of the virtual deformations (Equation [2]), W_i takes the form:

$$\begin{aligned} W_i &= -\int_L \left(\boldsymbol{R}\cdot\widehat{\boldsymbol{\gamma}} + \boldsymbol{M}\cdot\widehat{\boldsymbol{\chi}} + \boldsymbol{M}_\psi\cdot\widehat{\boldsymbol{\eta}}' + \boldsymbol{M}_s\cdot\widehat{\boldsymbol{\eta}}\right)\, dx \\ &= \int_L \Big(\boldsymbol{R}'\cdot(\widehat{\boldsymbol{v}}+\boldsymbol{x}\wedge\widehat{\boldsymbol{\theta}}) + \boldsymbol{M}'\cdot\widehat{\boldsymbol{\theta}} + (\boldsymbol{M}_\psi' - \boldsymbol{M}_s)\cdot\widehat{\boldsymbol{\eta}}\Big)\, dx \\ &\quad - \Big[\boldsymbol{R}\cdot\widehat{\boldsymbol{v}} + \boldsymbol{M}\cdot\widehat{\boldsymbol{\theta}} + \boldsymbol{M}_\psi\cdot\widehat{\boldsymbol{\eta}}\Big]_0^L \end{aligned} \quad [3]$$

where

$$\begin{array}{lllll} \boldsymbol{R} = \boldsymbol{r}(\boldsymbol{\sigma}\cdot\boldsymbol{x}) & \boldsymbol{M}_\psi = \langle \sigma_{xx}\psi^i \rangle \boldsymbol{x}_i & \\ \boldsymbol{M} = \overline{\boldsymbol{m}}(\boldsymbol{\sigma}\cdot\boldsymbol{x}) & \boldsymbol{M}_s = \langle \sigma_{xy}\psi^i_{,y} + \sigma_{xz}\psi^i_{,z} \rangle \boldsymbol{x}_i & \boldsymbol{x}_i \in \{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\} \end{array} \quad [4]$$

This defines the internal forces $\boldsymbol{R}, \boldsymbol{M}, \boldsymbol{M}_\psi$ and $\boldsymbol{M}_s$. $\boldsymbol{R}$=(N, T^y, T^z) and $\boldsymbol{M}$=(M^x, M^y, M^z) are the classical resultant and moment of the stress-vector $(\boldsymbol{\sigma}\cdot\boldsymbol{x})$: (N, T^y, T^z) are the axial and the shear forces, M^x is the torsional moment referring[2] to the shear center C, whereas (M^y, M^z) are the bending moment referring to the centroid G. The new ones $(\boldsymbol{M}_\psi, \boldsymbol{M}_s)$, are called[3] the *bimoment* vector and the *secondary internal force* vector, respectively. The components of $\boldsymbol{M}_\psi$ are denoted $(M_\psi^x, M_\psi^y, M_\psi^z)$ and those of $\boldsymbol{M}_s$ by (M_s^x, T_y^s, T_s^z), because M_s^x is homogeneous to a moment, and (T_s^y, T_s^z) are homogeneous to forces. The external virtual work is W_e= $\int_{S_0} \boldsymbol{H}_0\cdot\widehat{\boldsymbol{\xi}}\ dS$+ $\int_{S_L} \boldsymbol{H}_L\cdot\widehat{\boldsymbol{\xi}}\ dS$. Using Equation [1], W_e takes the form:

$$W_e = \boldsymbol{P}_0\cdot\widehat{\boldsymbol{v}}_0 + \boldsymbol{C}_0\cdot\widehat{\boldsymbol{\theta}}_0 + \boldsymbol{Q}_0\cdot\widehat{\boldsymbol{\eta}}_0 + \boldsymbol{P}_L\cdot\widehat{\boldsymbol{v}}_L + \boldsymbol{C}_L\cdot\widehat{\boldsymbol{\theta}}_L + \boldsymbol{Q}_L\cdot\widehat{\boldsymbol{\eta}}_L \quad [5]$$

where $(\boldsymbol{P}_I$=$\langle \boldsymbol{H}_I \rangle; \boldsymbol{C}_I$=$\overline{\boldsymbol{m}}(\boldsymbol{H}_I); \boldsymbol{Q}_I$=$\langle H_I^x\psi^x \rangle\boldsymbol{x}$+$\langle H_I^x\psi^y \rangle\boldsymbol{y}$+$\langle H_I^x\psi^z \rangle\boldsymbol{z})$ define, for the 1D theory, the external actions associated to the external surface force density $\boldsymbol{H}_I$, with $I \in \{0, L\}$.

2. It is common, for non symmetrical cross-section (in order to uncouple torsional and bending effects) to express the bending moments referring to the centroid while the torsional moment is referred to the shear center.
3. It is usual, in non-uniform torsional warping theories, to call M_ψ^x the *bimoment*, and M_s^x the *secondary torsional moment*. By analogy, we introduce the *bimoment vector* $\boldsymbol{M}_\psi$ and the *secondary internal force vector* $\boldsymbol{M}_s$.

Thanks to the principle of virtual work, Equations [3-5] provide the equilibrium equations and the boundary conditions:

$$\left.\begin{aligned} \boldsymbol{R}' &= \mathbf{0} \\ \boldsymbol{M}' + \boldsymbol{x} \wedge \boldsymbol{R} &= \mathbf{0} \\ \boldsymbol{M}'_\psi - \boldsymbol{M}_s &= \mathbf{0} \end{aligned}\right| \quad \begin{aligned} x = 0: \quad \{\boldsymbol{R}, \boldsymbol{M}, \boldsymbol{M}_\psi\}_0 &= -\{\boldsymbol{P}_0, \boldsymbol{C}_0, \boldsymbol{Q}_0\} \\ x = L: \quad \{\boldsymbol{R}, \boldsymbol{M}, \boldsymbol{M}_\psi\}_L &= \{\boldsymbol{P}_L, \boldsymbol{C}_L, \boldsymbol{Q}_L\} \end{aligned} \qquad [6]$$

Let $\boldsymbol{D}=(\boldsymbol{D}^\sigma, \boldsymbol{D}^\tau)$ denote the generalized strain vector and $\boldsymbol{T}=(\boldsymbol{T}^\sigma, \boldsymbol{T}^\tau)$ the corresponding generalized force vector defined by:

$$\begin{array}{ll} \boldsymbol{D}^\sigma=(\gamma_x, \chi_y, \chi_z, \eta'_x, \eta'_y, \eta'_z) & \boldsymbol{D}^\tau=(\chi_x, \eta_x, \gamma_y, \eta_y, \gamma_z, \eta_z) \\ \boldsymbol{T}^\sigma=(N, M^y, M^z, M^x_\psi, M^y_\psi, M^z_\psi) & \boldsymbol{T}^\tau=(M^x, M^x_s, T^y, T^y_s, T^z, T^z_s) \end{array} \qquad [7]$$

The 1D elastic constitutive relation can be written $\boldsymbol{T}=\boldsymbol{\Gamma}\boldsymbol{D}$ where $\boldsymbol{\Gamma}$ defines the structural rigidity operator. Using the matrix notation, the elastic strain energy for the 1D model of the beam is given by $W^{1D}_{el}(\boldsymbol{D}, \boldsymbol{D})=\frac{1}{2}\int_L [\boldsymbol{D}]^t[\boldsymbol{\Gamma}][\boldsymbol{D}]dx$. Besides, for the 3D problem, $\boldsymbol{\varepsilon}$ denoting the strain tensor associated to the displacement $\boldsymbol{\xi}(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{\eta})$ (Equation [1]), and using Hooke's law, the beam elastic strain energy can be written as

$$W^{3D}_{el}(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}) = \frac{1}{2}\int_L \langle E\varepsilon^2_{xx} + 4G(\varepsilon^2_{xy} + \varepsilon^2_{xz})\rangle dx \qquad [8]$$

Identifying the strain energies W^{3D}_{el} and W^{1D}_{el} allows to derive the rigidity operator $\boldsymbol{\Gamma}$. This identification shows that $\boldsymbol{\Gamma}$ can be written $\boldsymbol{\Gamma}=\begin{bmatrix} \boldsymbol{\Gamma}^\sigma & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Gamma}^\tau \end{bmatrix}$ and leads to the uncoupled relations $\boldsymbol{T}^\sigma=\boldsymbol{\Gamma}^\sigma\boldsymbol{D}^\sigma$ and $\boldsymbol{T}^\tau=\boldsymbol{\Gamma}^\tau\boldsymbol{D}^\tau$. The rigidity operators $\boldsymbol{\Gamma}^\sigma$ and $\boldsymbol{\Gamma}^\tau$ are associated to the axial stress and the shear stresses, respectively. Using the SV-warping functions properties (see details in (El Fatmi, 2007c; El Fatmi, 2007a), $\boldsymbol{\Gamma}^\sigma$ and $\boldsymbol{\Gamma}^\tau$ reduce to:

$$\boldsymbol{\Gamma}^\sigma = E\begin{bmatrix} A & 0 & 0 & 0 & 0 & 0 \\ & I_y & 0 & 0 & 0 & 0 \\ & & I_z & 0 & 0 & 0 \\ & & & I^{xx}_\psi & I^{xy}_\psi & I^{xz}_\psi \\ & & & & I^{yy}_\psi & I^{yz}_\psi \\ & & & & & I^{zz}_\psi \end{bmatrix}_{sym} \qquad [9]$$

$$\boldsymbol{\Gamma}^\tau = G\begin{bmatrix} I_x & J - I_x & z_cA & -z_cA & -y_cA & y_cA \\ & I_x - J & -z_cA & z_cA & y_cA & -y_cA \\ & & A & A_y - A & 0 & 0 \\ & & & A - A_y & 0 & 0 \\ & & & & A & A_z - A \\ & & & & & A - A_z \end{bmatrix}_{sym} \qquad [10]$$

with I_x=I_y+I_z+$A(y_c^2$+$z_c^2)$. $(A, A_y, A_z, I_y, I_z, J, y_c, z_c)$ are the classical cross-sectional constants (the area, the reduced areas, the moments of inertia, the torsional constant, and the components of the shear center C, respectively) and $I_\psi^{ij} = \langle \psi^i \psi^j \rangle$ define the cross-sectional warping constants and the *warping matrix* noted $\boldsymbol{I}_\psi$. The expressions of the strain (Equation [2]) and the inverse of the constitutive relations (Equations [9-10]) allow to express the normal and shear stresses with respect to the generalized stresses as follows:

$$
\begin{aligned}
\sigma_{xx}^{nuw} &= \overbrace{\frac{N}{A} + \frac{M^y}{I_y}\, z - \frac{M^z}{I_z}\, y}^{\boldsymbol{\sigma}_{xx}^{sv}} \\
&+ \frac{M_\psi^x}{\kappa_\sigma}\left[((I_\psi^{yy} I_\psi^{zz}\text{-}I_\psi^{yz\,2})\psi^x\text{+}(\text{-}I_\psi^{xy} I_\psi^{zz}\text{+}I_\psi^{xz} I_\psi^{yz})\psi^y\text{+}(I_\psi^{xy} I_\psi^{yz}\text{-}I_\psi^{xz} I_\psi^{yy})\psi^z)\right] \\
&+ \frac{M_\psi^y}{\kappa_\sigma}\left[((\text{-}I_\psi^{xy} I_\psi^{zz}\text{+}I_\psi^{xz} I_\psi^{yz})\psi^x\text{+}(I_\psi^{xx} I_\psi^{zz}\text{-}I_\psi^{xz\,2})\psi^y\text{+}(\text{-}I_\psi^{xx} I_\psi^{yz}\text{+}I_\psi^{xy} I_\psi^{xz})\psi^z)\right] \\
&+ \frac{M_\psi^z}{\kappa_\sigma}\left[((I_\psi^{xy} I_\psi^{yz}\text{-}I_\psi^{xz} I_\psi^{yy})\psi^x\text{+}(\text{-}I_\psi^{xx} I_\psi^{yz}\text{+}I_\psi^{xy} I_\psi^{xz})\psi^y\text{+}(I_\psi^{xx} I_\psi^{yy}\text{-}I_\psi^{xy\,2})\psi^z)\right] \\
&\text{with } \kappa_\sigma = I_\psi^{xx} I_\psi^{yy} I_\psi^{zz}\text{-}I_\psi^{xx} I_\psi^{yz\,2}\text{-}I_\psi^{xy\,2} I_\psi^{zz}\text{+}2I_\psi^{xy} I_\psi^{xz} I_\psi^{yz} I_\psi^{xz\,2} I_\psi^{yy}
\end{aligned}
\qquad [11]
$$

$$
\begin{aligned}
\boldsymbol{\tau}^{nuw} &= \overbrace{M^x\tau^x + T^y\tau^y + T^z\tau^z}^{\boldsymbol{\tau}^{sv}} \\
&+\frac{M_s^x}{\kappa_\tau}\Big[(A\text{-}A_y)(A\text{-}A_z)(I_x\boldsymbol{\tau}^x\text{-}\boldsymbol{x}\wedge\overline{\boldsymbol{X}})\text{-}A^2(y_c^2(A\text{-}A_y)\text{+}z_c^2(A\text{-}A_z))\boldsymbol{\tau}^x \\
&\qquad -z_c A(A\text{-}A_z)(A_y\boldsymbol{\tau}^y\text{-}\boldsymbol{y})\text{+}y_c A(A\text{-}A_y)(A_z\boldsymbol{\tau}^z\text{-}\boldsymbol{z})\Big] \\
&+\frac{T_s^y}{\kappa_\tau}\Big[\text{-}z_c A(A\text{-}A_z)(J\boldsymbol{\tau}^x\text{-}\boldsymbol{x}\wedge\overline{\boldsymbol{X}}) \\
&\qquad \text{+}((I_x\text{-}J)(A\text{-}A_z)\text{-}y_c^2A^2)(A\boldsymbol{\tau}^y\text{-}\boldsymbol{y})\text{-}z_c^2A^2(A\text{-}A_z)\boldsymbol{\tau}^y\text{-}y_c z_c A^2(A_z\boldsymbol{\tau}^z\text{-}\boldsymbol{z})\Big] \\
&+\frac{T_s^y}{\kappa_\tau}\Big[y_c A(A - A_y)(J\boldsymbol{\tau}^x - \boldsymbol{x}\wedge\overline{\boldsymbol{X}}) \\
&\qquad \text{-}y_c z_c A^2(A_y\boldsymbol{\tau}^y\text{-}\boldsymbol{y})\text{+}((I_x\text{-}J)(A\text{-}A_y)\text{-}z_c^2A^2)(A\boldsymbol{\tau}^z\text{-}\boldsymbol{z})\text{-}y_c^2A^2(A\text{-}A_y)\boldsymbol{\tau}^z\Big] \\
&\text{with } \kappa_\tau = (I_x\text{-}J)(A\text{-}A_y)(A\text{-}A_z)\text{-}A^2(y_c^2(A\text{-}A_y)\text{+}z_c^2(A\text{-}A_z))
\end{aligned}
\qquad [12]
$$

where σ_{xx}^{sv} and $\boldsymbol{\tau}^{sv}$ are the axial and the shear stresses of SV-solution (τ^x, τ^y, τ^z are the St-Venant-shears associated to unit torsional moment and shear forces, respectively). This result makes clear the additional contribution of the new internal forces $\boldsymbol{M}_\psi$ and $\boldsymbol{M}_s$ induced by the non-uniformity of warping.

3. Comments

– Equilibrium equations (Equation [6]), constitutive relation $\boldsymbol{T}$=$\boldsymbol{\Gamma D}$ (Equation [9-10]) and boundary conditions on S_0 and S_L form the 1D problem that defines the NUW-BT.

– For a cantilever beam submitted to shear-bending or torsion, the warping is restrained ($\boldsymbol{\eta}$=$\mathbf{0}$) for the built-in section, and free ($\boldsymbol{M}_\psi$=$\mathbf{0}$) in the other extremity; the warping is then non-uniform along the span.

– Due to warping, torsional and bending effects are coupled in the present NUW-BT, even if the torsional moment refers to the shear center C whereas the bending moments refer to the centroid G. For an arbitrary-CS this coupling effect is related to the three coupling components $(I_\psi^{xy}, I_\psi^{xz}, I_\psi^{yz})$ of the warping matrix and to the coordinates (y_c, z_c) of C. This coupling appears clearly in the constitutive relations (Equation [9-10]) and in the expressions of the stresses (Equations [11-12]). For bi-symmetrical-CS, one can show that (I_ψ^{xy}=I_ψ^{xz}=I_ψ^{yz}=y_c=z_c=0) and the torsional-flexural coupling vanishes.

– The lateral surface S_{lat} of the beam is free of loading. In Equation [12], the contribution of $\boldsymbol{M}_s$ to the shear is expressed with the SV-shears $(\boldsymbol{\tau}^x, \boldsymbol{\tau}^y, \boldsymbol{\tau}^z)$ and with the supplementary terms $(\boldsymbol{x} \wedge \overline{\boldsymbol{X}}), (\boldsymbol{y})$ and $(\boldsymbol{z})$. The SV-shears naturally vanish at the free edge of the section but the supplementary terms violate the "no shear" boundary conditions at the edge. Thus, for this theory founded on the displacement model (Equation [1]), the result on the shear distribution over the section is not quite satisfying.

– The application of the NUW-BT needs to previously know, for any given CS, all its constants (A, A_y, A_z, I_y, I_z, J, y_c, z_c) and in particular its SV-warping-functions (ψ^x, ψ^y, ψ^z) and shears $(\boldsymbol{\tau}^x, \boldsymbol{\tau}^y, \boldsymbol{\tau}^z)$. In such conditions, it is worth to note that we have just to compute the six scalars of the warping matrix $\boldsymbol{I}_\psi$, and for the stresses, the closed form results (Equations [11-12]) can be directly used without any additional computation.

– One can find in (El Fatmi, 2007a), the conditions for which the warping parameters can be linked to the classical cross-sectional strains, allowing to provide simplified versions of the present NUW-BT where the degrees of freedom are reduced.

4. Numerical applications

A representative set of CS is considered: solid-CS, thin-walled open/closed-CS, symmetric or not. The sections are denoted $\mathrm{S}_1, \mathrm{S}_2, \mathrm{S}_3$ and S_4; their dimensions are $(h \times h/2)$, the thickness for the thin-walled sections is $h/20$ and the elastic constants are E=200GPa and ν=0[4]. The SV-warping-functions, computed by *SECOPE*, are

4. ν is here chosen equal to zero just to be conform to the displacement model 1 which neglects the in-plane deformation due to Poisson's effects in the section. However, one can show that this theory may be used with $\nu \neq 0$, but using the good values of G ($\frac{E}{2(1+\nu)}$), A_y and A_z.

depicted in Figure 2. To illustrate the NUW-BT, the applications are devoted to the numerical analyzes of cantilever beams subjected to torsion or shear-bending (Figure 3-(a)-(b)). Numerical results are given for the 1D-structural behavior[5] and also for the 3D-stress distributions close to the built-in section: the stress predictions of the NUW-BT are compared to those obtained by 3D finite elements computations (3D-FEM). We give hereafter some significant results and one can find more detailed results in (El Fatmi, 2007b). In the following, $((\cdot)^{3D}$, $(\cdot)^{1D}$ and $(\cdot)^{sv}$ will denote quantities related to 3D-FEM, NUW-BT, and SV-BT, respectively.

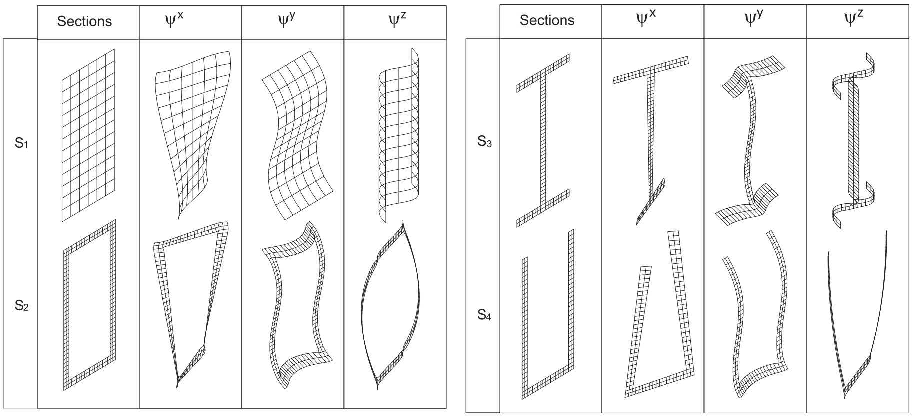

Figure 2. *SV-warping-functions corresponding to torsion and shear forces*

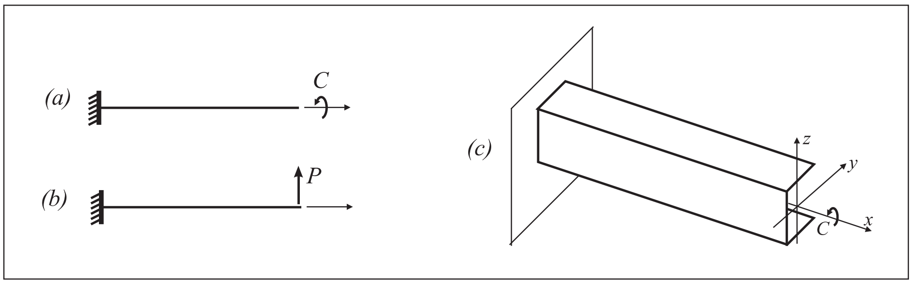

Figure 3. *Cantilever beams subjected to torsion or shear-bending*

Torsion. The beam is subjected to a unit tip torque C=1, and the aspect ratio $\frac{L}{h}$= 10. The two diagrams of Figure 4 depict, for $\mathrm{S}_1, \mathrm{S}_2, \mathrm{S}_3$, the distributions along the beam axis of the normalized 1D-quantities $\frac{\theta_x}{\theta_L^{sv}}$ and $\frac{\eta_x}{\eta_x(L)}$, respectively. The rotation θ_x is normalized by the SV-rotation of the end section θ_L^{sv}=$\frac{CL}{GJ}$. Except the region

5. It should be noted that the 1D-solution for the bi-symmetrical-CS are obtained analytically, but for the non-symmetrical-CS, A 1D-FEM has been used.

very close to the built-in section, the structural behaviors of solid-CS (S_1) and closed-CS (S_2) are similar with that of SV-solution. However, for an open-CS (S_3), warping effect is *not localized* and extends from the built-in section to the end of the beam.

Table 1. *Comparison between 3D-FEM ($(.)^{3D}$) and NUW-BT ($(.)^{1D}$) results on the extrem values of the axial stresses in the built-in section*

	$\sigma_{xx}^{3D}(\times h^{-2})$	$\sigma_{xx}^{1D}(\times h^{-2})$	$\left\lvert\frac{\sigma_{xx}^{1D}-\sigma_{xx}^{3D}}{\sigma_{xx}^{3D}}\right\rvert.100$
S_1	29.104	20.690	28.91
S_2	34.714	43.702	25.89
S_3	1499.8	1234.0	15.10

Table 1 compares 3D-FEM and NUW-BT results on the extreme values of the axial stress in the built-in section (S_0): qualitatively, the values of the axial stresses are of the same order of magnitude.

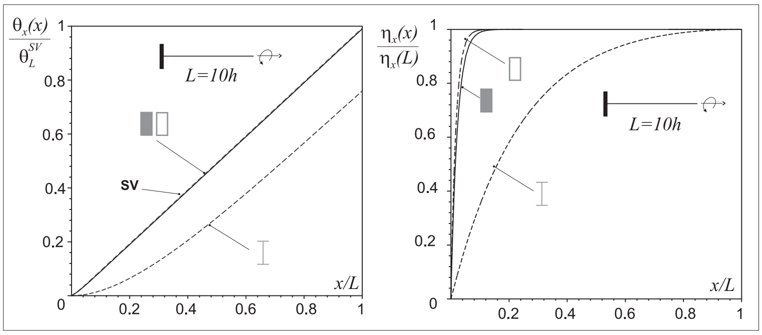

Figure 4. *Torsion: variations of twisting and warping along the span*

Figure 5 depicts, for S_1 and S_3 and by columns (a,b,c,d): (a)- the shear distribution $|\boldsymbol{\tau}|^{3D}$ on the built-in section S_0, indicating the values of the shear for two important points A and B of the section; (b)-the shear distribution $|\boldsymbol{\tau}|^{1D}$ on S_0, with the values of the shear for the same points A and B; (c)- the variations, starting from S_0, of the shears $|\boldsymbol{\tau}|^{3D}$ and $|\boldsymbol{\tau}|^{1D}$ normalized by $|\boldsymbol{\tau}|^{sv}$ for the particular point A where $|\boldsymbol{\tau}|^{sv}$ is maximum; (d)- the SV-shear distribution $|\boldsymbol{\tau}|^{sv}$, with the values of the shear for the same points A and B. For the solid-CS (S_1), in the built-in section, the extreme values of the shear are lower than those of St Venant $|\boldsymbol{\tau}|^{sv}$, and of the same order of magnitude with those of the axial stresses due to the restrained warping (see Table 1). In contrast to S_1, for the open-CS (S_3), the shear distribution in the built-in section is really different from that of St Venant and the magnitude order of the extreme values of the shear becomes negligible. More important, in the critical section, compared to the axial stress due to the restrained warping (see Table 1), the shear is negligible. For the torsion, these results allow to conclude that:

– the behaviors of the solid-CS and the closed-CS are similar: SV-solution is sufficiently precise to evaluate the twisting angle, but for the stress predictions close to the critical section, the restrained warping must be taken into account;

– in contrast, for an open-CS, SV-solution is no longer sufficient, warping effect is not localized and it is necessary to take into account the non-uniformity of the warping for both structural behavior and stress predictions.

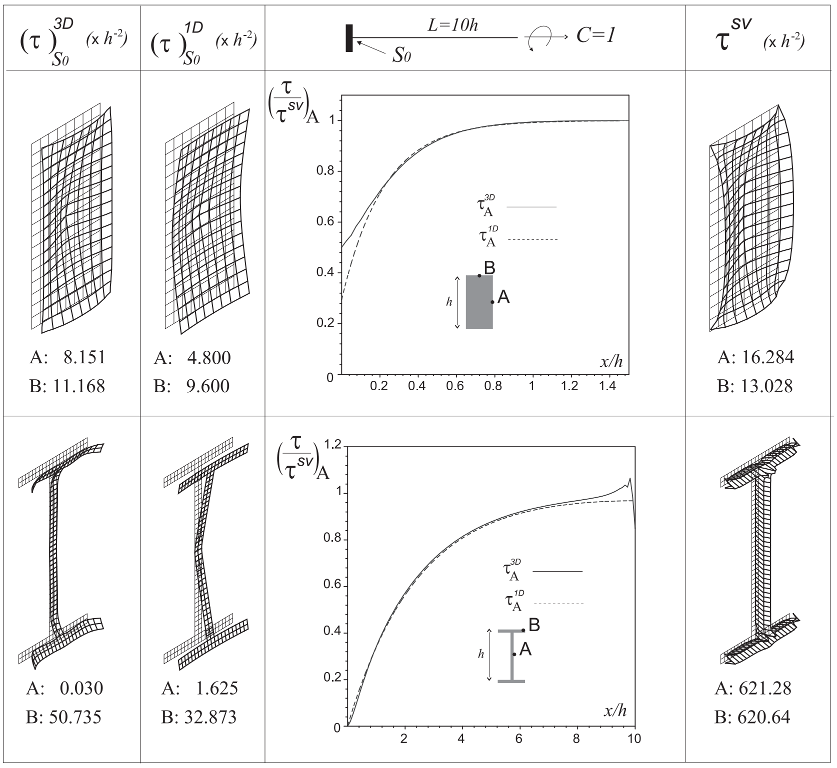

Figure 5. *Torsion of cantilever beams: shears distributions*

Shear-bending of a short beam. The cantilever beam is subjected to a tip transversal force (P=1), and the aspect ratio $\frac{L}{h}$ is chosen to be equal to 2.5. Figure 6 depicts, for S_1, S_2, S_3, the distributions along the beam axis of the normalized warping $\frac{\eta_y}{\eta_y(L)}$. The warpings effects remain very localized close to the built-in section (even if the case of the solid-CS (S_1) is a little different). It should be noted that the solution of the 1D-problem may be obtained in a closed form and the expression of the deflection is:

$$v_y(L)=\overbrace{\frac{PL^3}{3EI_z}+\frac{PL}{GA_y}}^{v_y^{sv}(L)}\left(1-\frac{A-A_y}{A}\frac{\tanh(\mathrm{K}^y L)}{\mathrm{K}^y L}\right)\ \text{with}\ \mathrm{K}^y=\sqrt{\frac{GA_y}{EI_\psi^{yy}}\frac{A-A_y}{A}}$$

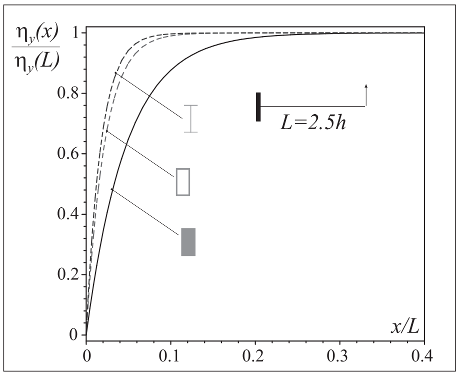

Figure 6. *Shear-bending: variations of warping along the span*

The simulations for the three sections and an aspect ratio $\frac{L}{h} \leq 2.5$, have shown that v_L^{sv} represents 99% of the total deflection $v_y(L)$. Table 2 compares 3D-FEM and NUW-BT results on the extreme values of the axial stresses in the built-in section. The results indicate that, for a short beam (L=2.5h), axial stresses σ_{xx}^{w} due to (the restrained) warping becomes important and may reach 50% of the axial stresses σ_{xx}^{sv} due to flexure.

Table 2. *Shear-bending: comparison of the axial stresses due to warping and flexure*

3D-FEM	σ_{xx}^{3D}	σ_{xx}^{sv}	$\left(\sigma_{xx}^{3D}\right)^w$	$\frac{\left(\sigma_{xx}^{3D}\right)^w}{\sigma_{xx}^{sv}}$ %
S_1	34.306	30.000	4.306	14.35
S_2	122.720	87.234	35.486	40.68
NUW-BT	σ_{xx}^{1D}	σ_{xx}^{sv}	$\left(\sigma_{xx}^{1D}\right)^w$	$\frac{\left(\sigma_{xx}^{1D}\right)^w}{\sigma_{xx}^{sv}}$ %
S_1	33.346	30.000	3.346	11.15
S_2	135.909	87.234	48.675	55.77

	$\left\lvert\frac{\sigma_{xx}^{1D}-\sigma_{xx}^{3D}}{\sigma_{xx}^{3D}}\right\rvert$ %
S_1	2.80
S_2	10.75

For the shear-bending, the structural behaviors of the three kinds of cross-section are similar, and the effect of the non-uniformity of the warping is very localized close to the critical section. The evaluation of the deflection can be done with SV-BT, and do not need a non-uniform warping theory. However, for a short beam and close to the critical section (as a built-in one), the stress prediction (especially the axial stress) must take into account the effect of the restrained warping, and use a non-uniform warping theory.

Flexural-torsional coupling. This coupling occurs for non-symmetrical-CS. Table 3 concerns the channel-CS (Figure 3-(c)) studied by (Kim *et al.*, 2005): for $\theta_x(L)$, the results are similar for the three theories (Vlasov-BT, Kim-BT[6], NUW-BT) and its magnitude is 90% lower than for uniform theory (SV-BT). The transversal displacement due to the flexural-torsional coupling computed by the present theory and that of Kim are very close. Both Kim-BT and NUW-BT find out a torsional-flexural coupling related to the non-symmetry of the cross-section but with different approaches. Kim-BT is built on a mixed approach (kinematic and static assumptions) and is written for thin-walled open/closed cross-sections, whereas NUW-BT is based on a fully kinematic approach and is supposed to be valid for any cross-section.

Table 3. *Coupling-effect: twisting angle θ_x and lateral displacement v_z*

	SV-BT	Vlasov-BT	Kim-BT	NUW-BT
$\theta_x(L)$	43.330	4.119	4.236	4.203
$v_z(L)$	-	-	2.163	2.369

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6. Kim-BT, written for open/closed thin-walled cross-section, is built on a mixed approach using the Hellinger-Reissner principle. This approach considers a kinematic model similar to that of NUW-BT, but where only the torsional warping function is considered, and introduces (thin-walled) Vlasov assumptions for the shear.