A generalized continuum approach to predict local buckling patterns of thin structures

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ABSTRACT. Macroscopic descriptions of instability pattern formation can be predicted by generic amplitude equations of Ginzburg-Landau type. A variant of this approach is presented, that permits to account for the coupling between local and global instabilities. The mean field and the amplitude of the fluctuations are governed by similar equations. The resulting model is a generalized continuum, where the macroscopic stresses are Fourier coefficients of the microscopic stresses. This new double scale description of cellular instabilities is applied to beam on an elastic foundation and to 3D nonlinear elasticity. We shall also discuss the behaviour of these new continuum models after a finite element discretisation.

RÉSUMÉ. L'évolution des instabilités spatio-temporelles peut se décrire macroscopiquement par des équations d'amplitude génériques de type Ginzburg-Landau. On établit une variante de cette approche, qui permet de prendre en compte des couplages entre instabilités locales et globales. Le champ moyen et la fluctuation obéissent à des équations similaires. Le modèle final est un milieu continu généralisé, où les contraintes macroscopiques sont des coefficients de Fourier de la contrainte microscopique. Cette nouvelle approche à deux échelles des instabilités cellulaires est présentée sur deux exemples : poutre sur fondation élastique et élasticité non linéaire 3D. On montrera également comment ces nouveaux modèles continus se comportent après une discrétisation par éléments finis.

KEYWORDS: Ginzburg-Landau equation, multi-scale analysis, local instability, local-global coupling, buckling, wrinkling.

MOTS-CLÉS : équation de Ginzburg-Landau, analyse multi-échelle, instabilités locales, couplage local-global, flambage, plissement.

DOI:10.3166/REMN.17.945-956 © 2008 Lavoisier, Paris

1. Introduction

In many cases of instability, the modal wavelength is very short, when compared with the size of the domain. Membrane wrinkling (Diaby *et al*, 2006) or thermo-convective instabilities (Wesfreid et Zaleski, 1984) are typical examples of these cellular instabilities. Such instabilities occur during the process of thin metal sheets by rolling, where the plastic deformation in the bite induces compressive residual stresses. These stresses generate sheet wrinkling, as depicted in Figure 1, which releases the compressive stresses and can affect the rolling process.

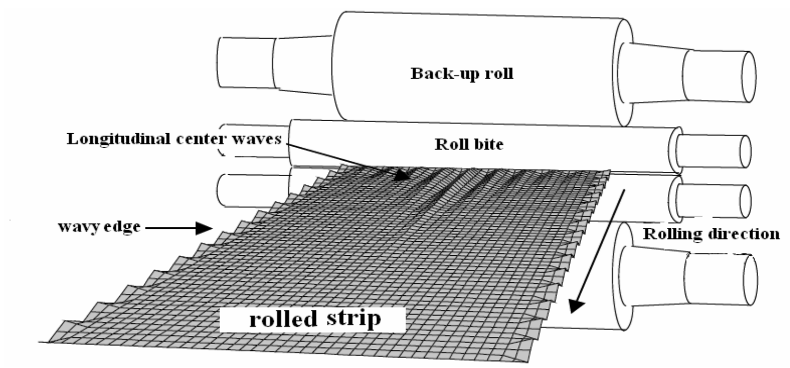


Figure 1. *Wrinkling patterns at bite exit*

Such phenomena can be modelized, either by direct simulations or by bifurcation analyses according to the Landau-Ginzburg theory. The first approaches are expensive if the local and global lengths are very different. Because they are based on restrictive assumptions on the response of the system, the second ones are limited to the vicinity of the bifurcation threshold.

A new approach is sketched in this paper. The model is a generalized continuum and the unknown fields include at least the mean value and the envelope of the fluctuations. The discretization of this model will also be studied.

2. Classical bifurcation analyses

The bifurcation theory allows to describe the evolution of a field $U(\mathbf{x}),\ \mathbf{x} \in \Omega \subset R^n$ as a function of a scalar parameter λ from a singular state $U_0(\mathbf{x}),\ \lambda_0$. In the case of a symmetric bifurcation and of a single mode $U_1(\mathbf{x})$, the bifurcated branch is parametrized by the amplitude of the deviation that is a real number denoted by a in the sequel:

$$U(\mathbf{x}) - U_0(\mathbf{x}) = aU_1(\mathbf{x}), \qquad \lambda - \lambda_0 = \lambda_2 a^2 . \tag{1}$$

In the case of a cellular instability, the amplitude $a(x)$ is complex to account for phase shift and it can vary slowly to account for amplitude modulations. The amplitude is governed by Landau-Ginzburg equation. For instance, let us consider a 2D case and let us assume that the pre-bifurcation state does not vary in the x-direction and that the bifurcation mode is harmonic in this direction. As it is well known, the states just after bifurcation are described by the formulae below, where q is the critical wavenumber, $\bar{a}$ is the complex conjugate of a and α_1, α_2 are scalar numbers ($i^2 = -1$)

$$U(x, y) - U_0(y) = [a(x)e^{iqx} + \bar{a}(x)e^{-iqx}]U_1(y) \tag{2}$$

$$\frac{d^2 a}{dx^2} + \alpha_1(\lambda - \lambda_0)a - \alpha_2 a^2 \bar{a} = 0 . \tag{3}$$

The latter analysis relies on a multi-scale asymptotic approach, where the amplitude is assumed to vary slowly when compared with e^{iqx}. For more details, see (Segel, 1969, Newell *et al*, 1969) for application to the Rayleigh-Bénard convection problem and to (Damil *et al,* 1986) for the buckling of long plates. The Landau-Ginzburg Equation [3] does not depend on the considered model and it is generic for any system having a « x→-x »symmetry (Iooss *et al*, 1989, Damil *et al,* 1992). The amplitude is only a multiplier of the bifurcation mode.

Another multi-scale approach is proposed in this paper, where slowly varying amplitudes are Fourier coefficients of the starting problem's unknowns.

3. *Generalized continuum media deduced from a two scale Fourier analysis*

We study phenomena such that the response of the system is the sum of a slowly varying mean field and a fluctuation that is nearly periodic in one spatial direction. As shown in Figure 2, at least two slowly varying functions are needed to model the phenomenon.

In this part, a general method is presented to deduce the equations satisfied by these slowly varying fields. All the unknowns of the models are sought in the form of Fourier series, whose coefficients vary slower than the harmonics:

$$U(x) = \sum_{m=-\infty}^{+\infty} U_m(x) e^{miqx} . \tag{4}$$

The present approach allows easily to describe the coupling between local and global buckling (Sridharan *et al*, 2001) or sandwich plates (Léotoing *et al*, 2002) or to predict the influence of wrinkling on the response of membranes (Diaby *et al*, 2006, Wong *et al*, 2006). The current work is motivated by the prediction of flatness defects at the exit of a rolling-mill.

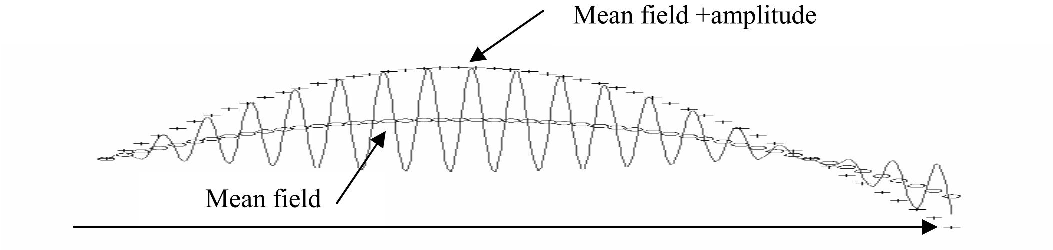


Figure 2. *At least two macroscopic fields are necessary to describe a nearly periodic response: the mean field and the amplitude of the fluctuation*

3.1. *First starting model: a nonlinear beam*

The proposed method is first applied to a classical nonlinear beam model which rests on an elastic foundation, where the constitutive law of the foundation is ($-g(v) = Cv + C_3 v^3$). The unknowns of the whole model are the axial displacement $u(x)$, the deflection $v(x)$ and the normal force $n(x)$:

$$\frac{dn}{dx} + f = 0, \qquad \frac{n}{ES} = \frac{du}{dx} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2, \qquad EI\frac{d^4 v}{dx^4} - \frac{d}{dx}(n\frac{dv}{dx}) - g = 0. \qquad [5]$$

3.2. *The associated generalized continuum*

One aims to establish a new macroscopic model, whose unknowns are the Fourier coefficients $U_m(x)$. Let us remark that coefficients of the harmonic 0 are real while other ones are complex, which doubles the number of degrees of freedom. The macroscopic model can be deduced from the microscopic one [5] simply by identifying the Fourier coefficients in each equation. Of course, the Fourier coefficients $U_m(x)$ are assumed to be constant over a period.

As an example, we first express the model obtained by considering only three terms in [4]: $U_0 \in R,\ U_1 \in C,\ U_{-1} = \overline{U_1}$. The coupling of all the harmonics is given in (Damil *et al* ,2006). The macroscopic model for the mean field is obtained from the harmonic 0 in each equation:

$$\frac{dn_0}{dx} + f = 0, \qquad \frac{n_0}{ES} = \frac{du_0}{dx} + \left(\frac{dv_0}{dx}\right)^2 + \left|\frac{dv_1}{dx} + iqv_1\right|^2, \qquad [6]$$

$$EI\frac{d^4 v_0}{dx^4} - \frac{d}{dx}\left(n_0 \frac{dv_0}{dx} + \overline{n_1}\left(\frac{d}{dx} + iq\right)v_1 + n_1 \overline{\left(\frac{d}{dx} + iq\right)v_1}\right)$$
$$+ Cv_0 + C_3\left(v_0^3 + 6v_0|v_1|^2\right) = 0. \qquad [7]$$

These equations look like the initial model, but nonlinearities induce some coupling between the mean field and the amplitude of the fluctuations. For instance the last term in [6-b] is positive and thus implies an elongation. If the beam is in compression, a local instability $v_1(x)$ can occur. In this case, the last term of [6-b] decreases the compressive stress. Hence this model permits a macroscopic description of the stress release due to local buckling in compressed thin sheets.

The equation governing the evolution of the fluctuation is obtained in the same way. If one supposes that the horizontal forces do not fluctuate ($f(x) = f_0(x)$), one establishes that the membrane stress and the horizontal displacement at order 1 are zero $n_1(x) = 0,\ \ u_1(x) = 0$. The equation for the fluctuation is then:

$$EI\left(\frac{d}{dx} + iq\right)^4 v_1 - \left(\frac{d}{dx} + iq\right)\left(n_0\left(\frac{d}{dx} + iq\right)v_1\right) + Cv_1 + 3C_3\left(v_0^2 v_1 + v_1^2 \overline{v_1}\right) = 0. \qquad [8]$$

Equation [8] is similar with the initial bending Equation [5-c]. There is a new coupling between mean fields and fluctuations. In particular a compressive macroscopic stress $n_0(x)$ can lead to local buckling, due to the second term in [8].

This Equation [8] is of 4th order. But if the assumptions of the asymptotic approach are applied:

$$\frac{d}{dx} = O(\sqrt{\varepsilon}), \quad v_1 = O(\sqrt{\varepsilon}), \quad \lambda - \lambda_0 = O(\varepsilon), \qquad [9]$$

one recovers the second order Landau-Ginzburg equation. Hence the proposed method is consistent with the asymptotic approach. One can consider [8] as a modified Landau-Ginzburg equation, that accounts from the starting model. But in the solution of [8], one can find boundary layers and rapidly varying responses that are cancelled in the asymptotic approach.

3.3. *Analysis of an eigenvalue problem*

To explain the nature of extended continuum introduced so far, consider the linear buckling problem of a long beam on a foundation, subjected to a uniform compression $n(x) = -\lambda$. One studies a non dimensional model, which is equivalent to choose $EI = 1$, $C = 1$. The bifurcation point λ_0 and the mode $v(x)$ are solutions of the linearized form of [5-c]:

$$\frac{d^4 v}{dx^4} + \lambda \frac{d^2 v}{dx^2} + v = 0. \quad [10]$$

In this example, analytical solutions are known. Disregarding boundary conditions, the smallest eigenvalue is $\lambda_0 = 2$ and the corresponding eigenmodes are $v(x) = e^{-ix}$ and $v(x) = e^{-ix}$. The critical wavenumber is $q = 1$.

We limit ourselves to the harmonic 1, *i.e.* to a simplified version of Equation [8]. Hence the associated extended model is:

$$\left(\frac{d}{dx} + i\right)^4 v_1 + \lambda \left(\frac{d}{dx} + i\right)^2 v_1 + v_1 = 0. \quad [11]$$

Because this model is linear, one checks easily that the Equations [10] and [11] are equivalent, via the relation $v(x) = v_1(x)e^{ix}$. Hence [11] has the same eigenvalue $\lambda_0 = 2$ and the corresponding modes are $v_1(x) = 1$, that is slowly varying, and $v_1(x) = e^{-2ix}$, that is varying rapidly and therefore is not consistent with the assumptions below. So, the generalized continuum model contains some slowly varying solutions, as expected, but it contains also other very oscillating solutions. Note that the latter ones are dropped by the asymptotic approach. Indeed, if the rules [9] are applied to Equation [11] and if it is truncated at order ε, one gets the linearized Landau-Ginzburg equation, that is:

$$4\frac{d^2 v_1}{dx^2} + (\lambda - 2)v_1 = 0. \quad [12]$$

One remarks that the solutions of [12] are varying slowly if the parameter λ is close to its critical value. Hence the Landau-Ginzburg approach is able to select the adequate solutions, according to an asymptotic criterion. In Part 4, we shall try to drop the oscillating solutions by discretising the model [11] by a coarse finite element mesh. In other words, the asymptotic low-pass filter will be replaced by a numerical one.

3.4. *Extension to 2D and 3D elasticity. Analysis of the constitutive law*

Let us now apply the same procedure in nonlinear elasticity with a linear stress-strain law. With the notations of (Cochelin *et al*, 2007), the 2[nd] Piola-Kichhoff stress $\{ s \}$ and the Green-Lagrange strain $\{ \gamma \}$ are related by a linear relationship, while the strain is related to the displacement gradient $\{\theta\}$ by a quadratic relationship:

$$\{ s \} = [D]\{ \gamma \}, \qquad \{ \gamma \} = [H]\{ \theta \} + \frac{1}{2}[A(\theta)]\{ \theta \}. \tag{13}$$

We seek nearly periodic responses that vary rapidly in one direction. This characteristic direction and the period are described by a wave vector $\mathbf{q} \in R^3$ that is assumed to be given. In practice, this vector comes from a linear stability analysis. Thus the vector $\{ \Lambda(\mathbf{x}) \}$, which includes displacement vector, its gradient, strain and stress tensors, is sought in the form of a Fourier series, whose coefficients $\{ \Lambda_m(\mathbf{x}) \}$ are varying slowly. For simplicity, we keep harmonics up to level 2:

$$\{ \Lambda(\mathbf{x}) \} = \sum_{m=-2}^{+2} \{ \Lambda_m(\mathbf{x}) \} e^{mi\mathbf{q}.\mathbf{x}}. \tag{14}$$

The latter procedure is applied to the constitutive law [13]. So we get a constitutive law for harmonic 0 (real number) and two constitutive laws for harmonics 1 and 2 (complex number), all these equations being coupled:

$$\begin{aligned} &[D]^{-1}\{ s_0 \} = \{ \gamma_0 \} = \\ &[H]\{ \theta_0 \} + \frac{1}{2}[A(\theta_0)]\{ \theta_0 \} + [A(\theta_{-1})]\{ \theta_1 \} + [A(\theta_{-2})]\{ \theta_2 \}, \end{aligned} \tag{15}$$

$$[D]^{-1}\{ s_1 \} = \{ \gamma_1 \} = [H]\{ \theta_1 \} + [A(\theta_{-1})]\{ \theta_2 \} + [A(\theta_0)]\{ \theta_1 \}, \tag{16}$$

$$[D]^{-1}\{ s_2 \} = \{ \gamma_2 \} = [H]\{ \theta_2 \} + [A(\theta_0)]\{ \theta_2 \} + \frac{1}{2}[A(\theta_1)]\{ \theta_1 \}. \tag{17}$$

Thus a generalized continuum model has been defined, that is a sort of superposition of several continua. Like every field of the model, the displacement is replaced by a generalized displacement that includes five Fourier coefficients for $m \in [-2,2]$. The constitutive law [15,16,17] can be merged in the following way:

$$\{ S \} = \left[D^{gen}\right]\{ \Gamma \}, \qquad \{ \Gamma \} = \left[H^{gen}\right]\{ \Theta \} + \frac{1}{2}\left[A^{gen}(\Theta)\right]\{ \Theta \}, \tag{18}$$

where the generalized stress $\{ S \}$, the generalized strain $\{ \Gamma \}$ and the generalized displacement gradient $\{ \Theta \}$ also include five Fourier coefficients. Note that the mth component of the displacement gradient is not the gradient of the mth component of the displacement, to account for rapid oscillations: in the same way the spatial derivative had been replaced by $\frac{d}{dx} + iq$ in the 1D case. The rule defining the Fourier components of a gradient vector is the following:

$$\{\nabla \mathbf{u}\}_m = \{\nabla(u_m)\} + mi\,[Q]\{u_m\} \qquad \text{where} \quad [Q] = \begin{bmatrix} \{q\} & 0 & 0 \\ 0 & \{q\} & 0 \\ 0 & 0 & \{q\} \end{bmatrix}. \qquad [19]$$

3.5. *Extended principle of virtual work*

One has obtained a constitutive law [18] for the extended elastic continuum that is similar to the initial one [13], but with a larger number of unknowns. In the same manner, we will define the principle of virtual work for the extended continuum. The previously defined technique with slowly varying Fourier coefficients can be applied to the equation of the balance of forces. The weak form of those equations is then the extended principle of virtual work. This weak form can also be deduced directly from the principle of virtual work of the initial problem, by using Parseval's identity. (Damil *et al*, 2006). The deduced principle of virtual work involves the Fourier coefficients of the stress and strain:

$$\int_\Omega \sum_{m=-2}^{+2} {}^t\{ \delta\gamma_{-m} \}\{ s_m \}\, d\Omega = \int_\Omega \sum_{m=-2}^{+2} {}^t\{ \delta u_{-m} \}\{ f_m \}\, d\Omega\,, \qquad [20]$$

where, for simplicity, the boundary terms have been omitted. The left hand side of [20] involves Fourier coefficients of stress and strain, i.e. macroscopic stresses and strains. One sees that the extended principle of virtual work has the same shape as in the initial model

$$\int_\Omega {}^t\{ \delta\Gamma \}\{ S \}\, d\Omega = \int_\Omega {}^t\{ \delta U \}\{ F \}\, d\Omega\,. \qquad [21]$$

It can be established that the Equations [18] [21] come from the stationarity of the following potential energy:

$$P^{gen}(U) = \frac{1}{2}\int_{\Omega} {}^{t}\{\Gamma(U)\}\left[D^{gen}\right]\{\Gamma(U)\}\, d\Omega - \int_{\Omega} {}^{t}\{U\}\{F\}\, d\Omega. \qquad [22]$$

4. Discretisation and numerical tests

4.1. *Finite element discretisation*

Because the generalized continuum model has the same form as the initial model, it can be discretized with the same set of shape functions. In the initial model, the displacement and its gradient can be related to nodal variables *via* two interpolation matrices $[N]$, $[G]$:

$$\{u(\mathbf{x})\}^{e} = [N]\{v\}^{e}, \qquad \{\nabla u(\mathbf{x})\}^{e} = [G]\{v\}^{e}. \qquad [23]$$

When applying this discretization principle to the generalized displacement and to the generalized gradient, one gets interpolation formulae similar to [23]. The first matrix, that interpolates the generalized displacement is block diagonal, since all the components are interpolated in the same way. The second matrix interpolates the displacement gradient and it is slightly more complicated because of the derivation rule [19]. The coupling between micro and macro scales appears at this level, via the wavenumber matrix $[Q]$:

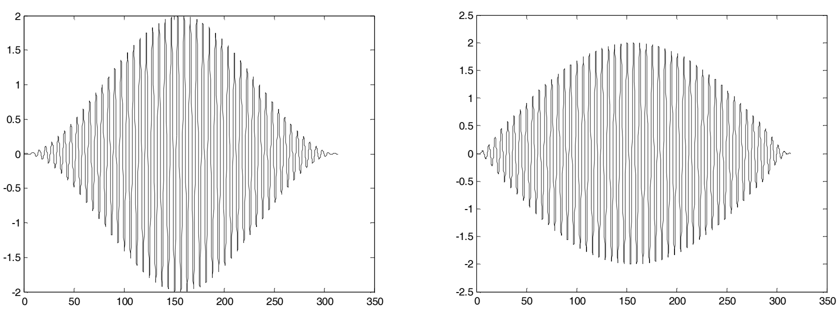
$$\{U(\mathbf{x})\}^{e} = \begin{bmatrix} N & 0 & 0 & 0 & 0 \\ 0 & N & 0 & 0 & 0 \\ 0 & 0 & N & 0 & 0 \\ 0 & 0 & 0 & N & 0 \\ 0 & 0 & 0 & 0 & N \end{bmatrix} \{V^{gen}\}^{e}, \qquad [24]$$

$$\{\Theta^{gen}(\mathbf{x})\}^{e} = \begin{bmatrix} [G] & 0 & 0 & 0 & 0 \\ 0 & [G] & -[Q][N] & 0 & 0 \\ 0 & [Q][N] & [G] & 0 & 0 \\ 0 & 0 & 0 & [G] & -2[Q][N] \\ 0 & 0 & 0 & 2[Q][N] & [G] \end{bmatrix} \{V^{gen}\}^{e}. \qquad [25]$$

4.2. *Numerical evaluation in a 1D eigenvalue problem*

Let us consider the linear beam buckling problem (see Section 3.3), described by the Equation [10], with a length $L = 100\pi$ and with clamped boundary conditions. The beam length is large, in such a way that the instability mode has about 50 cells. A direct analysis of [10] requires at least 400 Hermite finite elements. This direct analysis established that the first mode is a modulated oscillation, the amplitude having a sinusoidal shape. Incidentally, the Landau-Ginzburg equation, when associated with a Dirichlet boundary condition, predicts correctly this sinusoidal shape.

The equation of the macroscopic model [11] has been solved by the same interpolation principle, but with twice the number of degrees of freedom, because the envelope is described by a complex number. The mode is depicted on Figure 3a, with two macroscopic elements and assuming that $v_1(x)$ and its derivative are zero at both ends. This mode has not the expected sinusoidal shape, but Figure 3b shows that one comes near to this shape with ten macroscopic elements. However there are small differences with the reference solution near boundaries, because of boundary layer effects quoted in Section 3.2. Nevertheless a correct solution cannot be obtained with a very fine mesh. For instance with 100 macroscopic elements, one gets very oscillating solutions as predicted in Section 3.3.



a) Two macroscopic finite elements

(b) Ten macroscopic finite elements. The envelope has the true sinusoidal shape, with boundary effects

Figure 3. *Beam of length 100 π. The envelope v_1 and its derivative are zero at the ends*

However it is not necessary that the envelope and its derivative are set to zero at the boundary, even if these conditions are satisfied by the unknown $v(x)$ of the initial problem. As in many asymptotic approaches, the assumptions [4] are not necessarily valid up to the boundary. To solve this problem, it should be possible to apply the macroscopic model inside the domain, to apply the starting model near the boundary and to match the two solutions. One can also hope that the macroscopic model is more or less exact everywhere, as it is usual for instance in the homogenization theory. Let us consider clamped boundary conditions and assume that fluctuations

$$\mathrm{Re}\left(v_1(x)e^{iqx}\right) = v_1^R(x)\cos(qx) - v_1^I(x)\sin(qx) \quad [26]$$

vanish at $x = 0$, as well as its derivative. This implies that the real and imaginary parts of the envelope satisfy the two following boundary conditions:

$$v_1^R(0) = 0, \quad \frac{dv_1^R(0)}{dx} - qv_1^I(0) = 0. \quad [27]$$

If, in addition, the approximation [9] is taken into account, it appears that the amplitude $v_1(x)$ vanishes at the end, but the derivative does' nt. In the numerical experiment we now assume that $v_1(x)$ is zero, but no longer the derivative. As shown in Figure 4, the sinusoidal shape is recovered with a single macroscopic element. This implies that the macroscopic mesh is not related with the microscopic wavelength and the boundary conditions of the macroscopic model have to be settled carefully.

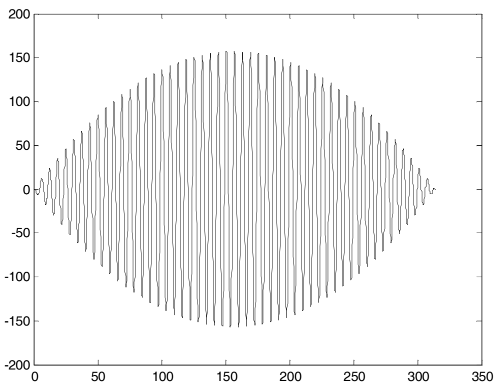


Figure 4. *Clamped beam of length 100 π. The envelope v_1 is zero at the boundary. The sinusoidal shape is recovered with only one macroscopic element*

5. Conclusions

A two scale approach to predict quasi-periodic instabilities has been presented, leading to a generalized continuum. The so defined macroscopic stresses are Fourier coefficients of the microscopic stress. Clearly, this approach is interesting when the modal wavelength is small with respect to the macroscopic wavelength. It seems much more manageable than Landau-Ginzburg methods, especially for discretization. The choice of boundary conditions for the macroscopic model is not an obvious task. We have not discussed here the number of harmonics to be accounted for; but at least five ones $0, \pm q, \pm 2q$ are required to recover the Landau-Ginzburg equation.

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