An enriched finite element approach for prescribed motions of thin immersed structures in a fluid

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ABSTRACT. *An Eulerian-Lagrangian fluid-structure coupling approach is presented. The method is dedicated when several thin structures are immersed in a fluid domain. Moreover, the structures may have large displacements. The finite element method is used for the space discretization. A fractional time scheme is used for time integration. The main point of the method is that the fluid mesh is fixed and is completely independent of the structure positions. In order to take into account the interface inside the fluid elements, new functions are added in the velocity and pressure fluid fields by using the extended finite element method X-FEM.*

RÉSUMÉ. *Une approche Euler-Lagrange pour le calcul couplé fluide-structure est présentée. La méthode est dédiée au cas de plusieurs structures minces immergées ayant de grands mouvements. La méthode des éléments finis est utilisée pour la discrétisation en espace. Le schéma d'intégration en temps est semi-implicite à pas fractionnés. Le point fort de la méthode réside dans le fait que le maillage fluide est fixe et indépendant de la position des structures. Afin de prendre en compte les discontinuités liées à la présence des interfaces dans le maillage fluide, les champs d'approximations de vitesse et de pression sont enrichis par l'ajout de nouvelles fonctions par la méthode des éléments finis étendus X-FEM.*

KEYWORDS: *fluide-structure interaction, Euler-Lagrange, thin immersed structures, large displacements, incompatible and overlapped meshes, enrichment (X-FEM), fractional time scheme.*

MOTS-CLÉS : *interaction fluide-structure, Euler-Lagrange, structures minces immergées, grands mouvements, maillages incompatibles, maillages recouvrants, enrichissement (X-FEM), schéma fractionné en temps.*

1. Introduction

This paper presents an approach to compute a fluid-structure coupled system in dynamics. The fluid is viscous, incompressible and can be with or with no flow. Several thin structures are immersed in the fluid. From the fluid point of view, they are seen as surfaces with no thickness (Figure 1). The structures can have large displacements. Although this first study is applied for an *a priori* known structure displacements the method can be extended to flexible structures.

The finite element method is used to discretize the fluid domain. The structures are localized in the fluid by level-sets. The time integration scheme is based on a fractional time scheme (Tralli *et al.*, 2005b; Tralli *et al.*, 2005a).

The arbitrary Lagrange-Euler (ALE) is classically used to treat this kind of problem. However, spurious mesh distorsions appear when the structures have large relative displacements.

The proposed method, based on the three following points, avoids this disadvantage:

1) an Eulerian description is used for the fluid while a Lagrangian description is used for the structure (prescribed structural displacements in this work), the method takes the advantages of both representations;

2) the fluid mesh is completely independent of the structure positions, this avoids remeshing strategies and mesh distorsions around the interfaces;

3) the fluid approximation fields are enriched by adding new functions around the fluid-structure interfaces in order to take into account the pression jump as well as the velocity gradient discontinuity, this enrichment is based on the so called extended finite element method (X-FEM) (Moës *et al.*, 1999).

A first application of the proposed method has been done in (Legay *et al.*, 2006) for a compressible fluid. The fluid was only on one side of the structure, therefore there was no need of enrichment in the fluid approximation. In other terms, only the two first points of the method were used. The X-FEM has already been applied to model evolving interfaces in a fluid domain: solidification problem (Chessa *et al.*, 2002), two-phase flow (Chessa *et al.*, 2003a; Walhorn *et al.*, 2005), bio-film membrane (Bordas, 2003).

Similar methods have been developped. Several meshless methods as SPH or EFG and more recently the natural element method (NEM) (Martinez *et al.*, 2003) allow to deal with large boundary displacements. Other methods use the idea of a fixed fluid mesh containing the structure: the immersed finite element method (Zhang *et al.*, 2004), the immersed boundary method (Peskin, 2002) and the fictitious domain methods (Glowinski *et al.*, 2001; Baaijens, 2001; Bertrand *et al.*, 1997). These methods do not use any enrichment in the fluid approximation.

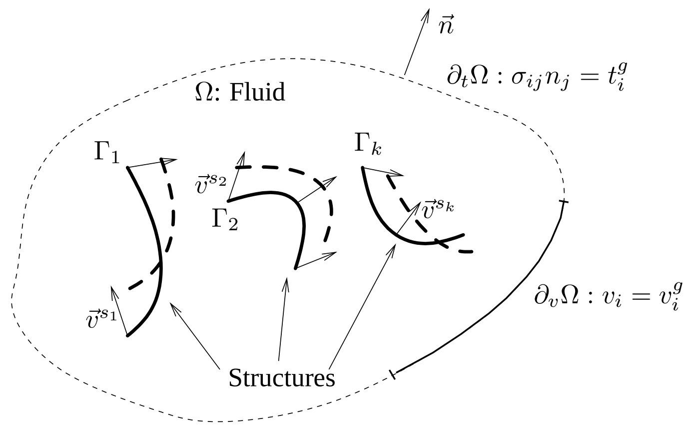

Figure 1. *Fluid-structure interaction problem where several thin structures are immersed in the fluid*

2. Problem description

2.1. *Strong form of the fluid equations*

The fluid (Ω domain, Figure 1) is incompressible and viscous. It has an Eulerian description. The momentum equations are

$$\rho\dot{v}_i + \rho v_{i,j}v_j - \tau_{ij,j} - \rho g_i = 0 \quad \text{in} \;\; \Omega \qquad [1]$$

where ρ is the density, v_i is the Eulerian velocity and g_i is the gravity. The stress tensor τ_{ij} is given by

$$\tau_{ij} = -p\delta_{ij} + 2\mu e_{ij} \qquad [2]$$

where p is the pressure and μ is the viscosity. The strain rate tensor e_{ij} is given by

$$e_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}). \qquad [3]$$

Boundary conditions in terms of velocity are

$$v_i - v_i^g = 0 \quad \text{on} \;\; \partial_v\Omega \qquad [4]$$

where v_i^g is the given velocity on the $\partial_v\Omega$ boundary. Boundary conditions in terms of traction are

$$\sigma_{ij}n_j - t_i^g = 0 \quad \text{on} \;\; \partial_t\Omega \qquad [5]$$

where n_i is the external unit normal vector to Ω and t_i^g is the given traction force on $\partial_t\Omega$ boundary. The incompressibility equation is

$$v_{i,i} = 0 \quad \text{in} \quad \Omega. \qquad [6]$$

2.2. *Strong form of the fluid-structure coupling equations*

Each k^{th} structure has its own *a priori* known displacement in this work, therefore the interface condition, for a non-slip interface, is

$$v_i - v_i^{s_k} = 0 \quad \text{on} \quad \Gamma_k \qquad [7]$$

where $v_i^{s_k}$ is the k^{th} structure known velocity and Γ_k is the interface known position between the fluid and the k^{th} structure. This condition becomes for a slip interface

$$v_i n_i^{s_k} - v_i^{s_k} n_i^{s_k} = 0 \quad \text{on} \quad \Gamma_k \qquad [8]$$

where $n_i^{s_k}$ is the unit normal vector to Γ_k.

2.3. *Weak form of the coupled system*

The weak form for the coupled system is: find v_i such that $\forall \delta v_i$,

$$\int_\Omega \delta v_i \, \rho\Big(\dot{v}_i + v_{i,j} v_j\Big) \mathrm{d}\Omega + \int_\Omega \delta v_{i,j} \tau_{ij} \mathrm{d}\Omega \\ - \int_\Omega \delta v_i \rho g_i \mathrm{d}\Omega - \int_{\partial\Omega} \delta v_i \tau_{ij} n_j \mathrm{d}S + \beta \sum_k \int_{\Gamma_k} \delta v_i \Big(v_i - v_i^{s_k}\Big) \mathrm{d}\Gamma_k = 0 \qquad [9]$$

with $v_{i,i} = 0$. This last condition is treated by a three step algorithm detailled in Section 3.3.

The interface velocity continuity is enforced by a penalty term (the last term of [9]) where β is an arbitrary large scalar. For a slip interface condition this penalty term becomes

$$\beta \sum_k \int_{\Gamma_k} \delta v_i n_i^{s_k} \left(v_j n_j^{s_k} - v_j^{s_k} n_j^{s_k} \right) \mathrm{d}\Gamma_k \qquad [10]$$

Note that this enforcement can be obtained by using distributed Lagrange multipliers as it has been done in (Kölke *et al.*, 2006).

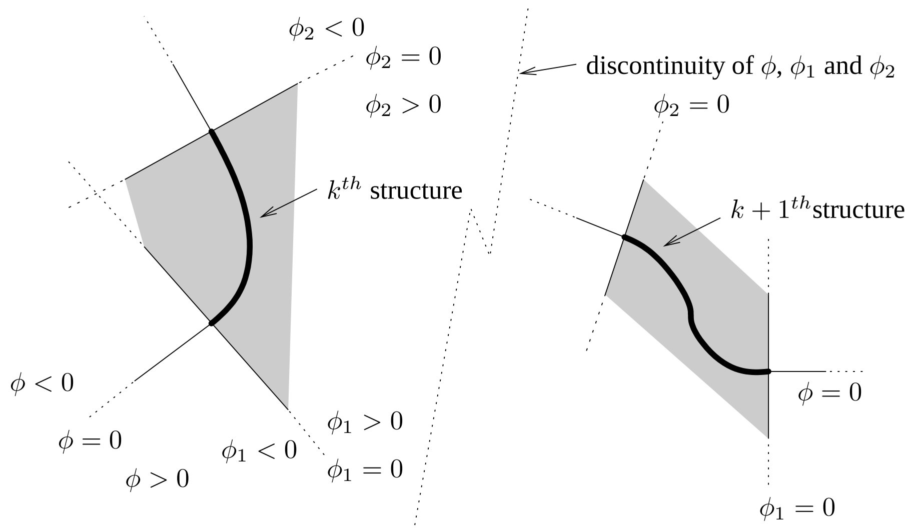

Figure 2. *Level-sets are used to localize a structure. The structure is on the zero level-set ϕ, in the ϕ_1 and ϕ_2 positive domain (gray domain). The same level-set functions can be used to localize several structures. These functions are discontinuous from a structure neighborhood to an other*

3. Discretization

3.1. *Space discretization*

The fluid discretization is based on the finite element method. A 9-node quadrilateral element is used for the veloticity field ($N_I^9(\underline{x})$ shape functions) while a 4-node quadrilateral element is used for the pressure field ($N_I^4(\underline{x})$ shape functions). The second element is based on the 4 corner nodes of the first. This *Q2-Q1* choice passes the LBB condition (Brezzi, 1974).

3.2. *Structure localizations*

In the fluid domain, the position of a structure is given by the zero level-set of a function $\phi(\underline{x}, t)$ (Figure 2) (Sethian, 1999). Such a function is for instance the signed distance to the interface. This function is updated by the *a priori* known structure position at each time. The function $\phi(\underline{x}, t)$ is positive on one structure side and negative on the other side.

In the case of an open structure with two tips, two other functions $\phi_1(\underline{x}, t)$ and $\phi_2(\underline{x}, t)$ are used to localize the tips. The two tip positions are such that $\phi_1(\underline{x}, t) = 0$

and $\phi_2(\underline{x},t) = 0$ with the condition $\phi(\underline{x},t) = 0$. The structure is in the domain where $\phi_1(\underline{x},t)$ and $\phi_2(\underline{x},t) = 0$ are both positives.

When several structures are immersed, they are all described by the three last level-sets $\phi(\underline{x},t)$, $\phi_1(\underline{x},t)$ and $\phi_2(\underline{x},t)$. These three functions are correct around each structure, that means they describe correctly the k^{th} structure, but they are discontinuous from a structure neighborhood to an other. However, when structure collisions are expected, each structure has to have its own set of three level-sets and numerical contact strategy has to be implemented as for instance in (Laure *et al.*, 2005).

The level-set functions are interpolated in the elements by using the quadratic approximation of the 9-node element.

3.3. *Time scheme*

The used time scheme has been developped recently. More details can be found in (Tralli *et al.*, 2005b) and (Tralli *et al.*, 2005a). It is based on a third order Runge-Kutta integration. The incompressibility equation is solved separately by a fractional time step strategy. The momentum equation are then solved by a semi-implicit scheme. The scheme can be decomposed into three steps:

1) equilibrium: solve momentum equation by a semi-implicit scheme with no taking into account the incompressibility, an intermediate velocity field v_i^* is obtained;

2) projection: under very general hypotheses, it is possible to split a generic vector field into the sum of a solenoidal field and an irrotational field: $v_i^* = v_i + \chi_{,i}$. The scalar field χ is an additional unknown potential;

3) correction: the end-of-step variables v_i and p are computed from v_i^* and χ.

4. Space approximation enrichment

4.1. *Partition of unity principle*

A partition of unity is a set of function $f_i(\underline{x})$ defined on Ω^{PU} such that their summation in Ω^{PU} is 1. This property allows to introduce any arbitrary functions $\psi(\underline{x})$ in the approximation space in Ω^{PU} (Melenk *et al.*, 1996) (Babuška *et al.*, 1997). The approximation of $g(\underline{x})$ can then be enriched in Ω^{PU} with an additional function $\psi(\underline{x})$,

$$g(\underline{x}) = \sum_j N_j(\underline{x}) G_j + \sum_i f_i(\underline{x}) \psi(\underline{x}) A_i^g \qquad [11]$$

where N_j is the shape function of the j^{th} node, G_j is the nodal unknown of $g(\underline{x})$ on the j^{th} node, and A_i^g is a new unknown associated to the i^{th} function of the partition of unity.

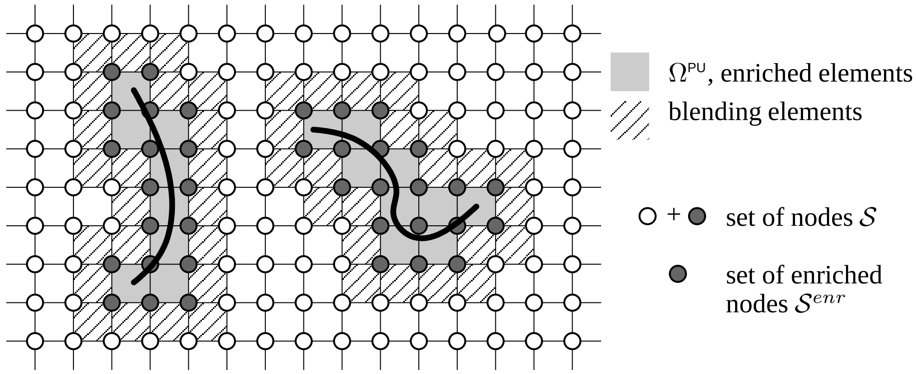

Figure 3. *Partition of unity support. The enriched domain is the set of all the fluid elements which are cut by the structures*

4.2. *Enriched fluid approximation*

4.2.1. *Partition of unity support* Ω^{PU}

The enriched domain Ω^{PU} is the set of all the fluid elements cut by the structures (Figure 3). These elements can be found by the knowledge of the three functions $\phi(\underline{x},t)$, $\phi^1(\underline{x},t)$ and $\phi^2(\underline{x},t)$ (Section 3.2).

The domain which is between the enriched and the non-enriched domain is the so called blending domain. A blending element may introduce spurious terms in the approximation. Strategies to correct these elements have been developped (Chessa *et al.*, 2003b). The chosen functions in this work for partition of unity, velocity and pressure approximations and for the enrichment do not introduce spurious terms as it is shown in (Legay *et al.*, 2005).

4.2.2. *Velocity field*

From a structure side to the other, velocity is continuous but its gradient is discontinuous (Figure 4). The approximation is enriched by a ramp like function, for instance by using the absolute value of the signed distance to the interface. The chosen partition of unity is the set of the 4 bilinear shape functions $N_j^4(\underline{x})$. The enriched velocity approximation in Ω^{PU} becomes

$$v_i = \sum_{I\in\mathcal{S}} N_I^9(\underline{x})V_{Ii} + \sum_{J\in\mathcal{S}^{enr}} N_J^4(\underline{x})(|\phi(\underline{x},t)| - |\phi(\underline{x}_J,t)|)A_{Ji}^v \qquad [12]$$

where $\mathcal{S}$ is the set of mesh nodes, V_{Ii} is the i^{th} component of the I^{th} node velocity, $\mathcal{S}^{enr}$ is the set of enriched nodes, $\underline{x}_J$ is the J^{th} node coordinates and A_{Ji}^v is an additional unknown.

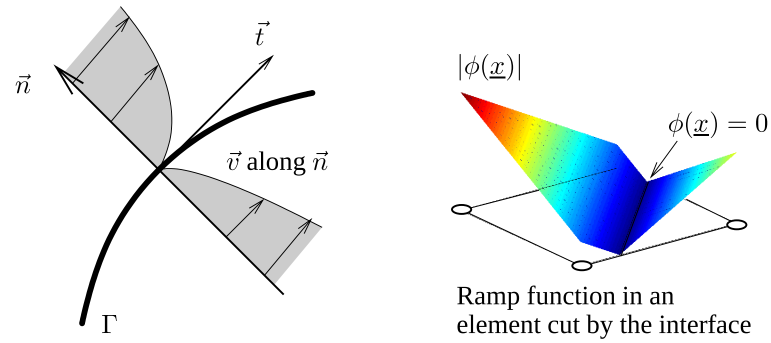

Figure 4. *Discontinuity of the velocity gradient from a structure side to the other. Ramp like enrichment of the velocity approximation*

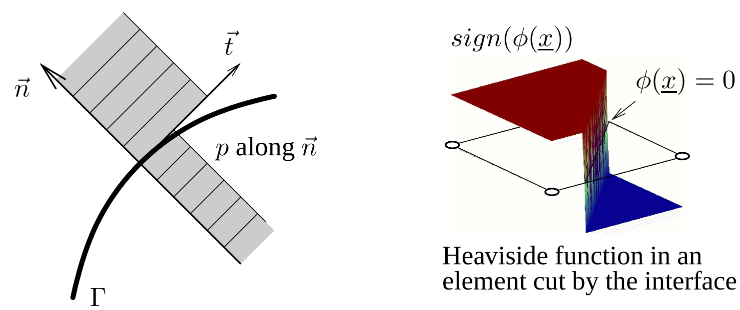

Figure 5. *Pressure discontinuity from a structure side to the other. Heaviside like enrichment of the pressure approximation*

4.2.3. *Pressure field*

From a structure side to the other, pressure is discontinuous (Figure 5). The approximation is enriched by a Heaviside like function, for instance by using the sign of the signed distance to the interface. As well as for the velocity enrichment, the chosen partition of unity is the set of the 4 bilinear shape functions $N_j^4(\underline{x})$. The enriched pressure approximation in Ω^{PU} is

$$p = \sum_{I \in \mathcal{S}} N_I^4(\underline{x}) P_I + \sum_{J \in \mathcal{S}^{enr}} N_J^4(\underline{x})(sign(\phi(\underline{x}, t) - sign(\phi(\underline{x}_J, t)) A_J^p \qquad [13]$$

where P_I is the I^{th} node pressure and A_J^p is an additional pressure unknown.

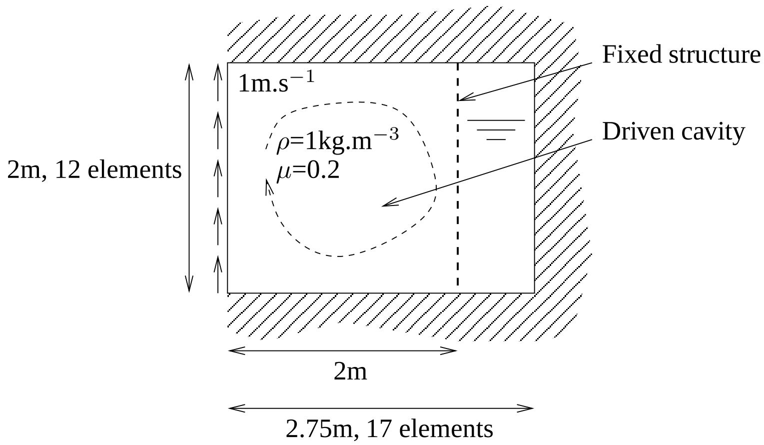

Figure 6. *Driven cavity problem*

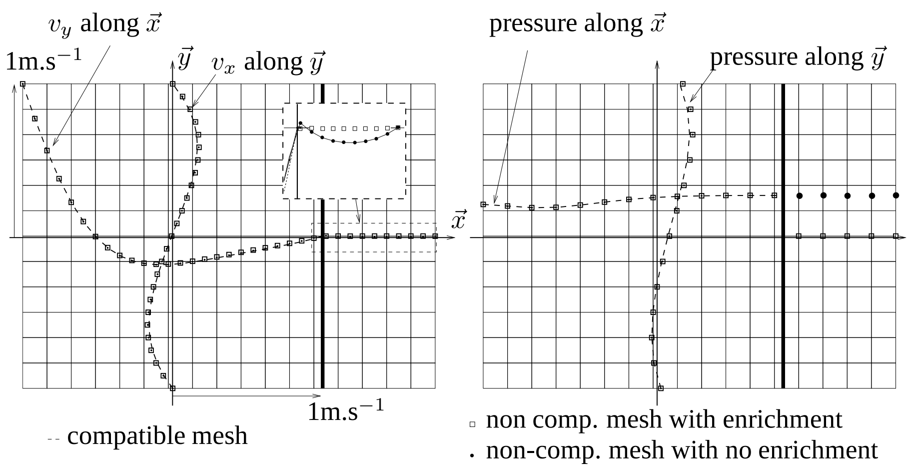

Figure 7. *Velocity and pressure profiles at t=1s for the driven cavity problem*

5. Applications

5.1. *Driven cavity*

The computational fluid domain is a rectangular separated by a fixed rigid structure (Figure 6). The structure is placed arbitrarily in the fluid mesh. The left part is a square in which the left boundary is driven in order to have the well known driven cavity problem. The right part has only fixed boundaries, it is expected to have no fluid velocity as well as a constant zero pressure.

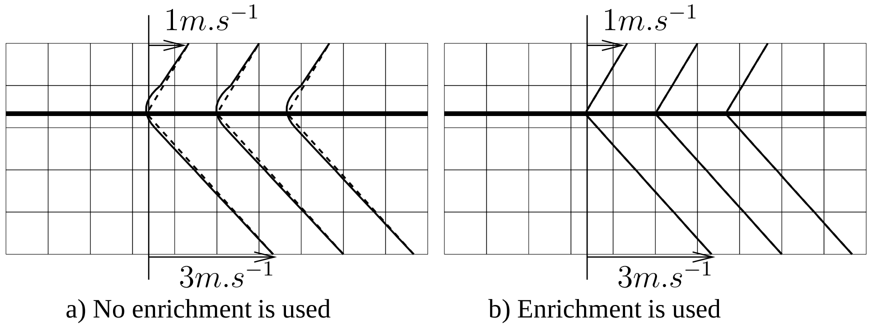

Figure 8. *Velocity profiles for the divided channel. The dotted line correspond to the exact velocity while the continuous line is the numerical one*

Figure 7 shows velocity and pressure along $\vec{x}$ and $\vec{y}$ for the stationnary state. Three results are compared: compatible mesh on the interface, non-compatible mesh with no enrichment, non-compatible mesh with enrichment. The three computations give almost the same results in the driven cavity part (left part). In the right part, the results show clearly that the enrichment is necessary to recover both zero velocity and pressure fields. This example shows that the method can decouple two fluid domains separated by an arbitrary located structure in the fluid mesh.

5.2. *Divided channel*

A fixed horizontal structure is immersed in a rectanguler fluid domain. The structure cut the fluid domain into two independent domains. The upper fluid boundary is driven with a 1 m.s^{-1} horizontal velocity while the lower one is driven with a 3 m.s^{-1} horizontal velocity. The expected velocity profile at the steady state is linear in each domain. The expected pressure is constant.

The results are shown on Figure 8 at steady state for both cases when the enrichment is not used (Figure 8a) and when the enrichment is used (Figure 8b). Velocity profiles are drawn for several sections. The velocity gradient discontinuity is well modeled when the enrichment is used and the exact solution is recovered while it is not when no enrichment is used.

5.3. *Translating piston in a channel*

A rigid piston is immersed in a one-dimensionnal channel (10 m $\times$ 1 m) filled by an inviscid fluid (Figure 9). It has a constant acceleration a_p = 0.1 m.s^{-2}. It can easily be shown that the pressure gradient in the fluid is equal to $p_{,x} = -\rho a_p$ where ρ = 2 kg.m^{-3} is the fluid density while the velocity along $\vec{x}$ is equal to the piston velocity

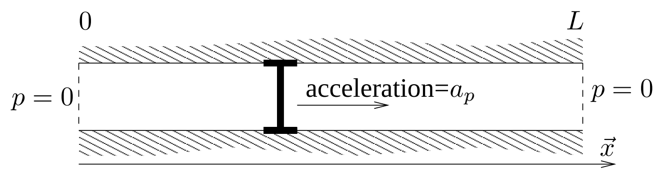

Figure 9. *Translating piston in a channel*

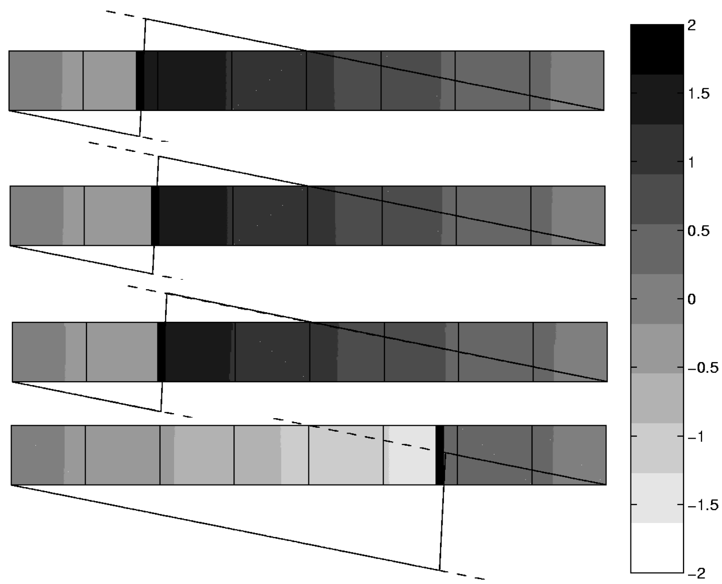

Figure 10. *Pressure field for the translating piston in a channel. The dotted line correspond to the exact pressure while the continuous line is the numerical one*

at each time. Since the pressure is imposed to be zero on the two ends of the channel, there is a large pressure jump from one side to the other one of the piston.

The pressure field is plotted on Figure 10 for several time steps. The piston moves within several fluid elements. The pressure jump through the piston is exactly catched by the Heaviside enrichment. The numerical velocity in the channel is not plotted since it is exactly the piston velocity at each time step.

5.4. *Fixed immersed structure in a driven cavity*

The fluid domain is a rectangular cavity, driven on the upper boundary (Figure 11). Figure 12 shows the streamlines at the stationnary state as well as the pressure field.

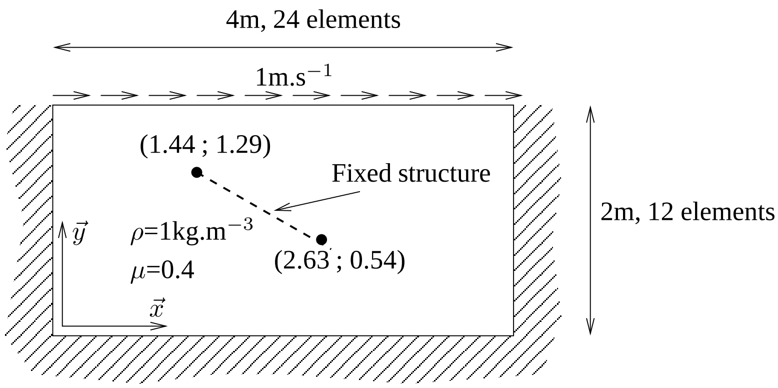

Figure 11. *Fixed structure in a driven cavity*

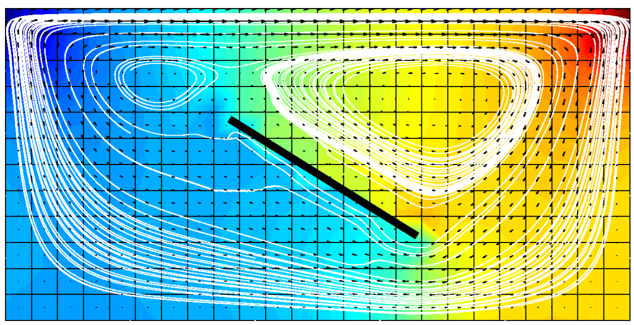

Figure 12. *Streamlines and pressure field for the fixed immersed structure problem at t=0.5s*

The pressure jump is well catched accross the structure. The streamlines are correct and go over the structure.

5.5. *Translating structure in a cavity*

A straight structure is immersed in a closed cavity (Figure 13). The structure is initially placed vertically in the left side of the cavity, it has an horizontal constant velocity. Figure 14 shows the streamlines as well as the pressure field for several time steps. The pressure jump accross the structure is well catched. The streamlines, represented for the relative velocity between fluid and structure, are correct.

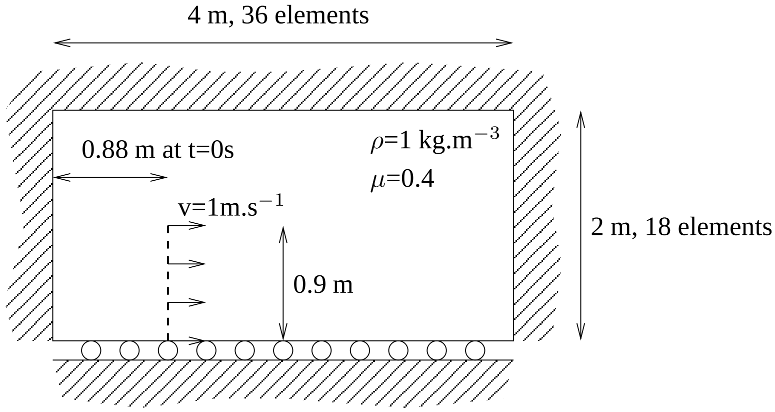

Figure 13. *Translating structure in a cavity*

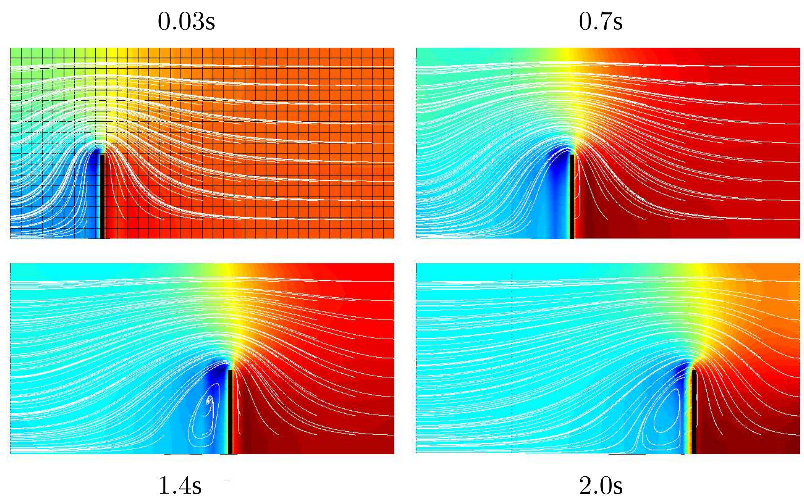

Figure 14. *Streamlines and pressure field for the translating structure problem. The streamlines are represented for the relative velocity between fluid and structure*

5.6. *Two rotating structures in a cavity*

Two straight structures rotate in a closed cavity (Figure 15). Figure 16 shows that the streamlines as well as the pressure field for several time steps are correct according to the structures positions.

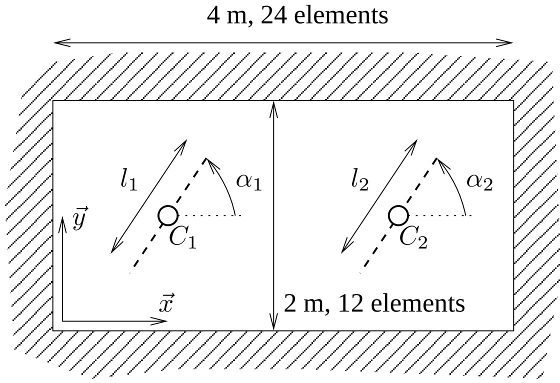

Figure 15. *Two rotating structures in a cavity*

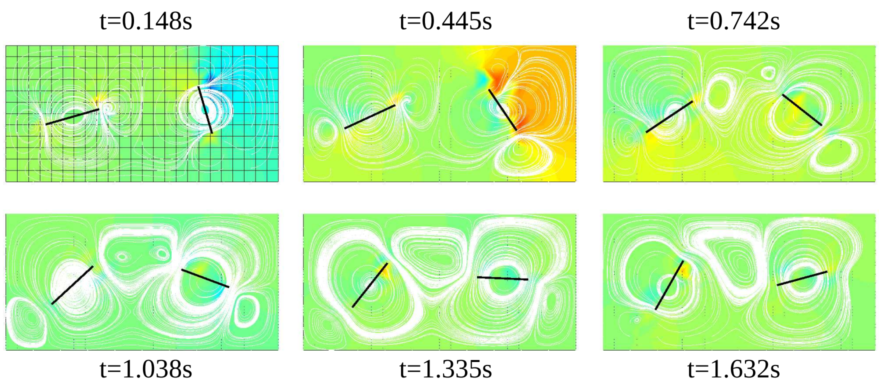

Figure 16. *Streamlines for the two rotating structures problem*

6. Conclusion

The developped method can deal with a fluid structure interaction problem where several thin structures are immersed. Moreover, the structure displacements can be large. This first study is done for an incompressible and viscous fluid where the structure velocities are known *a priori*.

The fluid mesh, using an Eulerian description, is fixed. The structures, localized by level-sets, have arbitrary positions in the fluid mesh. These two essential points avoid the mesh compatibility disadvantage along the interface.

In order to catch the pressure jump as well as the gradient velocity discontinuity accross the interface, the fluid approximation space is enriched by appropriate new functions. This partition of unity based approach is called the extended finite element method.

The applications are restricted in this article to low Reynolds number and laminar flows in order to avoid boundary layer effects. In the present approach, a fine mesh has to be used around the interface to take into account such an effect. The next step would be to use a partition of unity enrichment to model the boundary layer by introducing appropriate functions in the approximation around the interface with no need of refining the mesh.

The applications show that:

– the method can decouple arbitrary a fluid domain into two separated domains,

– the pressure and gradient velocity discontinuities are well modeled,

– the structures can have large displacements,

– the method can deal easily with several immersed structures, a numerical contact strategy has to be implemented if structure collisions occur.

Finally, the results show clearly the advantages of the proposed method. The next step is to validate the method with flexible structures which has been done in a space-time framework in (Kölke *et al.*, 2006).

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