# Diffuse approximation for field transfer in non linear mechanics

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ABSTRACT. *In this work, we propose a field transfer operator based on diffuse approximation. The key point of such an operator is to guarantee the conservation of relevant mechanical quantities related to the strutural state and to ensure the verification of some of the problem equations.*

RÉSUMÉ. *Dans ce travail, nous présentons un opérateur de transfert de champs basé sur les techniques d'approximation diffuse. Le point-clé de l'opérateur proposé est de conserver des grandeurs mécaniques relatives à l'état de la structure ainsi que d'assurer la vérification d'une partie des équations du problème.*

KEYWORDS: *remeshing technique, field transfer operator, non linear mechanics, diffuse approximation.*

MOTS-CLÉS : *remaillage, transfert de champs, mécanique non linéaire, approximation diffuse.*

## 1. Introduction

The high predictivity required for the computation by Finite Element Methods of the behavior of complex structures imposes to resort to adaptive remeshing techniques. it is especially the case when dealing with problems involving material and/or structural non linearities. In such cases, in order to ensure an acceptable quality of the solution it is necessary to adapt the domain discretisation to make it optimal as regards the representation of the solution.

Inherent to such adaptive remeshing techniques, the development of field transfer operators is a key point allowing to continue the calculation on a new mesh: the predictivity of the computation is then highly dependent on the effectiveness and reliability of the field transfer operator.

Ideally, a field tranfer operator has to provide the reconstruction, on a new mesh, of mecahnical fields verifying in a FE sense all the equations of the problem (Perić *et al.*, 1996): the equilibrium equation, the stress admissibility, the kinematic compatibility of the displacement field, boundary conditions... in order to guarantee the quality of the solution computed on the new mesh. field transfer operators based on Finite Element interpolations do not verify, in general, such conditions. The continuation of the computation can then become difficult, in a lot of cases, the succession of field transfer operations leading to non predictive solutions. It is all the more the case since we work with highly non linear material laws.

In this context, a main objective in the development of field transfer operators is to reach the best compromise between the computation cost and the quality of the solution. The field transfer proposed here aims at reconstructing the mechanical fields of interest, not only by using a simple approximation or interpolation, but also by ensuring the conservation of local and/or global mechanical quantities representative of the state of the structure such as, for instance, the dissipated energy, the strain energy or the stress admissibility.

In the following, we will focus on the presentation of the field transfer operator in the case of small perturbations for non linear material behaviors. The proposed operator is based on diffuse approximation techniques (Nayroles *et al.*, 1991; Breitkopf *et al.*, 2002).

In a first part, we present an overview of the continuum models considered in this work. In a second part, we focus on the presentation of the field transfer operator proposed for this type of models. Finally, in a third part, we give some numerical examples.

## 2. Continuum models

In order to define precisely the notations and the framework of this work, we present here a brief overview of the type of material laws used in this work. We consider the framework of the thermodynamics of continuum media and we consider more precisely associated material laws with internal variables. In this framework, we can define the evolution of the internal laws and the complementary Kuhn-Tucker conditions (also known as loading/un loading conditions) (also known as loading/unloading conditions) by appealing to the principle of maximum dissipation.

In the following, we give the main ingredients of the construction of such a model (Simo *et al.*, 2000). We consider only continuum models with isotropic hardening. The variables of the model are, then, decomposed in a classic way, into state variables and dual or associated variables summed up in Table 1.

**Table 1.** *Internal and associated variables of the model*

| state variables | associated variables |
|---|---|
| $\boldsymbol{\varepsilon}$ | $\boldsymbol{\sigma}$ |
| $\xi$ | $q$ |
| $\mathbf{v}$ | $\mathbf{A}$ |

where $\mathbf{v}$ is the vector composed of the internal variables other than the scalar variable associated to hardening $\xi$. $\mathbf{A}$ denotes the vector of dual variables associated to the state variables of vector $\mathbf{v}$, $\mathbf{A}$ is supposed to be written in terms of $\boldsymbol{\sigma}$.

For a classical plasticity model with isotropic hardening, $\mathbf{v} = \boldsymbol{\varepsilon}^p$ and $\mathbf{A} = \boldsymbol{\sigma}$. For an isotropic damage model (Mazars, 1984), $\mathbf{v} = \mathbf{D}$ ($\mathbf{D}$ denotes the compliance of the material), $\mathbf{A} = \frac{1}{2}\boldsymbol{\sigma} \otimes \boldsymbol{\sigma}$.

We consider here yield functions written in a generic form as:

$$\phi(\boldsymbol{\sigma}, q) = \sqrt{\boldsymbol{\sigma} : \mathcal{A} : \boldsymbol{\sigma}} - (\sigma_y - q) \quad [1]$$

where $\mathcal{A}$ is a symmetric definite positive forth order tensor. In those conditions, the evolution of internal varaibles can be written, by appealing to the principle of maximum dissipation as:

$$\dot{\xi} = \gamma \frac{\partial \phi(\sigma, q)}{\partial q} \text{ and } \dot{\mathbf{v}} = \gamma \frac{\partial \phi(\sigma, q)}{\partial \mathbf{A}(\boldsymbol{\sigma})} \quad [2]$$

The complementary loading/unloading conditions (or Kuhn-Tucker conditions) are written as:

$$\gamma \geq 0\ ,\ \phi(\sigma, q) \leq 0\ ,\ \gamma\phi(\sigma, q) = 0 \quad [3]$$

where $\gamma$ is a Lagrange multiplier.

The expression of the instantaneous dissipation can, then, be written as:

$$\mathcal{D}^{\text{inst}} = \dot{\mathbf{v}} \cdot \mathbf{A}(\boldsymbol{\sigma}) + \dot{\xi} q \quad [4]$$

and the total dissipation between time $t = 0$ and time $t$:

$$\mathcal{D}^{\text{tot}} = \int_0^t \mathcal{D}^{\text{inst}}\, \text{dt} \quad [5]$$

Even if the hypothesis considered seem to be limitative, this framework allows to deal with a large range of material laws used in numerical simulations, such as plasticity models driven by von Mises criterion or isotropic damage laws (Mazars, 1984).

## 3. Field transfer strategy

For material laws without history variables, the continuation of the computation can be performed by the knowledge on the new mesh of the displacement field. For non linear material laws with history variables, the continuation of the computation requires the reconstruction on the new discretisation of the displacement field but also of the internal variables fields defining the local state of the structure.

In the case of the continuum models such as the ones presented in Section 2, the continuation of the computation requires the projection on the new discretization of:

– the scalar internal variable $\xi$ defining the state of the material;

– the stress state;

– and , finally, the displacement field.

The field transfer operator has, then, to provide the reconstruction of those three fields. Moreover, in order to facilitate the continuation of the computation after the field transfer, the projection operator proposed guarantees:

– the conservation of the total dissipated energy from the old discretization to the new one;

– the admissibility of the reconstructed stress field as regards the yield function $\phi(\boldsymbol{\sigma}, q)$;

– the verification for the reconstructed stress field of the equilibrium equation;

– the kinematic compatibility and the boundary conditions for the displacement field.

It has to be noticed that, in order to write correctly the stress admissibility constraint it is necessary to reconstruct an additional variable: $\gamma$ (see Equation [3]) to which the conservation of the instantaneous dissipation is imposed. The field transfer operator can then be decomposed into three steps:

– step 1: reconstruction of the scalar variables $\xi$ and $\gamma$ in order to determine, on each point of the new discretization, the material state and its evolution. This step is performed by imposing the conservation of total and instantaneous dissipations;

– step 2: reconstruction of the stress field by imposing the stress admissibility (defined from the previous step) and local equilibrium equation;

– step 3: reconstruction of the displacement field by imposing boundary conditions and kinematic compatibility equation. In the following, we detail each of those steps.

### 3.1. *Step 1: transfer of the internal variables and of their evolution*

As described previously, the first step of the transfer consists in the reconstruction on the new discretization, of the two scalar variables $\xi$ (variable associated to hardening defining the anelastic state of the material) and $\gamma$ (Lagrange multiplier defining the evolution of the material state).

This reconstruction is performed by using a diffuse approximation projection (Nayroles *et al.*, 1991; Breitkopf *et al.*, 2002). Given $\mathbf{x}$ a point of the new discretization where the reconstruction of the previous scalar variables is needed (we will develop in the following considering only the reconstruction of the scalar variable $\xi$, an identical strategy is used for the reconstruction of $\gamma$). The variable $\xi$ is known on a cloud of points composed of the Gauss points denoted as $\mathbf{x}^{g,old}$ of the old discretization. The scalar filed $\xi^{\text{new}}$ to be transfered on $\mathbf{x}$ is approximated locally on a basis of approximation of any a priori degree (we work here with a degree one basis) as:

$$\xi^{\text{new}} = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \mathbf{p}^T(\mathbf{x})\mathbf{a} \qquad [6]$$

where $x$ et $y$ denote the coordinates in the reference system centered on $\mathbf{x}$. The $\mathbf{a}$ component on the basis of approximation $\mathbf{p}$ is then determined by solving the optimisation problem: *Find* $\mathbf{a}$ *such that* $J_{\mathbf{x}}(\mathbf{a})$ *be minimal with:*

$$J_{\mathbf{x}}(\mathbf{a}) = \frac{1}{2} \sum_{i \in V(\mathbf{x})} W(\mathbf{x}_i^{g,old}, \mathbf{x}) \| \mathbf{p}^T(\mathbf{x}_i^{g,old} - \mathbf{x})\mathbf{a} - \xi^{\text{old}}(\mathbf{x}_i^{g,old}) \|^2 \qquad [7]$$

$V(\mathbf{x})$ denotes a neighborhood of the point of approximation $\mathbf{x}$, this neighborhood has to contain a sufficient number of Gauss points $\mathbf{x}_i^{g,old}$ of the old discretization in order to ensure the unisolvance of the previous optimisation problem (we chose here $V(\mathbf{x})$ such that it contains at least 4 Gauss points of the old discretization).

The function $W(\cdot, \mathbf{x})$ corresponds to a weight function being 1 on $\mathbf{x}$ and 0 outside of $V(\mathbf{x})$. The choice of this function allows to ensure any continuity order of the approximated field. This function can be interpreted as the contribution of point $\mathbf{x}_i^{g,old}$ for the approximation on point $\mathbf{x}$. The solution of the optimisation problem [7] can be written for every point $\mathbf{x}$ as:

$$\mathbf{a} = \mathbf{A}^{-1}\mathbf{B} \qquad [8]$$

with:

$$\begin{aligned} \mathbf{A} &= \sum_{i \in V(\mathbf{x})} W(\mathbf{x}_i^{g,old}, \mathbf{x})\, \mathbf{p}(\mathbf{x}_i^{g,old} - \mathbf{x})\, \mathbf{p}^T(\mathbf{x}_i^{g,old} - \mathbf{x}) \\ &\text{and} \\ \mathbf{B} &= \sum_{i \in V(\mathbf{x})} W(\mathbf{x}_i^{g,old}, \mathbf{x})\, \mathbf{p}^T(\mathbf{x}_i^{g,old} - \mathbf{x})\, \xi^{\text{old}}(\mathbf{x}_i^{g,old}) \end{aligned} \qquad [9]$$

The variable $\xi^{\text{new}}$ on point $\mathbf{x}$ is then given by:

$$\xi^{\text{new}} = \mathbf{p}^T(\mathbf{0})\,\mathbf{a} \quad [10]$$

From the variable $\xi^{\text{new}}$ reconstructed on each Gauss point of all the elements of the new discretization, it is possible to determine, for each element of the new mesh, the value of the total dissipated energy per unit area, we denote $\mathcal{D}_e^{\text{new}}$ this quantity. The equivalent mean quantity $\mathcal{D}^{\text{old}}$ can be evaluated on the old mesh. In order to limit the diffusion inherent to the transfer, elements for which $\mathcal{D}_e^{\text{new}} \leq c * \mathcal{D}^{\text{old}}$ are considered as undamaged and the variable $\xi^{\text{new}}$ is set to zero for this element ($c$ corresponds to a yield defined a priori). The final point of this step consists in the conservation of the total dissipation. This is performed by renormalizing the variable $\xi^{\text{new}}$. The associated variable $q^{\text{new}}$ can then be deduced on each point of the new discretization.

A strictly identical strategy is used for the reconstruction of the variable $\gamma^{\text{new}}$, the instantaneous dissipation is considered instead of the total one.

At the end of this step of the field reconstruction, the fields $\xi$ and $\gamma$ are known on each point of the new discretization, defining the state of the material and its evolution. The knowledge on each point of the variable $\gamma$ also allows to define the stress admissibility from the loading/unloading conditions.

### 3.2. *Step 2: reconstruction of the stress field*

The second step of the field transfer operator proposed consists of the the reconstruction of the stress field on the new discretization by imposing the stress admissibility and the local equilibrium equation verification. We have to treat two different zones:

– the zone for which the state of the material is evolving. This zone is defined by $\gamma \neq 0$, the stress admissibility is then written as $\phi(\boldsymbol{\sigma}, q) = 0$;

– the zone for which the state of the material is not evolving. This zone is defined by $\gamma = 0$, the stress admissibility is then written as $\phi(\boldsymbol{\sigma}, q) \leq 0$.

#### 3.2.1. *Treatment of the local equilibrium*

The verification of the local equilibrium equation for the stress field rebuilt on the new mesh is ensured by the choice of the polynomial basis of approximation (Villon *et al.*, 2002). The latter is chosen so that the reconstructed stress field verifies, locally around the point $\mathbf{x}$, $\mathbf{div}\,\boldsymbol{\sigma} = 0$. The stress field reconstructed is then approximated on $\mathbf{x}$ by:

$$\boldsymbol{\sigma}^{\text{new}} = \begin{bmatrix} 1 & 0 & 0 & x & 0 & y & 0 \\ 0 & 1 & 0 & 0 & x & 0 & y \\ 0 & 0 & 1 & -y & 0 & 0 & -x \end{bmatrix} \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_7 \end{bmatrix} = \mathbf{P}^T(\mathbf{x})\boldsymbol{\theta} \quad [11]$$

The local equilibrium equation is then verified in the diffuse derivative sense (the evolution of $\boldsymbol{\theta}$ with $\mathbf{x}$ is not taken into account). $\boldsymbol{\theta}$, component of $\boldsymbol{\sigma}^{\text{new}}$ on the polynomial basis of approximation $\mathbf{P}$, is here a vector of dimension 7.

3.2.2. *Treatment of the stress admissibility*

The reconstruction of the stress field on the new discretization can be performed by imposing the stress admissibility given as $\phi(\sigma, q) \leq 0$ in non evolving zones and $\phi(\sigma, q) = 0$ in evolving zones.

Noting that the first step of the field transfer operator has provided the definition on each point of the new mesh of the associated variable $q$, the stress admissibility is written only on the stress field $\boldsymbol{\sigma}$.

A first technique of reconstruction of the stresses consists of imposing the previous stress admissibility constraint to the function to optimize. The optimization problem can, then, be written as:

*Find $\boldsymbol{\theta}$ such that $J_{\mathbf{x}}(\boldsymbol{\theta})$ be minimal under the constraint of admissibility with:*

$$J_{\mathbf{x}}(\boldsymbol{\theta}) = \frac{1}{2} \sum_{i \in V(\mathbf{x})} W(\mathbf{x}_i^{g,old}, \mathbf{x}) \| \mathbf{P}^T(\mathbf{x}_i^{g,old} - \mathbf{x})\boldsymbol{\theta} - \boldsymbol{\sigma}^{old}(\mathbf{x}_i^{g,old}) \|^2 \qquad [12]$$

where the stress admissibility constraint is written as $\phi(\boldsymbol{\sigma}^{\text{new}}, q^{\text{new}}) = 0$ if $\gamma^{\text{new}} \neq 0$ and $\phi(\boldsymbol{\sigma}^{\text{new}}, q^{\text{new}}) \leq 0$ if $\gamma^{\text{new}} = 0$

Noting that for the zone defined by $\gamma = 0$ in which the state of the material is not evolving, if the minimization constraint is active, solving the previous optimization problem is similar to solving the problem on the zone defined by $\gamma \neq 0$, only two cases have to be treated:

– resolution of the optimization problem without constraint;
– resolution of the optimization problem under the constraint; $\phi(\boldsymbol{\sigma}^{\text{new}}, q^{\text{new}}) = 0$.

We are then brought to the resolution of a quadratic optimization problem under equality constraint. Moreover, noting that the stress admissibility $\phi(\boldsymbol{\sigma}^{\text{new}}, q^{\text{new}}) = 0$ can be written under a quadratic form, the determination of $\boldsymbol{\theta}$ can be reduced to the resolution of a quadratic optimization problem under a quadratic constraint.

As we have seen in Section 2, Equation [1], the yield functions considered can be written as:

$$\phi(\boldsymbol{\sigma}^{\text{new}}, q^{\text{new}}) = \sqrt{\boldsymbol{\sigma} : \mathcal{A} : \boldsymbol{\sigma}} - (\sigma_y - q^{\text{new}}) \qquad [13]$$

In the following, we denote $\mathbf{C}$ the matrix form of the fourth order tensor $\mathcal{A}$. The matrix $\mathbf{C}$ is symmetric of rank 3 ($\mathcal{A}$ is symmetric definite poisitive). The stress admissibility constraint can then be written on $\mathbf{x}$ as:

$$(\boldsymbol{\sigma}^{\text{new}})^T \cdot \mathbf{C} \cdot \boldsymbol{\sigma}^{\text{new}} = (\sigma_y - q^{\text{new}})^2 \qquad [14]$$

With the approximation of $\boldsymbol{\sigma}^{\text{new}}$ chosen (equation [11]), we can rewrite the stress admissibility as:

$$\boldsymbol{\theta}^T \cdot \mathbf{P}(\mathbf{0}) \cdot \mathbf{C} \cdot \mathbf{P}^T(\mathbf{0}) \cdot \boldsymbol{\theta} = (\sigma_y - q^{\text{new}})^2 \qquad [15]$$

Denoting:

$$\mathcal{C} = \mathbf{P}(\mathbf{0}) \cdot \mathbf{C} \cdot \mathbf{P}^T(\mathbf{0}) / (\sigma_y - q^{\text{new}})^2 \qquad [16]$$

$\mathcal{C}$ is a matrix of dimension $(7 \times 7)$ of rank 3. The stress admissibility can then be written as:

$$\boldsymbol{\theta}^T \cdot \mathcal{C} \cdot \boldsymbol{\theta} = 1 \qquad [17]$$

In the following, we detail the resolution of the quadratic optimization problem under quadratic constraint:

*Find $\boldsymbol{\theta}$ so that:*

$$\min_{\boldsymbol{\theta}^T \mathcal{C} \boldsymbol{\theta} = 1} J_{\mathbf{x}}(\boldsymbol{\theta}) = \frac{1}{2} \sum_{i \in V(\mathbf{x})} W(\mathbf{x}_i^{g,old}, \mathbf{x}) \| \mathbf{P}^T(\mathbf{x}_i^{g,old} - \mathbf{x}) \boldsymbol{\theta} - \boldsymbol{\sigma}^{old}(\mathbf{x}_i^{g,old}) \|^2 \qquad [18]$$

This problem can be rewritten:

$$\min_{\boldsymbol{\theta}^T \mathcal{C} \boldsymbol{\theta} = 1} \frac{1}{2} \boldsymbol{\theta}^T \mathbf{A} \boldsymbol{\theta} - \mathbf{b}^T \boldsymbol{\theta} \qquad [19]$$

where:

$$\mathbf{A} = \sum_{i \in V(\mathbf{x})} W(\mathbf{x}_i^{g,old}, \mathbf{x})\, \mathbf{P}^T(\mathbf{x}_i^{g,old} - \mathbf{x}) \mathbf{P}(\mathbf{x}_i^{g,old} - \mathbf{x})$$
$$\text{and} \qquad [20]$$
$$\mathbf{b} = \sum_{i \in V(\mathbf{x})} W(\mathbf{x}_i^{g,old}, \mathbf{x})\, \mathbf{P}^T(\mathbf{x}_i^{g,old} - \mathbf{x})\, \boldsymbol{\sigma}^{\text{old}}(\mathbf{x}_i^{g,old})$$

The matrix $\mathcal{C}$ being of rank 3, it is similar (considering the transformation matrix $\mathcal{P}$) to a diagonal matrix $\mathcal{D}$ written:

$$\mathcal{D} = \begin{bmatrix} \mathbf{0}_{4,4} & \mathbf{0}_{4,3} \\ \mathbf{0}_{3,4} & \boldsymbol{D} \end{bmatrix} \qquad [21]$$

where $\boldsymbol{D}$ is a diagonal matrix of dimension $(3 \times 3)$ and $\mathbf{0}_{i,j}$ denotes the null matrix of dimension $(i \times j)$.

In the following, we denote $\zeta = \mathcal{P}^T\boldsymbol{\theta} = \zeta_1 + \zeta_2$ where $\zeta_1 = \mathcal{P}_1^T\boldsymbol{\theta}$ is the projection of $\boldsymbol{\theta}$ on $\mathrm{Ker}\,\mathcal{C}$ and $\zeta_2 = \mathcal{P}_2^T\boldsymbol{\theta}$ is the projection of $\boldsymbol{\theta}$ on $\mathrm{Ker}\,\mathcal{C}^\perp$. With this decomposition of $\zeta$, the problem [19] can be rewritten as:

$$\left\{\begin{array}{l}\min\limits_{\zeta}\left(\dfrac{1}{2}\zeta_1^T\mathcal{P}_1^T\mathbf{A}\mathcal{P}_1\zeta_1 + \dfrac{1}{2}\zeta_2^T\mathcal{P}_2^T\mathbf{A}\mathcal{P}_2\zeta_2\right.\\ \qquad\qquad\left. + \zeta_1^T\mathcal{P}_1^T\mathbf{A}\mathcal{P}_2\zeta_2 - \mathbf{b}^T\mathcal{P}_1\zeta_1 - \mathbf{b}^T\mathcal{P}_2\zeta_2\right)\\ \text{under the constraint} \quad \zeta_2^T\boldsymbol{D}\zeta_2 = 1\end{array}\right. \qquad [22]$$

In order to simplify the notations, we will denote $\mathcal{A}_{ij} = \mathcal{P}_i^T\mathbf{A}\mathcal{P}_j$ with $i = 1,2$ and $j = 1,2$ and $b_i = \mathcal{P}_i^T\mathbf{b}$. The optimality system associated to the optimization problem [22] is then written, by denoting $\lambda$ the Lagrange multiplier associated with the admissibility constraint:

$$\left\{\begin{array}{l}\mathcal{A}_{11}\zeta_1 + \mathcal{A}_{22}\zeta_2 = b_1\\ \mathcal{A}_{12}\zeta_1 + (\mathcal{A}_{22} + \lambda\boldsymbol{D})\,\zeta_2 = b_2\\ \zeta_2^T\boldsymbol{D}\zeta_2 = 1\end{array}\right. \qquad [23]$$

Noting that $\mathcal{A}_{11}$ is invertible ($\mathbf{A}$ is symmetric definite positive), $\zeta_1$ can be deduced from the first equation, it remains then to solve the system:

$$\left\{\begin{array}{l}\left(\tilde{\mathcal{A}} + \lambda\boldsymbol{D}\right)\zeta_2 = \tilde{b}\\ \zeta_2^T\boldsymbol{D}\zeta_2 = 1\end{array}\right. \qquad [24]$$

with $\tilde{\mathcal{A}} = \mathcal{A}_{22} - \mathcal{A}_{12}\mathcal{A}_{11}^{-1}\mathcal{A}_{12}$ and $\tilde{b} = b_2 - \mathcal{A}_{12}\mathcal{A}_{11}^{-1}b_1$.

The matrix $\tilde{\mathcal{A}}$ is symmetric definite positive, $\boldsymbol{D}$ is symmetric, we can so reduce simultaneously those two matrices: we can conclude to the existence of a rotation matrix $\mathcal{R}$ such that: $\mathcal{R}^T\tilde{\mathcal{A}}\mathcal{R} = \mathcal{I}$ and $\mathcal{R}^T\boldsymbol{D}\mathcal{R} = \tilde{\mathcal{D}}$ where $\tilde{\mathcal{D}}$ is a diagonal matrix. Denoting $\eta = \mathcal{R}^T\zeta_2$, the resolution of the previous problem is reduced to the resolution of the following problem:

$$\left\{\begin{array}{l}\left(\mathcal{I} + \lambda\tilde{\mathcal{D}}\right)\eta = \mathcal{R}^T\tilde{b}\\ \eta^T\tilde{\mathcal{D}}\eta = 1\end{array}\right. \qquad [25]$$

$\tilde{\mathcal{D}}$ being diagonal, the determination of $\eta$ is performed by the resolution of an algebraic equation of degree 6. The resolution of such an equation is carried out by using the root finder algorithm involving the companion matrix. $\boldsymbol{\theta}$ is computed from $\eta$ by considering transformation matrices.

At the end of this step, the Gauss points variables needed for the continuation of the computation are known.

### 3.3. *Step 3: reconstruction of the displacement field*

The reconstruction of the displacement field, final step of the transfer operator proposed, is performed by imposing:

– the verification, in a strong sense, essential boundary conditions;

– the verification at best of the kinematic compatibility condition.

The displacement field $\mathbf{u}^{\text{new}}$ is then searched in the Finite Elements subspace associated to the new discretization as the solution of the following optimization problem under constraint:

$$\min_{\mathbf{u}_{|\Gamma}=\bar{\mathbf{u}}} \frac{1}{2}\left(w_1\|\mathbf{u}^{\text{new}} - p_{AD}(\mathbf{u}^{\text{old}})\|^2 + w_2 h^2\|\boldsymbol{\nabla}^{\text{s}}\mathbf{u}^{\text{new}} - \boldsymbol{\varepsilon}^{\text{new}}\|^2\right) \quad [26]$$

$p_{AD}(\mathbf{u}^{\text{old}})$ denotes the projection by diffuse approximation of the displacement field known on the old discretization on the new discretization. The coefficients $w_1$ and $w_2$ allow to fix the quality with which the two conditions: "closeness of the displacement rebuilt by diffuse approximation" and "verification of the kinematic compatibility condition" are verified. $h$ corresponds to the mean length of elements and $\bar{\mathbf{u}}$ corresponds to the displacement prescribed on the boundary $\Gamma$ of the domain.

The problem [26] is then solved in the finite element subspace associated to the new discretization. The determination of the displacement field on the new mesh consists of the determination of the nodal displacement vector.

## 4. Applications

We present here some results obtained for a damage model. We compare the results obtained by the proposed transfer operator with the ones obtained from a standard transfer based on FE interpolation shape functions (Perić *et al.*, 1996).

### 4.1. *Continuum model used*

We briefly present here very the different ingredients of the material model used for the different tests presented in the following. The model used is an isotropic damage model (Mazars, 1984). This model is of the form presented in Section 2. The couples of state variables of the model are, in the case considered here:

– $(\boldsymbol{\varepsilon}, \boldsymbol{\sigma})$: strain and stress field;

– $(\xi, q)$: internal variable and dual variable associated to hardening;

– $(\mathbf{D}, \mathbf{Y})$: material compliance tensor (eventually damaged) and dual variable. $\mathbf{Y}$ is written as: $\frac{1}{2}\boldsymbol{\sigma} \otimes \boldsymbol{\sigma}$.

The evaluation of the variable $\xi$ alone provides the determination of the compliance $\mathbf{D}$ and of the dual variable associated with hardening: $q = -K\xi$ where $K$ is the hardening modulus. The yield function $\phi(\boldsymbol{\sigma}, q)$ used is written as (Brancherie *et al.*, 2004):

$$\phi(\boldsymbol{\sigma}, q) = \underbrace{\sqrt{\boldsymbol{\sigma} : \mathbf{D}^e : \boldsymbol{\sigma}}}_{\|\boldsymbol{\sigma}\|_{\mathbf{D}^e}} - \frac{1}{\sqrt{E}}(\sigma_f - q) \leq 0 \quad [27]$$

where $\mathbf{D}^e$ denotes the compliance of the undamaged material (that is the inverse of the Hooke tensor), $E$ is the Young modulus and $\sigma_f$ the elastic limit of the material.

### 4.2. *Coherence of the proposed field transfer operator*

In order to verify the coherence of the field transfer operator, a transfer is performed considering a new discretization equal to the old one and the computation is continued from the rebuilt field. The test considered is a traction test on a notched beam (see Figure 1).

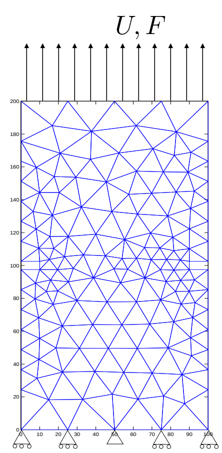
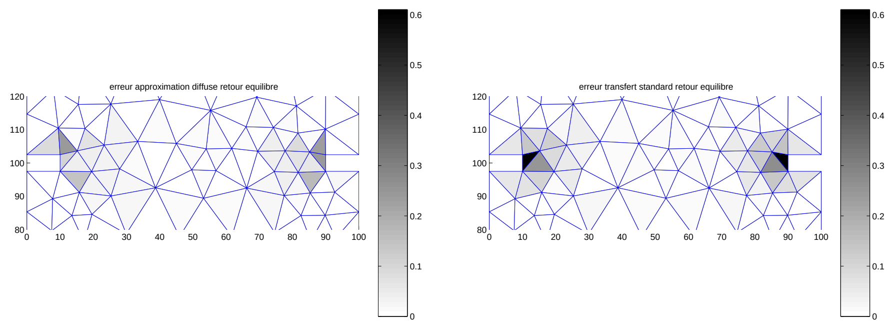



**Figure 1.** *Traction test on a notched beam*

We present here the results obtained performing a standard transfer and the proposed transfer compared to the one obtained from a direct computation. The results in terms of the error on the damage field obtained after the transfer operation and the equilibrium recovery and at the end of the computation compared to the damage field obtained from the direct calculation are given on Figures 2 and 3.

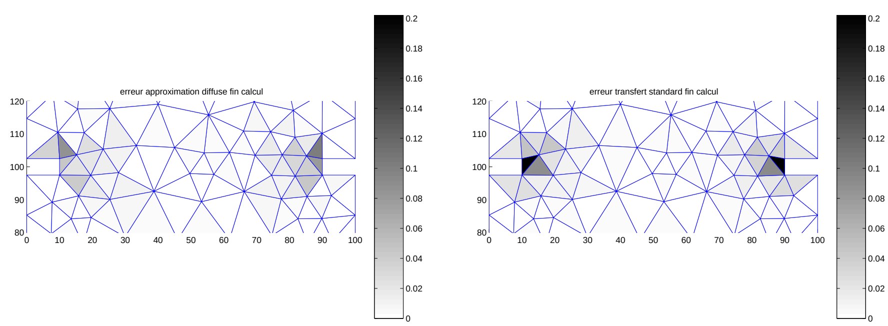
We can note that the proposed transfer reaches error levels in terms of the predicted damage field much lower than the one obtained locally by using a standard transfer. The standard transfer suffers from numerical diffusion, this point could lead to a bad prediction of the evolution of the damage state when continuing the calculation.

The global results given in terms of load/prescribed displacement curve are given on Figure 4. We can note, in this case, that global results for the given test are quite close to the one obtained by direct computation after the proposed transfer as well as after the standard transfer even if the standard transfer seems to be less effective.




(a) Error on $\xi$ after the proposed transfer (b) Error on $\xi$ after a standard transfer

**Figure 2.** *Map of the error on the variable $\xi$ after transfer and equilibrium recovery*




(a) Error on $\xi$ after the proposed transfer (b) Error on $\xi$ after a standard transfer

**Figure 3.** *Map of the error on the variable $\xi$ at the end of the computation*

### 4.3. *Comparison with a standard field transfer operator*

The test considered in this part is the same as previously used for testing the coherence of the field transfer operator. In this example, two remeshing of the domain are performed during the loading process. The remeshing is, in this case, performed when the evolution of the state of the material reaches a limit value.

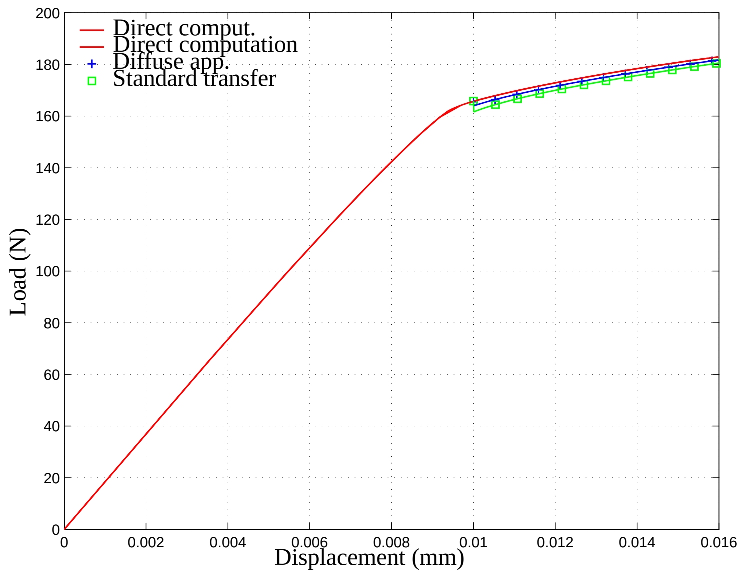
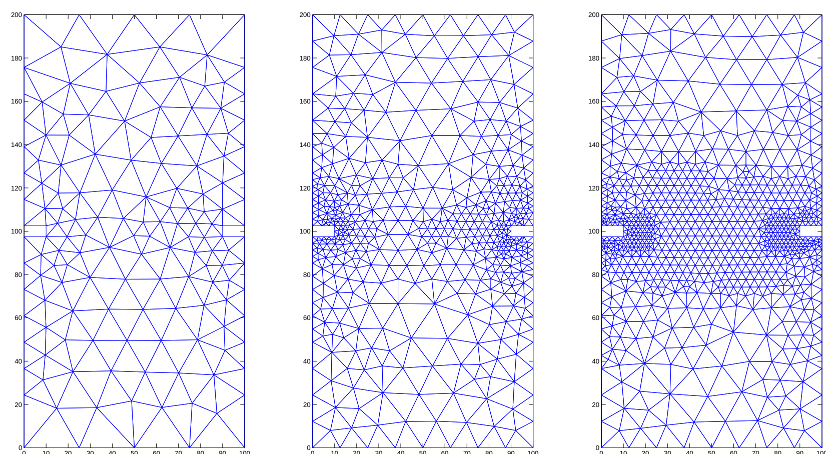


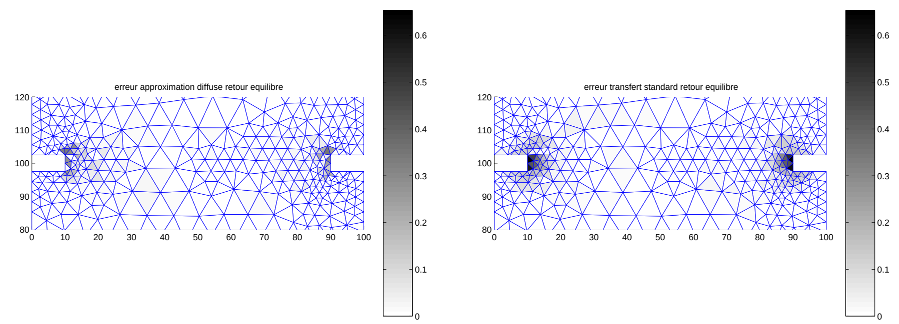

**Figure 4.** *Load /displacement response*



(a) Original mesh (b) First remeshing mesh (c) Second remeshing mesh

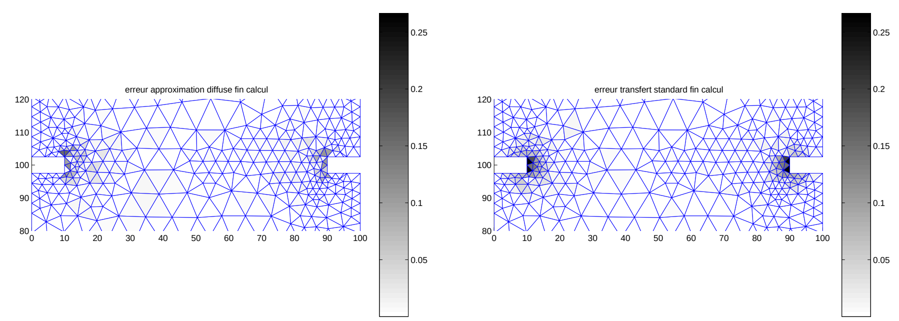
**Figure 5.** *Different discretizations used for the computation*

The indicator of remeshing is based on the value of the dissipation at each point (which is a scalar quantity defining the evolution of the state of the material). The new element length map associated to an optimized discretization is defined by considering the latter quantity and by imposing locally an element length ensuring a uniform instantaneous dissipated energy per element. The Figure 5 represents the different meshes used for each step of the computation. We compare here the results obtained after the proposed field transfer operator and after a field transfer based on FE interpolation shape functions (Perić *et al.*, 1996) with the results obtained by direct computation (computation carried out on the chosen discretization from the beginning of the loading process). The Figure 6 gives the results in terms of the error on the damage variable compared with the solution obtained from a direct computation for the first remeshing and the first transfer operation. The amount of error is calculated just after transfer and at the end of the computation that is when the second remeshing is decided.




(a) Error on $\xi$ after the proposed transfer

(b) Error on $\xi$ after a standard transfer

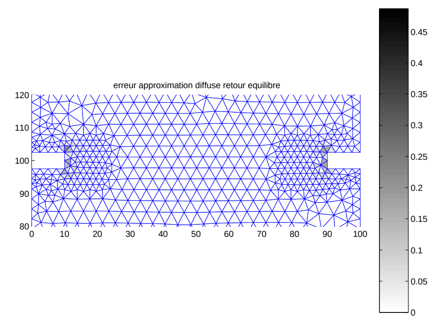



(c) Error on $\xi$ at the end of the computation (proposed transfer)

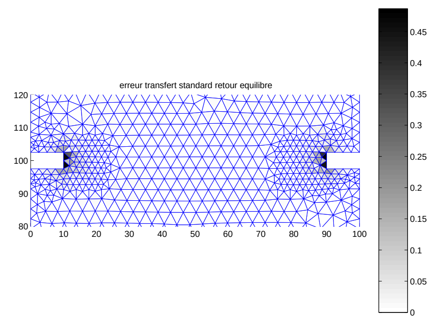
(d) Error on $\xi$ at the end of the computation (standard transfer)

**Figure 6.** *Error map on the variable $\xi$ after transfer and at the end of the computation for the proposed strategy and after a standard transfer for the fi rst remeshing*

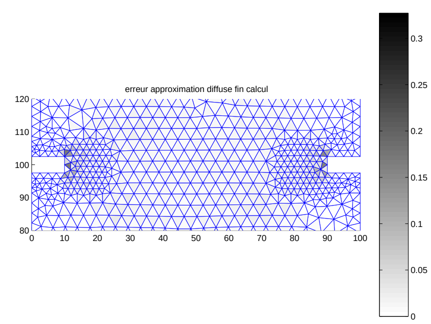
The same types of results are given for the second remeshing on Figure 7. We can note that the quality of the local results here evaluated on the value of the damage variable on each point of the structure is better after the proposed transfer then after a standard transfer. We note in particular a numerical diffusion with the standard transfer which is brought up to the end of the computation, this diffusion is limited in the case of the proposed transfer.



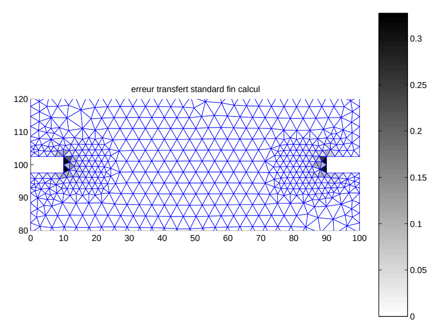

(a) Error on $\xi$ after the proposed transfer




(b) Error on $\xi$ after a standard transfer




(c) Error on $\xi$ at the end of the computation (proposed transfer)

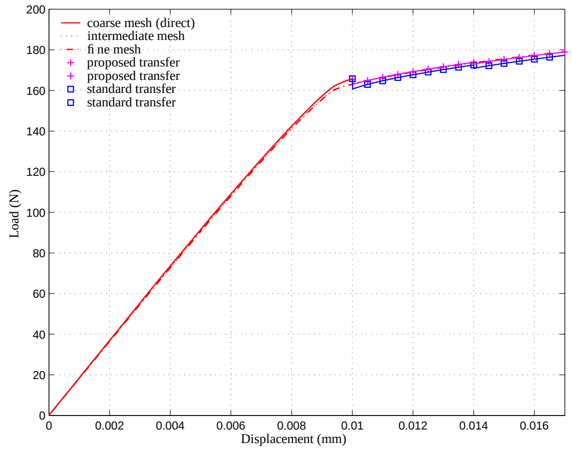



(d) Error on $\xi$ at the end of the computation (standard transfer)

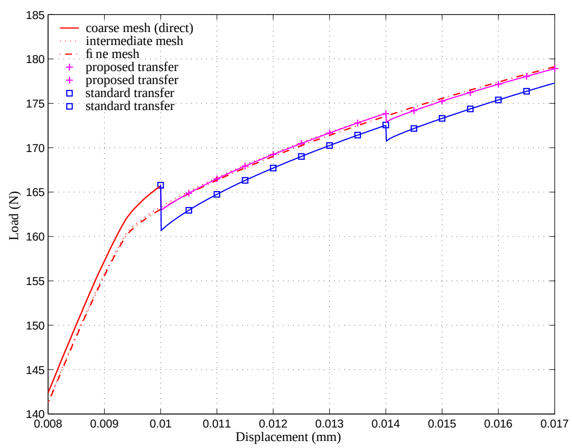
**Figure 7.** *Error map on the variable $\xi$ after transfer and at the end of the computation for the proposed strategy and after a standard transfer for the second remeshing*

Moreover, the error level obtained in the most sensible zones of the structure (here the notch) by the proposed transfer are much lower than by the standard transfer, allowing to envisage the use of such a technique for the local description of crack propagation.

Some global results given in terms of load/displacement curves are presented on Figure 8. We can note that the proposed transfer gives global results very close to the results obtained by direct computation for the intermediate and fine meshes. The solution is not being degraded all along the remeshing process as it is the case when using a standard field transfer operator.




(a) Load/displacement response




(b) Zoom on the two remeshing

**Figure 8.** *Load/displacement response*

## 5. Conclusion

The field transfer operator proposed is based on the diffuse approximation techniques. The key point of the proposed operator is to ensure the conservation of mechanical quantities representative of the state of the structure such as the total or instantaneous dissipated energy but also the verification of the problem equations such as stress admissibility, local equilibrium equation or kinematic compatibility of the displacement field with the state variables. The comparisons carried out between the results obtained after a standard transfer and the proposed field transfer operator have shown that the quality of the results is better in terms of global results (load/displacement curve) as well as local variables (internal variables). Moreover, the proposed strategy is applicable to a large range of materials in numerical simulations. In the presented work, the remeshing operation is driven by a criterion based on the evolution of the internal variables, a remeshing criterion based on the evaluation of the error by the diffuse approximation techniques could also be considered.

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