An algorithm for computing the critical state of unilateral buckling of thin plates

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ABSTRACT. *When modelling the buckling phenomenon of thin plates in presence of an obstacle, we obtain a variational inequality with two unknowns that are the buckling load and the corresponding buckling mode. Using the finite elements method, the continuous problem is approximated by a discrete problem. Then, an algorithm for computing the buckling critical load and the corresponding unilateral buckling mode of the plate is suggested. The last part of the paper is devoted to some numerical results obtained for the same rectangular plate but three different kinds of obstacle.*

RÉSUMÉ. *On s'intéresse, dans ce papier, au flambement unilatéral d'une plaque mince en présence d'un obstacle. Après approximation par éléments finis, un algorithme est proposé pour calculer la charge critique de flambement ainsi que le mode correspondant. La fin du papier est consacrée à des résultats numériques obtenus pour la même plaque rectangulaire, et trois différents types d'obstacle.*

KEYWORDS: *unilateral buckling, buckling critical load, buckling mode, finite element method, algorithm, eigenvalue problem.*

MOTS-CLÉS : *flambement unilatéral, charge critique de flambement, mode de flambement, méthode d'éléments finis, algorithme, problème aux valeurs propres.*

1. Introduction

Consider a thin plate of thickness 2ε occupying a two-dimensional open set ω. Assume that it is supported on the whole of its edge γ, clamped on a part γ_0 of its edge whose Lebesgue measure is not zero, and simply supported on γ/γ_0. Furthermore, the plate is subjected to a one-parameter plane compressive load λ on another part γ_1 of its edge (see figure 1).

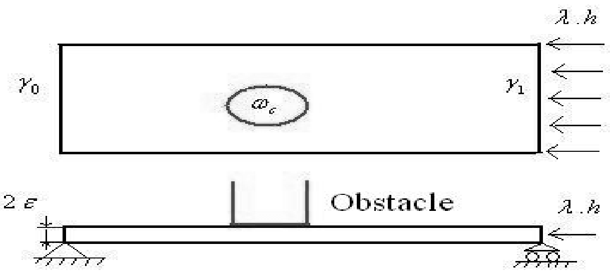

Figure 1. *A rectangular plate is in presence of an obstacle*

It is well known, if there are no obstacles in the neighbourhood of the plate, that there exists an increasing sequence of strictly positive real numbers (λ_n) escaping to the infinity and such that each load $\lambda_n.h$ involves an instability of the plate manifested by a large vertical displacement (see (Ayadi *et al.*, 1990; Ciarlet *et al.*, 1977; Destuynder, 1990; Timoshenko, 1966)). Such a physical displacement is proportional to the so-called buckling mode of the plate corresponding to the so-called buckling load $\lambda_n.h$.

In this paper, we suppose that the plate is in presence of a rigid fixed plane obstacle that lies just above it (see figure 1). The contact between the plate and the obstacle is supposed to be without friction. If the obstacle is not initially in contact with the plate, the problem becomes classical, in the sense that it has the same solution as the linear one. The unilateral buckling and even the unilateral post-buckling of thin plates have been tackled and investigated by many authors since the late seventies. Let us cite, in particular, (Cimetière, 1980, 1985; Do, 1975, 1976), and (Riddell, 1977). In this paper, we do not claim deeply studying the unilateral buckling problem. We just contribute to compute numerically the critical state of unilateral buckling problem. We so concentrate on developing a new algorithm in order to avoid the numerical instabilities of the penalization method.

The paper is organized as follows. The second section is devoted to the description of the unilateral buckling model. In the third section the adequate framework, so that the critical state of unilateral buckling exists (see (Do, 1975, 1976; Riddell, 1977)), is recalled. Then, in the fourth section, the approximation of the continuous problem, by the finite element method, leads to a discrete "nonlinear eigenvalue problem" that is mathematically well posed. The fifth section, where our contribution is the most significant, is devoted to both construction and convergence justification of the suggested algorithm. In order to test our algorithm and validate the unilateral buckling model proposed, the same rectangular plate and three different kinds of obstacle are considered in the sixth section. Some related numerical results, obtained with the Mindlin's plate model and the Macneal's finite element, are then exhibited. For each kind of obstacle, first, is given the unilateral buckling mode; then, the shape of the obstacle as well as the sections of the buckling modes on the largest principal axis of the plate are plotted in the same figure.

2. Mathematical modelling of unilateral buckling

When taking into account the unilateral contact condition and considering a nonlinear elastic constitutive law, we obtain a very difficult mathematical problem (see (Ciarlet, 1986), (Ciarlet et *al.,* 1977; Duvaut et *al.,* 1972)). Nevertheless, we know a particular solution to the latter. It is the linear elasticity solution, obtained with linearized strains, for which the vertical displacement is zero and the plane displacements are solution to the following variational equation:

$$\sum_{\alpha,\beta,\nu,\mu=1}^{2} \int_{\omega} E_{\alpha\beta\nu\mu} \frac{\partial u_\nu^p}{\partial x_\mu} \frac{\partial v_\alpha}{\partial x_\beta} d\omega = \lambda \sum_{\alpha=1}^{2} \int_{\gamma_1} H_\alpha v_\alpha d\gamma \;\; \forall v \in V , \qquad [1]$$

where $E_{\alpha\beta\nu\mu}$ is the membrane stiffness tensor for the linear elastic constitutive law (depending on Young's modulus and the Poisson's ratio),

$$H_\alpha = \int_{-\varepsilon}^{\varepsilon} h_\alpha dx_3,\; \alpha \in \{1,2\},$$

and

$$V = V = \left\{ v \in H^1(\omega)^2 : v = 0 \quad on \quad \gamma_0 \right\}.$$

The following usual assumptions are made.

(i) $H_\alpha \in L^2(\gamma_1),\; \alpha \in \{1,2\}$,

(ii) $E_{\alpha\beta\nu\mu} \in L^\infty(\omega)$ *for all* $(\alpha,\beta,\nu,\mu) \in \{1,2\}^4$,

(iii) $\exists \;\; k > 0$ *such that* $\sum_{\alpha,\beta,\nu,\mu=1}^{2} E_{\alpha\beta\nu\mu} \theta_{\alpha\beta} \theta_{\nu\mu} \geq k \sum_{\alpha,\beta=1}^{2} \left(\theta_{\alpha\beta}\right)^2$, for all symmetric tensor θ of order two, that is the ellipticity property. It is shown in (Ciarlet, 1986) and in (Duvaut *et al.*, 1972) that problem [1] admits a unique solution $u^p \in V$. Moreover, the tensor of membrane efforts is, (see for instance (Ayadi *et al.*, 1990; Ayadi, 1993; Ciarlet *et al.*, 1977)), expressed by:

$$n^p_{\alpha\beta} = \sum_{\nu,\mu=1}^{2} E_{\alpha\beta\nu\mu} \frac{\partial u^p_\nu}{\partial x_\mu} = \lambda . n^h_{\alpha\beta}, \ \ \alpha, \beta \in \{1,2\}.$$

Looking for a non-trivial solution to the nonlinear problem described above, we need the linearizing technique: set $u = u^p + w$ and show that the deflection w_3 of the plate is solution to the following inequality:

$$\sum_{\alpha,\beta,\nu,\mu=1}^{2} \int_\omega D_{\alpha\beta\nu\mu} \frac{\partial^2 w_3}{\partial x_\nu \partial x_\mu} \frac{\partial^2 (v - w_3)}{\partial x_\alpha \partial x_\beta} d\omega \geq \lambda \sum_{\alpha,\beta=1}^{2} \int_\omega n^h_{\alpha\beta} \frac{\partial w_3}{\partial x_\alpha} \frac{\partial (v - w_3)}{\partial x_\beta} d\omega \qquad [2]$$

for all admissible deflection *v*, where $D_{\alpha\beta\nu\mu}$ denotes the bending rigidity tensor of the plate. The other components w_1 and w_2, of the displacement w, are related to the deflection w_3 by Kirchhoff-Love formulae (see (Ciarlet *et al.*, 1977)).

3. Mathematical framework and existence results

Let us start by defining the adequate framework used in this paper so that the problem [2] admits at least one solution.

$$H = H^1_0(\omega),$$
$$W = \left\{ v \in H^2(\omega) : v = 0 \ on \ \gamma \ and \ \frac{\partial v}{\partial \nu} = 0 \ on \ \gamma_0 \right\},$$
$$K = \{ v \in W : v \leq 0 \ \ in \ \ \omega_c \},$$

ω_c being a subset of ω where the contact between the obstacle and the plate could occur.

$$a(u,v) = \sum_{\alpha,\beta,\nu,\mu=1}^{2} \int_\omega D_{\alpha\beta\nu\mu} \frac{\partial^2 u}{\partial x_\nu \partial x_\mu} \frac{\partial^2 v}{\partial x_\alpha \partial x_\beta} d\omega,$$
$$b(u,v) = \sum_{\alpha,\beta=1}^{2} \int_\omega n^h_{\alpha\beta} \frac{\partial u}{\partial x_\alpha} \frac{\partial v}{\partial x_\beta} d\omega .$$

The Sobolev spaces H and W are respectively equipped with the following norms:

$$|u|_{1,\omega} = \left(\int_\omega \sum_{\alpha=1}^{2} \left(\frac{\partial u}{\partial x_\alpha} \right)^2 \right)^{\frac{1}{2}}, \quad \|u\|_{2,\omega} = \left(\int_\omega \sum_{|\alpha| \le 2} \left(\partial^{|\alpha|} u \right)^2 \right)^{\frac{1}{2}}.$$

The bilinear forms b and a are obviously continuous in the spaces H and W respectively. Assume now that the bending rigidity tensor $D_{\alpha\beta\nu\mu}$ satisfies the ellipticity property (iii) so that the bilinear form a is coercive (see (Ciarlet et *al*., 1977), (Destuynder, 1990; Duvaut et *al*., 1972)). That means there exists a positive constant α such that

$$a(v,v) \ge \alpha \|v\|_{2,\omega}^2, \ \textit{for all } v \in W.$$

Within the framework defined above, problem [2] is mathematically well posed as stated by the following theorem.

Theorem 3.1. *There exist* $\lambda > 0$ *and nonzero vector* $w_3 \in K$ *such that*

$$a(w_3,\ v - w_3) \ge \lambda b(w_3,\ v - w_3) \ \forall v \in K. \qquad [3]$$

Moreover, λ *is the minimum of the Rayleigh quotient over the closed convex cone* K *, which is realized on* w_3 *:*

$$\lambda = \min_{v \in K - \{0\}} \frac{a(v,v)}{b(v,v)} = a(w_3, w_3). \qquad [4]$$

Proof: See (Riddell, 1977).

4. Numerical approximation

The discrete unilateral buckling problem

We shall approximate the nonlinear eigenvalue problem [3] by using a conformal finite element method. More precisely, because the functions of the space W are at least continuous in $\overline{\omega}$, we should need, for instance, continuous differentiable finite element schemes. Let then W_h be a finite dimensional subspace of the space W and let K_h be a nonempty closed convex cone of W_h. The discrete "nonlinear eigenvalue problem", supposed to approach the continuous one [3], consists in finding pairs $(\lambda_h, w_{3h}) \in IR^+ \times (K_h - \{0\})$ such that

$$a\left(w_{3h}, v_h - w_{3h}\right) \geq \lambda_h b\left(w_{3h}, v_h - w_{3h}\right), \quad for\, all\, v_h \in K_h . \qquad [5]$$

Equivalently, if we let φ_k, $1 \leq k \leq N$, denotes a basis in the space W_h, the discrete problem consists in finding $\lambda_h \in IR^+$ and $U \in C - \{0\}$ such that

$$\left(A_h U, V - U\right) \geq \lambda_h \left(B_h U, V - U\right), \quad for\, all\; V \in C, \qquad [6]$$

where (.,.) denotes the inner product in IR^N, C denotes a non-empty closed cone convex subset of IR^N, and the symmetric and positive definite matrices A_h and B_h have respectively for expressions $a\left(\varphi_k, \varphi_l\right)$ and $b\left(\varphi_k, \varphi_l\right)$, $1 \leq k, l \leq N$. In fact, we are not interested in all pairs $\left(\lambda_h, U\right) \in IR^+ \times \left(C - \{0\}\right)$ satisfying the variational inequality [6]. We only would like to focus on the smallest eigenvalue and the corresponding mode satisfying problem [6], which are respectively the approximated buckling critical load and the approximated first buckling mode. Consequently, the smallest eigenvalue, denoted by λ_{1h}, is expressed by the minimizing problem:

$$\lambda_{1h} = \min_{v_h \in K_h - \{0\}} \frac{a\left(v_h, v_h\right)}{b\left(v_h, v_h\right)} = \min_{V \in C - \{0\}} \frac{\left(A_h V, V\right)}{\left(B_h V, V\right)} . \qquad [7]$$

It is shown in (Dixmier, 1981) that the minimum of problem [7] exists and occurs at both

$$U = \left(u_i\right)_{i=1,N} \in C - \{0\} \; and \; w_{3h} = \sum\nolimits_{i=1}^{N} u_i \varphi_i \in K_h - \{0\}$$

which also are solutions to problems [6] and [5] respectively (Ciarlet, 1978).

REMARK. —Observe that, in general, the convex set K_h is not a subset of the convex set K. So, we cannot compare the approximated buckling critical load λ_{1h} to the exact buckling critical load λ.

5. The numerical algorithm

5.1. *The nonlinear eigenvalue problem to be solved*

Consider two real, symmetric and positive definite matrices A and B. The nonlinear eigenvalue problem we are dealing with is: Find the smallest $\lambda > 0$ and $u \in \left(IR_-^i \times IR^j\right) - \{0\}$ such that

$$\left(Au, v-u\right) \;\geq\; \lambda\left(Bu, v-u\right) \;\forall v \in \left(IR_{-}\right)^{i} \times IR^{j}, \qquad [8]$$

where $i + j = N \in IN^{*}$.

Let $R(A,B)$ be the mapping, *Rayleigh quotient,* associated with matrices *A* and *B*:

$$R\left(A,B\right): v \in IR^{N} - \left\{0\right\} \rightarrow R\left(A,B\right)\left(v\right) = \frac{\left(Av, v\right)}{\left(Bv, v\right)}.$$

Then, solving problem [8] is equivalent to solving the minimizing problem:

$$\lambda = \min_{v \in \left(IR_{-}\right)^{i} \times IR^{j} - \left\{0\right\}} R\left(A,B\right)\left(v\right). \qquad [9]$$

It is well known, (see (Dixmier, 1981)), that problem [9] has at least one solution $u \in \left(IR_{-}\right)^{i} \times IR^{j}$ such that $\left\|u\right\|_{2} = 1$.

The main goal of the paper is: how to compute the approximated unilateral buckling critical load λ and the corresponding approximated unilateral buckling mode u at minimum cost? After showing that the solution of problem [9] is also solution to a certain linear eigenvalue problem, and being inspired by an idea of H. Ben Dhia (see (Ben Dhia et *al.,* 2004)), an algorithm, which consists in computing the smallest eigenvalue of each of a finite number of linear eigenvalue problems, is suggested. This means, at least, that the convergence of the algorithm is guaranteed. Then, there is a direct relation between the number of mesh nodes in the contact region ω_{c} and that of linear eigenvalue problems to be solved. Indeed, the smaller the former is, the smaller the later will be. Moreover, the computation of these problems may be done in parallel if we dispose of parallel machines.

5.2. *The linear problem associated with the nonlinear one*

In order to give a very interesting characterization of the pair $\left(\lambda, u\right)$, solution to problem [9], we need the following definitions and notations:

$$I\left(u\right) = \left\{k \in \left\{1, 2, ..., i\right\}; u_{k} = 0\right\},$$

$$p\left(u\right) = card\left(I\left(u\right)\right),$$

$$E\left(u\right) = \left\{x \in \left(IR_{-}\right)^{i} : x_{k} = 0 \;\; \textit{if and only if} \;\; k \in I\left(u\right)\right\},$$

$A(u)$ denotes the sub-matrix of A obtained by eliminating rows A_k and columns A^k, $k \in I(u)$, while u^* denotes the sub-vector of u obtained by eliminating the null components. Thus, we have

$$u^* \in \left(IR_-^*\right)^{i-p(u)} \times IR^j .$$

For all $0 \le p \le i$, $\wp_p\left(\{1,2,...,i\}\right)$ denotes the set of all subsets, containing p elements chosen in the set $\{1,2,...,i\}$. It is convenient to recall that $\wp_p\left(\{1,2,...,i\}\right)$ is finite and its cardinal is equal to C_i^p .

Lemma 5.1. *The pair* $\left(\lambda, u^*\right)$ *is solution to the unidentified linear eigenvalue problem*:

$$\lambda = \min_{v \in (IR)^{N-p(u)}-\{0\}} R\big(A(u), B(u)\big)(v) . \qquad [10]$$

Proof: From the double inequality:

$$R\big(A(u), B(u)\big)\left(u^*\right) \ge \min_{v \in \left(IR_-^*\right)^{i-p(u)} \times IR^j} R\big(A(u), B(u)\big)(v) ,$$

And

$$\min_{v \in (IR_-)^i \times IR^j - \{0\}} R(A,B)(v) \le \min_{v \in E(u) \times IR^j} R(A,B)(v) ,$$

we deduce the new minimizing problem:

$$\lambda = \min_{v \in (IR_-^*)^{i-p(u)} \times IR^j} R\big(A(u), B(u)\big)(v) .$$

Because $\left(IR_-^*\right)^{i-p(u)} \times IR^j$ is an open subset of $IR^{N-p(u)}$, it follows, (see (Ciarlet, 1982)), the first Euler's optimality condition:

$$R'\big(A(u), B(u)\big)\left(u^*\right) = 0 \iff A(u)u^* = \lambda B(u)u^* ,$$

as well as the second optimality condition:

$$R''\big(A(u), B(u)\big)\left(u^*\right)(w,w) \ge 0 \text{ for all } w \in IR^{N-p(u)} .$$

But the above second optimality condition can be otherwise expressed as follows:

$$R\big(A(u),B(u)\big)\big(u^{*}\big) \le R\big(A(u),B(u)\big)(w),\ \textit{for all}\ w\in IR^{N-p(u)}.$$

Consequently, the pair $\left(\lambda,u^{*}\right)$ satisfies the linear eigenvalue problem [10].

□

REMARK. —As long as the buckling mode u is unknown, the linear eigenvalue problem [10] is unidentified. This is why we are going to solve a finite number of identified linear eigenvalue problems.

Theorem 5.2. *In order to solve problem* [9], *according to lemma 5.1, we should solve a finite number of identified linear problems of type* [10]. *This number is at most equal to* 2^i .

Proof**:** We can easily show the following partition:

$$\left(IR_{-}\right)^{i}\times IR^{j}=\left(\bigcup_{p=0}^{i} E(p)\right)\times IR^{j},$$

where

$$E(p)=\left\{x\in\left(IR_{-}\right)^{i}: \textit{the number of null components is exactly equal } p\right\}.$$

But each set $E(p), 0\le p\le i$, is in its turn union of a finite number of sets as exemplify the following formula:

$$E(p)=\bigcup_{M(p)\in\wp_{p}(\{1,2,\dots,i\})} E\big(p,M(p)\big),$$

where

$$E\big(p,M(p)\big)=\left\{v\in E(p):\ v_{k}=0 \textit{ if and only if } k\in M(p)\right\}.$$

Thus, the minimizing problem [9] rewrites as follows:

$$\lambda=\min_{0\le p\le i}\left(\min_{M(p)\in\wp_{p}(\{1,2,\dots,i\})}\left(\min_{v\in E(p,M(p))\times IR^{j}} R(A,B)(v)\right)\right).$$

According to lemma 4.1, there exists an optimal pair $\left(p^*, M\left(p^*\right)\right)$ such that

$$\lambda = \min_{v \in (IR)^{N-p^*} - \{0\}} R\left(A\left(p^*, M\left(p^*\right)\right), B\left(p^*, M\left(p^*\right)\right)\right)(v),$$

where $A\left(p^*, M\left(p^*\right)\right)$ denotes the sub-matrix of A obtained by eliminating p^* rows A_k and p^* columns A^k, $k \in M\left(p^*\right)$. Because the optimal pair $\left(p^*, M\left(p^*\right)\right)$ is ignored, we shall search for the solution of problem [9] by solving a certain number of linear eigenvalue problems among the following:

$$\begin{cases} 0 \le p \le i, \\ M(p) \in \wp_p\left(\{1,2,...,i\}\right), \\ \lambda\left(p, M(p)\right) = \min\limits_{v \in (IR)^{N-p} - \{0\}} R\left(A\left(p, M(p)\right), B\left(p, M(p)\right)\right)(v), \end{cases} \qquad [11]$$

whose number is equal to $C_i^0 + C_i^1 + ... + C_i^i = 2^i$. □

In order to give more details, regarding the number of linear eigenvalue problems to be solved, we need the following definition.

Definition 5.3. Let, for all $0 \le p \le i$, **M(p)** be the set of all subsets $M(p) \in \wp_p\left(\{1,2,...,i\}\right)$ for which

$$\lambda\left(p, M(p)\right) = R\left(A\left(p, M(p)\right), B\left(p, M(p)\right)\right)(x) \textit{ such that } x \in \left(IR_-^*\right)^{i-p} \times IR^j,$$

$$\lambda(p) = \min_{M(p) \in \mathbf{M(p)}} \lambda\left(p, M(p)\right).$$

REMARK. —Although the convergence of penalized problems is theoretically justified, the penalization method still remains unstable. Indeed, because the penalty parameter is supposed to approach zero, the penalization method yields very ill-conditioned matrices and consequently leads to inaccurate results. In order to avoid and overcome these numerical instabilities, a direct method is suggested. Its greatest advantage is, in my knowledge to the penalization method (Ayadi, 1990), the accuracy of its results.

The reader might think that the large number of linear eigenvalue problems [11] to be solved, which may reach 2^i, is a drawback of the algorithm. However, this number can be noticeably reduced. Indeed, according to several computations, we have observed that if $\lambda(p)$ and $\lambda(q)$ exist, for $0 \le p, q \le i$, then we have:

$$p \le q \Rightarrow \lambda(p) \le \lambda(q).$$

How to interpret that very interesting result? From the mechanical point of view, if the number p of fixed points in the contact region ω_c gets larger, the thin plate gets more rigid and therefore, the buckling critical load gets larger. Besides, the linear problems [11] can always be handled independently if parallel machines are available. Indeed, our algorithm represents a typical example of parallel computations.

5.3. *A flow chart of the algorithm*

Hereafter the different practical stages of the algorithm:

$\lambda = +\infty$ *and* $q = 0,$

For $p = 0$ *to* i

For $m = 1$ *to* C_i^p

$\lambda(p,m(p)) = R(A(p,m(p)),B(p,m(p)))(w^*)$

If $\left(w^* \in \left(IR_-^*\right)^{i-p} \times IR^j\right)$

$\lambda = \min(\lambda, \lambda(p,m(p)))$

If $(\lambda = \lambda(p,m(p)))$, then $q = w^*$

End if

End if

End

If $(q \ne 0)$ then the algorithm has converged. Break

End if

End

6. Numerical results

In order to test our algorithm and validate our numerical unilateral buckling model, three numerical tests corresponding to the same rectangular plate but three different kinds of obstacle are achieved (see figure 2 below).

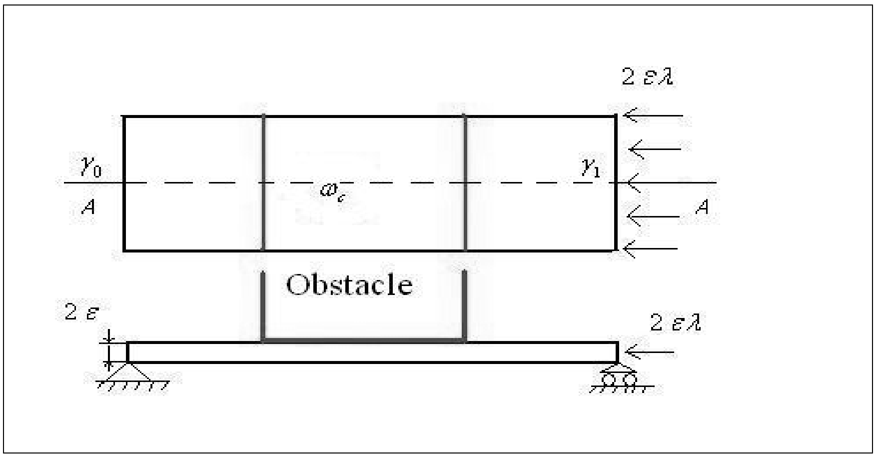

Figure 2. *The rectangular plate is in presence of the obstacle*

Occupying the two-dimensional domain $\omega =]-0.1, 0.1[\times]-0.05, 0.05[$, the plate is simply supported on the whole of its edge and compressed on the part γ_1 of its edge by a uniform load $2\varepsilon\lambda$. Furthermore, the plate is supposed to have a thickness $2\varepsilon = 0.006m$ and to be made of a homogenous and isotropic material whose mechanical features are: the Young's modulus E = 1.000e+09Pa, and the Poisson's ratio $\nu = 0.3$.

In order to minimize the number of degrees of freedom, the Kirchhoff-Love's plate model is replaced by the Mindlin's plate model. The later satisfies the exact three-dimensional boundary conditions, but does not allow representing three-dimensional singularities. Boundary layer models based on Kirchhoff-Love theory, at the opposite, permit to point out the existence of such stress singularities (see (Davet et *al.,* 1985)). Fortunately, the two theories are suitable for computing the critical state of buckling. The Mindlin's plate model involves, as unknowns, the deflection w_3 and the two rotations θ_1 and θ_2 of the mid plane of the plate, which are related by the formulae (see (Ciarlet et *al.,* 1977), (Destuynder, 1990)):

$$\theta_\alpha = -\frac{\partial w_3}{\partial x_\alpha}, \alpha = 1, 2.$$

Because the Mindlin's model involves at most first order partial derivatives, a continuous finite element is used. This is the quadrangle with four nodes, known as Macneal's finite element (see (Macneal, 1978)). Let now (Q_h) be a mesh of the plate by regular quadrangles. Then the space W is approximated by the finite dimensional subspace:

$$W_h = \left\{ (v_{3h}, r_{1h}, r_{2h}) \in \left(C^0(\varpi) \right)^3 : v_{3h|Q}, r_{\alpha h|Q} \in Q_1[X,Y] \ for\, all\, Q \in (Q_h) \ and\ "B.C.S" \right\}$$

where "*B.C.S*" is the abbreviation of boundary condition satisfied, and

$$Q_1[X,Y] = \{ Q_1 = a_{00} + a_{10}X + a_{01}Y + a_{11}XY \}$$

is the set of polynomials which are linear with respect to each variable. We likewise approximate the convex set:

$$K = \left\{ (v_3, r_1, r_2) \in W : v_3(x,y) \le 0 \ \ for\ \ all\ \ (x,y) \in \omega_c \right\}$$

by

$$K_h = \left\{ (v_{3h}, r_{1h}, r_{2h}) \in W_h : v_{3h}(x_i, y_j) \le 0,\ for\, all\, mesh\, node\, (x_i, y_j) \in \varpi_c \right\}.$$

For a mesh made up of 400 squares, we obtain the following numerical results.

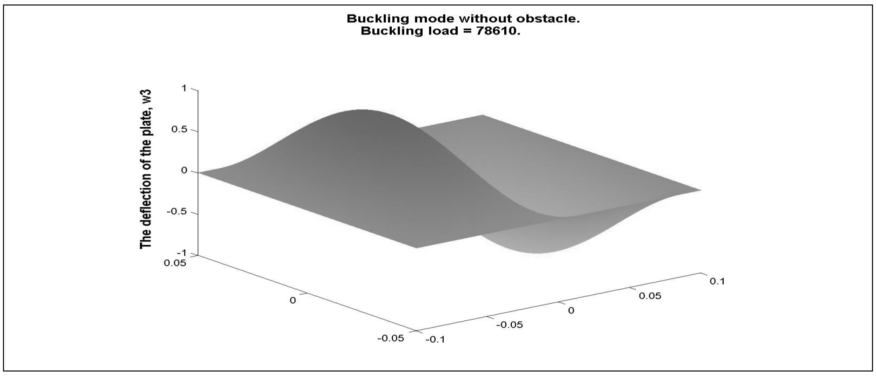

Figure 3. *The buckling mode of the plate in the absence of any obstacle*

Consider now the first kind of obstacle occupying the domain:

$$\omega_{c1} = [-0.04, 0.04] \times [-0.05, 0.05].$$

The figure 4 shows the unilateral buckling mode of the plate in presence of an obstacle involving the contact domain $\omega_c = \omega_{c1}$. Observe that the buckling mode respects the unilateral contact conditions. The buckling critical load is noticeably greater than that obtained in the absence of the obstacle.

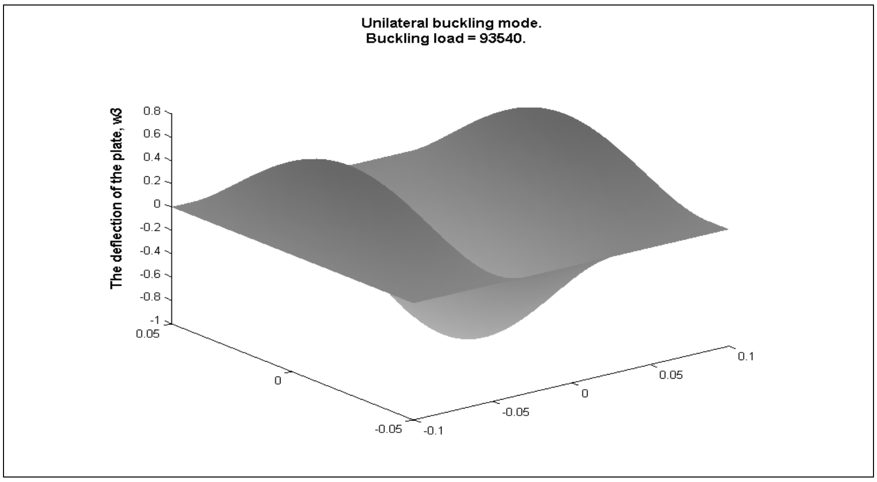

Figure 4. *The unilateral buckling mode of the plate in the case:* $\omega_c = \omega_{c1}$

In the figure 5 below are plotted three curves showing the sections *A* of the buckling modes (in the absence and in the presence of the obstacle) of the plate as well as the section *A* of the obstacle (see figure 2). These curves show the position of the plate in relation to the obstacle in both cases: without obstacle and in the presence of the obstacle. It is very clear that there is not interpenetration of material, which is the goal of this work.

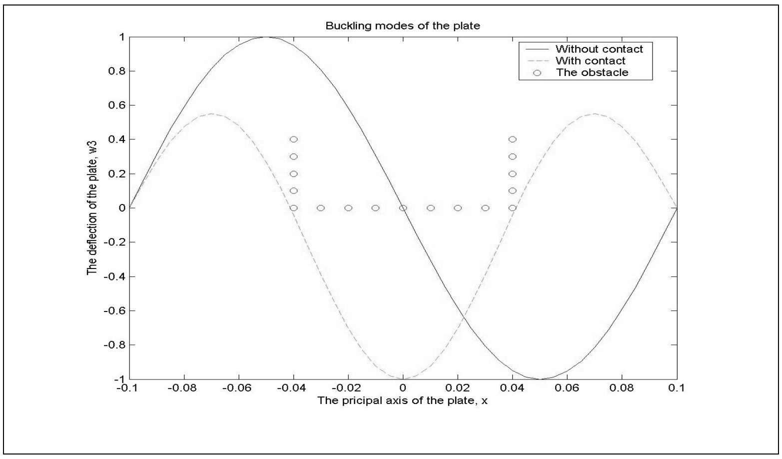

Figure 5. *The sections A of the buckling modes of the plate in the case:* $\omega_c = \omega_{c1}$

The second kind of obstacle, occupying the domain

$$\omega_{c2} = [-0.04, 0.04] \times [-0.02, 0.02],$$

is then considered.

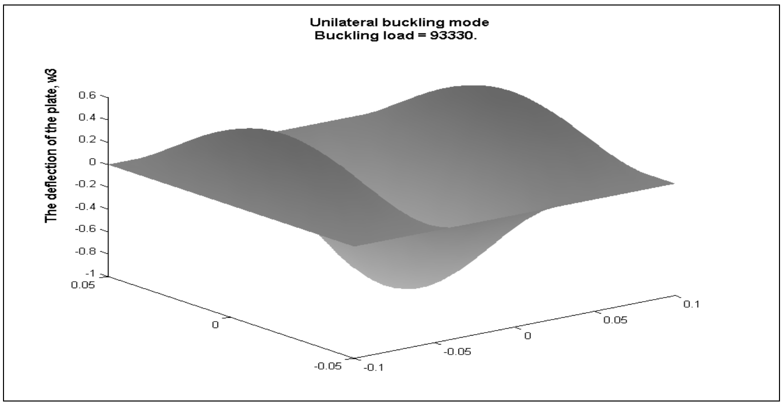

Figure 6. *The unilateral buckling mode of the plate in the case:* $\omega_c = \omega_{c2}$

The noticeable changing of the obstacle has slightly affected the shape of the unilateral buckling mode as well as the buckling critical load.

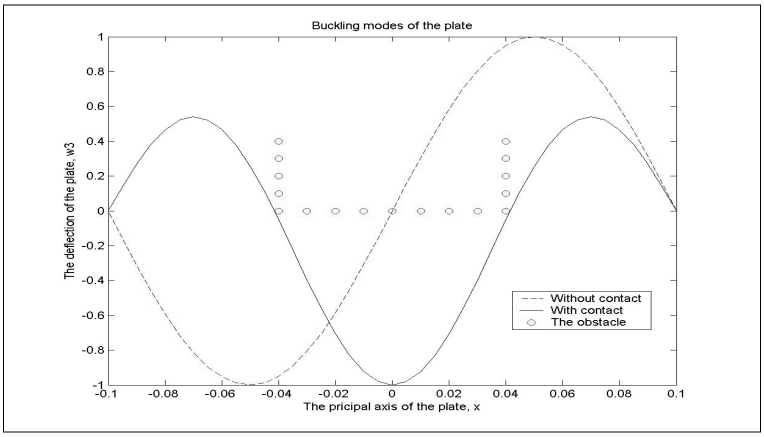

Figure 7. *The sections A of the buckling modes of the plate in the case:* $\omega_c = \omega_{c2}$

Finally, the third kind of obstacle occupying the domain $\omega_c = \overline{\omega}$ (all the plate is a contact zone) yields:

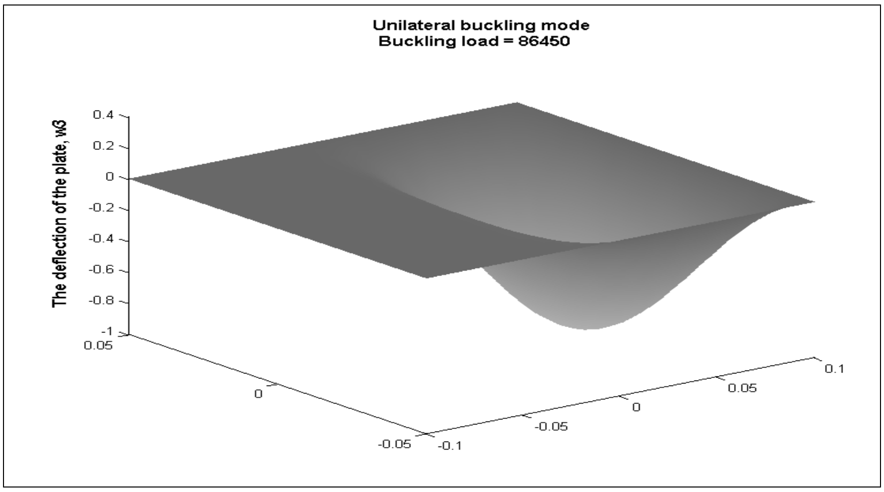

Figure 8. *The unilateral buckling mode of the plate in the case:* $\omega_c = \overline{\omega}$

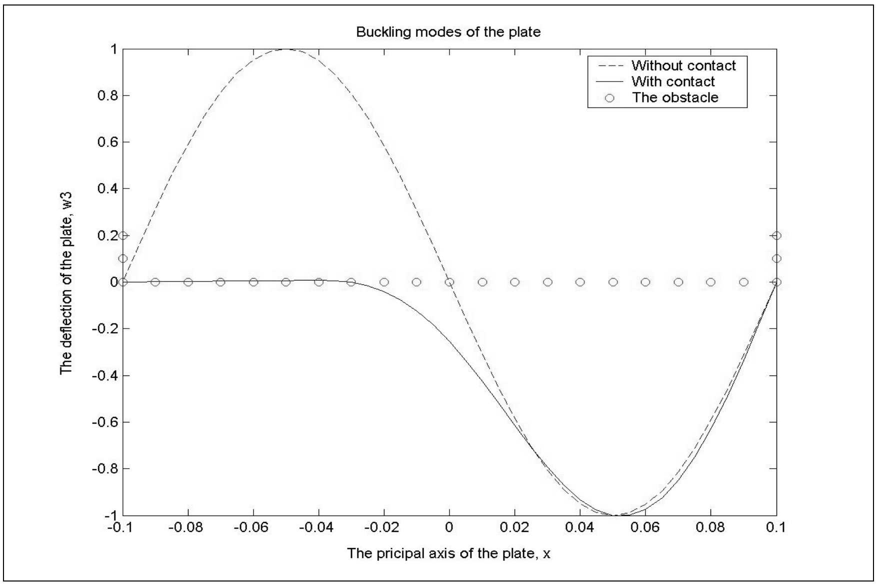

Figure 9. *The sections A of the buckling modes of the plate in the case:* $\omega_c = \overline{\omega}$

7. Conclusion and perspectives

Because methods based on the penalization technique are unstable, in the sense that the unilateral buckling mode completely changes by slightly changing the penalty parameter, a direct method is suggested. Implemented with the Mindlin's plate model and the Macneal's finite element, the algorithm we are proposing has allowed to obtain the buckling critical load and the corresponding unilateral buckling mode with a good accuracy and a reasonable cost. Indeed, it consists to solve a finite number of linear problems of the same kind. To each problem corresponds a couple of real, symmetric and positive definite matrices A and B and we have to compute the smallest eigenvalue and the correspondent eigenvector for the generalized eigenvalue problem $Au = \lambda Bu$. Fortunately, all these problems can be handled independently by using parallel machines.

The engineering application of the work reported herein is motivated by the fact that: in several mechanical situations, the slender structure is not at all tolerated to buckle. Therefore, the buckling phenomenon must be very well understood by the constructors.

The future works could deal with unilateral buckling of equilibrium states of shells, which involves sufficiently large difficulties both from the theoretical and the numerical points of view, such as curvature and stiffness problems.

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