Probabilistic nonparametric model of impedance matrices

Application to the seismic design of a structure

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ABSTRACT. *Economic and legal pressures on the structural engineers force them to consider uncertainty in the domains interacting, through boundary impedances, with their design structure. A probabilistic model of this impedance is constructed around a mean hidden state variables model using a nonparametric method. This mean model is constructed using only deterministic tools. The methodology is applied to the design of a gas tank on a layered soil.*

RÉSUMÉ. *Des facteurs économiques et réglementaires poussent les ingénieurs à prendre en compte les incertitudes existant dans les domaines en interaction, via des impédances de frontière, avec les structures qu'ils modélisent. Un modèle probabiliste de ces impédances est construit par une méthode non paramétrique, autour d'un modèle moyen à variables d'état cachées identifié à partir de calculs déterministes. L'approche est appliquée au dimensionnement sismique d'une cuve de stockage de gaz sur sol stratifié.*

KEYWORDS: *nonparametric model, impedance, probabilistic mechanics, uncertainties*

MOTS-CLÉS : *modèle non paramétrique, impédance, mécanique probabiliste, incertitudes*

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1. Introduction

In aeronautics, hydrodynamics and geodynamics, engineers have to deal with unbounded domains - atmosphere, sea or soil - interacting through boundary impedance matrices with the structures they are designing (Wolf 1985). More generally, domain decomposition techniques make use of impedance matrices when the entire Finite Element (FE) model of an engineering system is too large to be built all at once. Let us consider an unbounded domain Ω, coupled through boundary Γ to a structure (figure 1). The impedance of Ω through Γ will be denoted $\boldsymbol{Z}$. At the continuous level, it is the classical Dirichlet-to-Neumann operator, and the impedance matrix once numerical approximation is applied. It links - for the local harmonic boundary value problem defined on Ω at a frequency ω- the displacement vector $\boldsymbol{u}$ and the stress vector $\boldsymbol{t}$, defined on a given basis of interface functions on the boundary Γ.

$$\boldsymbol{Z}(\omega)\boldsymbol{u}(\omega) = \boldsymbol{t}(\omega).$$

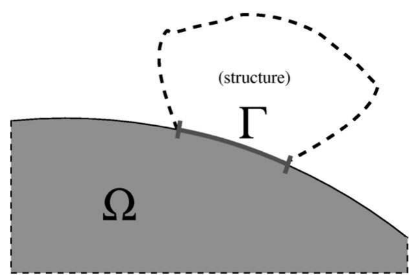


Figure 1. *Uncertain unbounded domain Ω coupled to a structure through a boundary* Γ

The balancing of security and economic issues require that the engineers be able to compute, as precisely as possible, those impedance matrices. Unfortunately, that required accuracy is often out of reach. In soil mechanics, for example, the scarcity of the available data on the mechanical parameters, their spatial variability, the errors introduced by the measuring procedures, and the important errors due to the simplistic models used (Favre 1998), make the achievement of an exact estimation of an impedance matrix illusory. In that case, a probabilistic model has to be constructed and the probability density function of the impedance - rather than a single mean value - estimated.

In that purpose, many different stochastic methods have been developed (Schuëller 1997). They all share the same characteristic that they try to identify the uncertainty on the parameters of the problem, and propagate that uncertainty to the response of the system through the resolution of a system of stochastic differential equations. The most classical of those parametric methods is the Stochastic FE method (Cornell 1971, Ghanem *et al.* 1991) which works fine for the construction of the probabilistic model of the impedance of a bounded domain but cannot be extended to unbounded domains. Even when coupled with the (deterministic) Boundary Element (BE) method (Savin *et al.* 2002), only a bounded subdomain is considered to have uncertain characteristics.

In this article, a method is presented to construct the probabilistic model of an impedance matrix, avoiding the construction of the probabilistic model of the dynamic stiffness matrix of the unbounded domain. The link between the value of the parameters of the mechanical model and the value of the impedance matrix is therefore not explicitely given, and the parametric methods cannot be used. A nonparametric method, recently introduced (Soize 1999, Soize 2000), is chosen. It is based on the maximum entropy principle (Jaynes 1957), constrained only by the unquestionable information on the model.

The causality of the impedance matrix being one of these constraints, a mean model has to be constructed which enforces it (*cf.* section 2). The principle of the nonparametric method of random uncertainties in linear structural dynamics is then briefly recalled, followed by the construction of the probabilistic model for the impedance matrix (*cf.* section 3). The required identification of the mean model from experimental or computational results is then described (*cf.* section 4). Finally, this construction is applied for the design under seismic loading of a gas tank resting on a pile foundation (*cf.* section 5).

2. Mean model of the impedance matrix

As any physical quantity, the impedance matrix must verify, in the time domain, the causality condition. It states only the natural law that no effect should take place without a cause, or mathematically written:

$$\boldsymbol{u}(t) = \boldsymbol{0}\,, \forall t < 0 \Rightarrow (\hat{\boldsymbol{Z}} * \boldsymbol{u})(t) = \boldsymbol{0}\,, \forall t < 0,$$

where $t \rightarrow \hat{\boldsymbol{Z}}(t)$ is the inverse Fourier transform of the impedance matrix in the frequency domain $\omega \rightarrow \boldsymbol{Z}(\omega)$. Any model, mean or probabilistic, for the impedance matrix should enforce that relation. In the frequency domain, three methods may be used: the Kramers-Kronig relations (Kronig 1926, Kramers 1927); the expansion of the impedance matrix on a basis of Hardy functions (Pierce 2001); or the construction of the impedance matrix on an underlying system ensuring causality (Chabas *et al.* 1987).

2.1. *Kramers-Kronig relations*

The first method was originally used in electromagnetic problems and relates the real and imaginary part of any causal quantity in the frequency domain. In the case of the impedance matrix it states that

$$\Re\{\boldsymbol{Z}(\omega)\} = \frac{1}{\pi} \oint_{\mathbb{R}} \frac{\Im\{\boldsymbol{Z}(\omega')\}}{\omega - \omega'} d\omega'.$$

where $\Re\{\boldsymbol{Z}\}$ and $\Im\{\boldsymbol{Z}\}$ are, respectively, the real and imaginary part of the impedance matrix, and $\oint$ stands for Cauchy's integral. This relation is still used in experimental physics where the imaginary part of a quantity can sometimes be measured independently from its real part. Here, it is not constructive as no information is available on $\Im\{\boldsymbol{Z}\}$.

2.2. *Expansion on a basis of Hardy function*

Another possibility is to expand the impedance matrix on a basis of functions that are known to span the entire space of causal functions: the Hardy functions space $\mathbb{H}$. As the family $(\omega \to e_n(\omega))_{n\geq 0}$, defined, for $\omega \in \mathbb{R}$, by

$$e_n(\omega) = \frac{1}{\sqrt{\pi}}\left(\frac{1}{i\omega - 1}\right)\left(\frac{i\omega + 1}{i\omega - 1}\right)^n,$$

is an orthonormal basis of $\mathbb{H}$, the impedance matrix can be expanded, for $\omega \in \mathbb{R}$, in

$$\boldsymbol{Z}(\omega) = -\omega^2 \boldsymbol{Z}_{-2} + i\omega \boldsymbol{Z}_{-1} + \boldsymbol{Z}_0 + \sum_{n\geq 0} \boldsymbol{Z}_{n+1} e_n(\omega),$$

where $\boldsymbol{Z}_n$ is the n^{th} coordinate - frequency-independent - of the pseudo-differential part of $\boldsymbol{Z}$ in the basis $(e_n)_{n\geq 0}$. Unfortunately, the convergence rate of the approximation $\sum_{n=0}^{N} \boldsymbol{Z}_n e_n$ of $\boldsymbol{Z}$ for increasing values of N is not known, and the *a priori* unknown signature of the coordinates $\boldsymbol{Z}_n$ would require the construction of complicated sets of random matrices at the hour of using the nonparametric method.

2.3. *Hidden state variables model*

Finally, the impedance matrix can be constructed on an underlying system ensuring that the causality property is verified. It is sought with the same structure as the impedance matrix of a mechanical system whose vibrations in the time domain are governed by a second-order differential equation with constant coefficients. For some systems - a bounded linear elastic system, for example - this approach corresponds exactly to the classical modelling: the impedance is the condensation on the n_Γ degrees of freedom (DOFs) of the boundary of the $n \times n$ dynamic stiffness matrix $\boldsymbol{A}$ written as

$$\boldsymbol{A}(\omega) = -\omega^2 \boldsymbol{M} + i\omega \boldsymbol{D} + \boldsymbol{K}. \quad [1]$$

where $\boldsymbol{M}$, $\boldsymbol{D}$ and $\boldsymbol{K}$ are the real, frequency-independent, matrices of mass, damping and stiffness. $\boldsymbol{M}$ is in $\mathbb{M}_n^+(\mathbb{R})$, the set of all $n \times n$ real positive-definite matrices and $\boldsymbol{D}$ and $\boldsymbol{K}$ are in $\mathbb{M}_n^{+0}(\mathbb{R})$, the set of all $n \times n$ real positive semi-definite matrices.

In the general case, this approach defines an approximation pattern for the impedance matrix and, although the notation will be kept, $\boldsymbol{A}$, $\boldsymbol{M}$, $\boldsymbol{D}$ and $\boldsymbol{K}$ are not the actual dynamic stiffness, mass, damping and stiffness matrices. Likewise, the variables that appear in this model are related to the real DOFs of the physical system only indirectly. This model of the impedance matrix will therefore be called a hidden state variables model.

The bloc-decomposition of the dynamic stiffness matrix on the n_Γ DOFs of the boundary and the n_h hidden state variables,

$$\boldsymbol{A}(\omega) = \begin{bmatrix} \boldsymbol{A}_\Gamma(\omega) & \boldsymbol{A}_c(\omega) \\ \boldsymbol{A}_c^T(\omega) & \boldsymbol{A}_h(\omega) \end{bmatrix}, \qquad [2]$$

leads to an impedance matrix in the form:

$$\boldsymbol{Z}(\omega) = \boldsymbol{A}_\Gamma(\omega) - \boldsymbol{A}_c(\omega)\boldsymbol{A}_h^{-1}(\omega)\boldsymbol{A}_c^T(\omega), \qquad [3]$$

and the bloc-decomposition corresponding to equation [2] for the mass, damping and stiffness matrices leads to, with identification to equation [1],

$$\begin{aligned} \boldsymbol{A}_\Gamma(\omega) &= -\omega^2\boldsymbol{M}_\Gamma + i\omega\boldsymbol{D}_\Gamma + \boldsymbol{K}_\Gamma, \\ \boldsymbol{A}_c(\omega) &= -\omega^2\boldsymbol{M}_c + i\omega\boldsymbol{D}_c + \boldsymbol{K}_c, \\ \boldsymbol{A}_h(\omega) &= -\omega^2\boldsymbol{M}_h + i\omega\boldsymbol{D}_h + \boldsymbol{K}_h. \end{aligned}$$

where $\boldsymbol{M}_\Gamma \in \mathbb{M}^+_{n_\Gamma}(\mathbb{R})$, $\boldsymbol{D}_\Gamma, \boldsymbol{K}_\Gamma \in \mathbb{M}^{+0}_{n_\Gamma}(\mathbb{R})$, $\boldsymbol{M}_h \in \mathbb{M}^+_{n_h}(\mathbb{R})$, $\boldsymbol{D}_h, \boldsymbol{K}_h \in \mathbb{M}^{+0}_{n_h}(\mathbb{R})$ and $\boldsymbol{M}_c, \boldsymbol{D}_c, \boldsymbol{K}_c \in \mathbb{M}_{n_\Gamma,n_h}(\mathbb{R})$.

Equation [3] can be rewritten in the following form:

$$\boldsymbol{Z}(\omega) = \frac{\boldsymbol{N}(\omega)}{d(\omega)}, \qquad [4]$$

where $\omega \mapsto \boldsymbol{N}(\omega)$ and $\omega \mapsto d(\omega)$ are two polynomials with constant coefficients (matricial for $\boldsymbol{N}$ and scalar for d). The degrees of $\boldsymbol{N}$ and d verify $\deg \boldsymbol{N} = \deg d + 2$. The values of the matrices $\boldsymbol{M}$, $\boldsymbol{D}$ and $\boldsymbol{K}$ of this mean model for the impedance matrix can be identified from computational or experimental results (*cf.* section 4).

3. The nonparametric method

The nonparametric method was originally developped in linear structural dynamics (Soize 1999, Soize 2000, Soize 2001a) with applications in vibrations and transient elastodynamics, and was extended to nonlinear dynamical systems (Soize 2001b) and to the medium frequency range (Soize 2003). The coupling of structures with different levels of uncertainty has also been considered in (Soize *et al.* 2003, Chebli *et al.* 2004) and a nonparametric-parametric approach has been presented in (Desceliers *et al.* 2003) to model each source of uncertainty with the most appropriate method. Hereafter are only recalled the main ideas, with no proof. The reader should refer to (Soize 1999, Soize 2000) for more details.

The main concept of this method is to identify, for each problem, the unquestionable information, and to use the maximum-entropy principle to derive a probabilistic model using only that available information. This information is scarcer than that used in the parametric methods and includes for example, in linear structural dynamics, the positive-definiteness of the mass matrix and the existence of the moments of the inverse of that matrix. Here the available information is composed of the causality of the impedance matrix, which is enforced by the hidden state variables model that was chosen, and the classical information available for the mass, damping and stiffness matrices. This information consists in their signature, their square-integrability, the integrability of their inverse and their mean value.

More precisely, let $\boldsymbol{B}_n$ (be it $\boldsymbol{M}$, $\boldsymbol{D}$ or $\boldsymbol{K}$) be a $n \times n$ random matrix such that:

1) $\boldsymbol{B}_n$ is a random matrix with values in $\mathbb{M}_n^+(\mathbb{R})$, almost surely;

2) $\boldsymbol{B}_n$ is a second-order random variable: $E\{\|\boldsymbol{B}_n\|_F^2\} < +\infty$;

3) The mean value $\underline{\boldsymbol{B}_n}$ of $\boldsymbol{B}_n$ is given in $\mathbb{M}_n^+(\mathbb{R})$: $E\{\boldsymbol{B}_n\} = \underline{\boldsymbol{B}_n}$;

4) $\boldsymbol{B}_n$ is such that: $E\{\ln(\det \boldsymbol{B}_n)\} = \nu \ , \ \nu < +\infty$;

where $\|\boldsymbol{B}_n\|_F = (\mathrm{tr}\{\boldsymbol{B}_n \boldsymbol{B}_n^*\})^{1/2}$ is the Frobenius norm of $\boldsymbol{B}_n$, $E\{\cdot\}$ is the mathematical expectation, and the fourth condition is that which enforces the integrability of the inverse of $\boldsymbol{B}_n$. Using the maximum entropy principle, the probability density function $p_{\boldsymbol{B}_n}$ of $\boldsymbol{B}_n$, constrained only by this information, can be calculated analytically, with respect to a measure $\tilde{d}\boldsymbol{B}_n$ on $\mathbb{M}_n^S(\mathbb{R})$, the set of all $n \times n$ symmetric real matrices.

The parameter ν which is introduced in this fourth condition is usually replaced by a dispersion parameter δ which arises naturally in the computation of $p_{\boldsymbol{B}_n}$ of $\boldsymbol{B}_n$ and is the actual dispersion on $\boldsymbol{B}_n$ (Soize 2000).

$$\delta = \left(\frac{E\{\|\boldsymbol{B}_n - \underline{\boldsymbol{B}_n}\|_F^2\}}{\|\underline{\boldsymbol{B}_n}\|_F^2} \right)^{1/2} . \qquad [5]$$

The estimation of δ depends on the type of information available:

– when no objective information is known about δ, a sensitivity analysis must be performed with δ as the parameter and its value estimated depending on the level of stochastic fluctuations (level of uncertainty);

– when experimental data is available, mathematical statistics give the value of δ;

– when a parametric model has been constructed in the low-frequency range, where data uncertainties are, in general, more important than model uncertainties, δ can be estimated through statistics on the first eigenfrequency;

– when the uncertain system pertains to a class of systems for which δ has already been studied, the same value can be re-used.

A method was devised to compute efficiently Monte-Carlo trials of such a random matrix $\boldsymbol{B}_n$, given its mean value $\underline{\boldsymbol{B}_n}$ and a dispersion parameter δ. It can also be shown that, if no correlation is explicitely introduced as a constraint in the maximum entropy method between the elements of a set of random matrices, then they are independent variables. This proves that the matrices $\boldsymbol{M}$, $\boldsymbol{D}$ or $\boldsymbol{K}$, each one with its own mean value ($\underline{\boldsymbol{M}}$, $\underline{\boldsymbol{D}}$ and $\underline{\boldsymbol{K}}$) and dispersion parameter ($\delta_{\boldsymbol{M}}$, $\delta_{\boldsymbol{D}}$ and $\delta_{\boldsymbol{K}}$) can be drawn independently. For each triplet of Monte-Carlo trials $[\boldsymbol{M}, \boldsymbol{D}, \boldsymbol{K}]$, a realization of the dynamic stiffness matrix $\boldsymbol{A}$ is computed using equation [1] and, finally, a realization of the impedance matrix $\boldsymbol{Z}$ is obtained with equation [3].

The construction of this probabilistic model of the impedance matrix therefore requires the knowledge of the mean values $\underline{\boldsymbol{M}}$, $\underline{\boldsymbol{D}}$ and $\underline{\boldsymbol{K}}$ and dispersion parameters $\delta_{\boldsymbol{M}}$, $\delta_{\boldsymbol{D}}$ and $\delta_{\boldsymbol{K}}$. The identification of the dispersion parameters is described in (Arnst *et al.* 2005, Soize 2005, Ratier *et al.* 2005), and that of the mean matrices is described in the next section.

4. Identification of the matrices $\underline{\boldsymbol{M}}$, $\underline{\boldsymbol{D}}$, $\underline{\boldsymbol{K}}$ of the mean model

The identification of the mean values $\underline{\boldsymbol{M}}$, $\underline{\boldsymbol{D}}$, $\underline{\boldsymbol{K}}$ of the mass, damping and stiffness matrices in the hidden state variables model of the impedance matrix is performed in two steps:

1) The algebraic form [4] of $\boldsymbol{Z}$ is considered, and the value of the coefficients of the polynomials $\boldsymbol{N}$ and d are sought so as to minimize an error function between $\boldsymbol{Z} = \boldsymbol{N}/d$ and a given impedance matrix $\tilde{\boldsymbol{Z}}$, that was either measured experimentally or computed. The first step is then an interpolation of a given (matricial) function by a (matrix) rational function. Since the degree of $\boldsymbol{N}$ and d is *a priori* unknown, an iteration on the number of hidden state variables has to be set to obtain an approximation sufficiently accurate.

2) Given the coefficients of the polynomials $\boldsymbol{N}$ and d in [4], the second step consists in finding the mass, damping and stiffness frequency-independent matrices $\underline{\boldsymbol{M}}$, $\underline{\boldsymbol{D}}$, $\underline{\boldsymbol{K}}$ giving rise in equations [1-3] to such coefficients. No approximation is performed in that step, although, as will be seen, more than one solution may arise.

The interesting feature of that methodology is that it separates the problem into one very generic approximation problem that can be solved by virtually any of many existing methods (Guillaume *et al.* 1996, Allemang *et al.* 1998, Pintelon *et al.* 2001), and one more specific identification problem that does not involve any approximation. Particularly, the error function of step 1 can be adapted to the type of mean impedance available: experimental or computational. For the purpose of the example in this article (*cf.* section 5), a linear least squares approximation with orthonormal polynomial vectors was used but it will not be described here and the reader is refered to (Pintelon *et al.* 2004, Bultheel *et al.* 1995). For the remainder of this section, only step 2 of the identification will be considered.

Let $\mathbf{\Phi}$ an orthogonal $n_h \times n_h$ real frequency-independent matrix, $\boldsymbol{\mathcal{M}}_c$ a $n_\Gamma \times n_h$ real frequency-independent matrix and $\boldsymbol{U}$ the $n \times n$ real frequency-independent matrix, defined by

$$\boldsymbol{U} = \begin{bmatrix} \boldsymbol{I}_{n_\Gamma} & -\boldsymbol{\mathcal{M}}_c \\ \mathbf{0}_{n_\Gamma,n_h} & \mathbf{\Phi} \end{bmatrix} \qquad [6]$$

where $\boldsymbol{I}_{n_\Gamma}$ is the $n_\Gamma \times n_\Gamma$ real identity matrix, and $\mathbf{0}_{n_\Gamma,n_h}$ the $n_\Gamma \times n_h$ real null matrix. It is obvious from equation [3], that the sets of matrices $[\underline{\boldsymbol{M}}, \underline{\boldsymbol{D}}, \underline{\boldsymbol{K}}]$ and $[\boldsymbol{U}\underline{\boldsymbol{M}}\boldsymbol{U}^T, \boldsymbol{U}\underline{\boldsymbol{D}}\boldsymbol{U}^T, \boldsymbol{U}\underline{\boldsymbol{K}}\boldsymbol{U}^T]$ lead to the same impedance matrix, and therefore that they are equivalent sets if only the impedance is given.

Starting from a set $[\underline{\boldsymbol{M}}, \underline{\boldsymbol{D}}, \underline{\boldsymbol{K}}]$, if $\mathbf{\Phi}$ is chosen as the matrix of the eigenvectors of the generalized eigenvalue problem for $\boldsymbol{M}_h$ and $\boldsymbol{K}_h$, normalized with respect to $\boldsymbol{M}_h$ (by hypothesis, $\mathbf{\Phi}$ also diagonalizes $\boldsymbol{D}_h$), and $\boldsymbol{\mathcal{M}}_c = \boldsymbol{M}_c$, then we have

$$\boldsymbol{U}\underline{\boldsymbol{M}}\boldsymbol{U}^T = \begin{bmatrix} \boldsymbol{m}_\Gamma & \mathbf{0}_{n_h,n_\Gamma} \\ \mathbf{0}_{n_\Gamma,n_h} & \boldsymbol{I}_h \end{bmatrix}, \qquad [7]$$

and

$$\boldsymbol{U}\underline{\boldsymbol{D}}\boldsymbol{U}^T = \begin{bmatrix} \boldsymbol{d}_\Gamma & \boldsymbol{d}_c \\ \boldsymbol{d}_c^T & \boldsymbol{d}_h \end{bmatrix}, \boldsymbol{U}\underline{\boldsymbol{K}}\boldsymbol{U}^T = \begin{bmatrix} \boldsymbol{k}_\Gamma & \boldsymbol{k}_c \\ \boldsymbol{k}_c^T & \boldsymbol{k}_h \end{bmatrix}, \qquad [8]$$

where $\boldsymbol{d}_h$ and $\boldsymbol{k}_h$ are diagonal matrices which, as is customary in structural vibrations, are written in terms of eigenfrequencies and modal damping $\boldsymbol{d}_h = \mathrm{diag}(2\zeta_k\omega_k)_{1\le k\le n_h}$ and $\boldsymbol{k}_h = \mathrm{diag}(\omega_k^2)_{1\le k\le n_h}$. Since for any matrix $\boldsymbol{U}$ in the form of equation [6], the sets $[\underline{\boldsymbol{M}}, \underline{\boldsymbol{D}}, \underline{\boldsymbol{K}}]$ and $[\boldsymbol{U}\underline{\boldsymbol{M}}\boldsymbol{U}^T, \boldsymbol{U}\underline{\boldsymbol{D}}\boldsymbol{U}^T, \boldsymbol{U}\underline{\boldsymbol{K}}\boldsymbol{U}^T]$ are equivalent, we freely choose to perform the identification on a set in the form of equations [7-8]. The impedance matrix can then be written

$$\boldsymbol{Z}(\omega) = -\omega^2\boldsymbol{m}_\Gamma + i\omega\boldsymbol{d}_\Gamma + \boldsymbol{k}_\Gamma - \sum_{k=1}^{n_h} \frac{(i\omega\boldsymbol{d}_c + \boldsymbol{k}_c)(i\omega\boldsymbol{d}_c + \boldsymbol{k}_c)^T}{-\omega^2 + 2i\zeta_k\omega_k\omega + \omega_k^2} \qquad [9]$$

On the other hand, the matricial rational function $\omega \to \boldsymbol{N}(\omega)/d(\omega)$ can be expanded in a unique pole-residue form:

$$\boldsymbol{Z}(\omega) = \frac{\boldsymbol{N}(\omega)}{d(\omega)} = -\omega^2\boldsymbol{R}_{-2} + i\omega\boldsymbol{R}_{-1} + \boldsymbol{R}_0 + \sum_{k=1}^{2n_h} \frac{\boldsymbol{R}_k}{i\omega - p_k}. \qquad [10]$$

In the general case, the poles and the residue are complex, but can be paired as they are present with their complex conjugate. Let us denote two elements of a pair with α and β, so that the $\boldsymbol{R}_k^\alpha + \boldsymbol{R}_k^\beta$ and $\boldsymbol{R}_k^\alpha p_k^\beta + \boldsymbol{R}_k^\beta p_k^\alpha$ are real. Equations [9-10] yield obvious identifications for $\boldsymbol{m}_\Gamma$, $\boldsymbol{d}_\Gamma$, $(\omega_k)_{1\le k\le n_h}$ and $(\zeta_k)_{1\le k\le n_h}$ and lead to the following system of equations for the $\boldsymbol{k}_c^k$, the $\boldsymbol{d}_c^k$ and $\boldsymbol{k}_\Gamma$:

$$\begin{cases} \boldsymbol{d}_c^k\boldsymbol{k}_c^{kT} + \boldsymbol{k}_c^k\boldsymbol{d}_c^{kT} - 2\zeta_k\omega_k\boldsymbol{d}_c^k\boldsymbol{d}_c^{kT} = -(\boldsymbol{R}_k^\alpha + \boldsymbol{R}_k^\beta) & ,\ 1 \le k \le n_h \\ \boldsymbol{k}_c^k\boldsymbol{k}_c^{kT} - \omega_k^2\boldsymbol{d}_c^k\boldsymbol{d}_c^{kT} = (\boldsymbol{R}_k^\alpha p_k^\beta + \boldsymbol{R}_k^\beta p_k^\alpha) & ,\ 1 \le k \le n_h \\ \boldsymbol{k}_\Gamma + \sum_{k=1}^{n_h} \boldsymbol{d}_c^k\boldsymbol{d}_c^{kT} = \boldsymbol{R}_0 & \end{cases}$$

The first $2n_h$ equations can be solved by diagonalization and, finally $\boldsymbol{k}_\Gamma$ can be computed from the knowledge of the $\boldsymbol{d}_c^k$.

5. Seismic design of a gas storage tank on a layered soil

The method presented in this paper is applied in this section to the seismic design of a concrete gas storage tank set on a circular rigid superficial foundation on a layered soil. The tank is 80 meters-wide and 38 meters-high and is modeled deterministically. The soil is constituted of a 50 meters-deep soft layer (mass density $\rho = 2000kg/m^3$, Young's modulus $E = 5.33 \times 10^9 N/m^2$, Poisson coefficient $\nu = 0.33$ and hysteretic damping coefficient $\beta = 0.001$) on top of a stiffer half-space ($\rho = 2500kg/m^3$, $E = 6.0 \times 10^9 N/m^2$, $\nu = 0.33$ and $\beta = 0.001$). The mean soil impedance matrix is computed using the BE method (Miss3D program (Clouteau 2003)). The tank is modeled using the FE method (figure 2). The Frequency Response Function (FRF) of the horizontal displacement of the highest point of the structure for a unit plane shear wave excitation propagating from infinity is considered. The real and imaginary parts of that FRF are drawn in dashed line on figure 3.

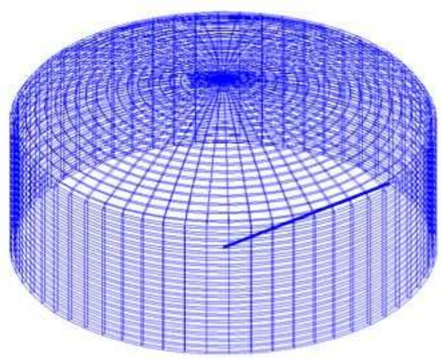
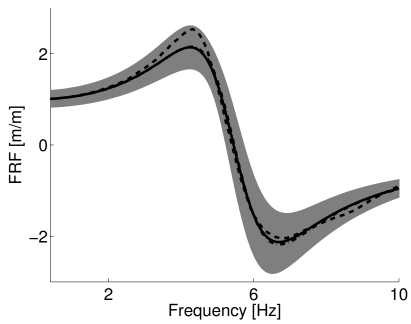
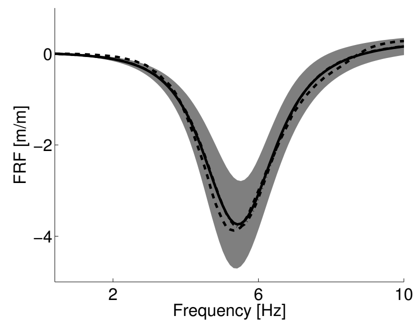


Figure 2. *FE model of the gas storage tank*

The hidden state variables model of the mean impedance matrix is constructed, yielding a correct approximation of the BE result with only one hidden variable, and the corresponding FRF is drawn on figure 3 in dash-dotted line (it is almost perfectly covered by the solid line). Finally, the probabilistic model of the impedance matrix is approximated using 1000 Monte-Carlo trials (the mean and the covariance of the impedance matrix converge after a few hundred trials) for equal dispersion parameters for the mass, damping and stiffness matrices: $\delta_{\boldsymbol{M}} = \delta_{\boldsymbol{D}} = \delta_{\boldsymbol{K}} = 0.1$. They have been chosen here by analogy with previous works on bounded structures, but should in the future be identified directly form experiments. For each impedance matrix, the corresponding displacement of the top of the building is computed and drawn on figure 3.



(a) Real part



(b) Imaginary part

Figure 3. *Real and imaginary parts of the FRF of the highest point of the tank to a unit plane shear wave excitation: FE-BE deterministic model (dashed line); Mean hidden variables model (dash-dotted line); envelope of the Monte-Carlo trials (grey patch); mean of the trials (solid line)*

6. Conclusion

The method presented in this paper allows to construct a nonparametric probabilistic model of the soil impedance matrix that takes into account both the data uncertainties and the modelling errors. The only required knowledge is a mean impedance matrix and a set of dispersion parameters that can be identified from experiments (Arnst *et al.* 2005, Soize 2005) or that one can vary in a parametric study (Ratier *et al.* 2005). The mean impedance matrix can be either computed or measured, and is approximated by a hidden variables model that ensures its causality. The way to draw the realizations of the random impedance matrix is given and the response statistics are computed by the Monte-Carlo method. Although the method was presented in the case of seismic engineering, it is useful for a very broad range of applications: any problem involving an unbounded domain is eligible. It may prove interesting even for large bounded domain, as the reduction of the analysis to boundary impedance matrices reduces the computational costs compared to classical parametric methods where the entire uncertain domain has to be modeled. The application showed the applicability of the method for an industrial design problem.

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