On improved rectangular finite element for plane linear elasticity analysis

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ABSTRACT. Based on the strain approach, with the customary two displacement d.o.f at each corner, a rectangular element is developed for the general plane linear elasticity problems. The adoption of independent linear variations for direct strains and an independent constant shearing strain proves to be effective in improving the accuracy of the elements. However any singularity is eliminated by the use of local axes optimally oriented. From the numerical examples of a beam under bending and shear, by using the concept of static condensation it is concluded that the exact solutions can be obtained. Several numerical examples are presented to show that the present element has high accuracy, excellent computational efficiency.

RÉSUMÉ. Un élément rectangulaire basé sur l'approche en déformation, possédant les deux d.d.l essentiels pour chacun des nœuds sommets, est développé pour la résolution des problèmes d'élasticité linéaire plane en général. L'adoption de variations linéaires pour les composantes normales de la déformation et une déformation de cisaillement constante permettent d'améliorer la précision des éléments. Cependant toute singularité est éliminée par l'utilisation d'un repère local orienté d'une manière optimale. A partir d'exemples numériques d'une poutre soumise à la flexion et au cisaillement et en utilisant le concept de condensation statique, on conclut que des solutions exactes peuvent êtres obtenues. Quelques applications ont été abordées et ont permis de mettre en évidence la fiabilité et la précision du présent élément.

KEYWORDS: rectangular element, two-dimensional elasticity, strain approach, membrane, static condensation.

MOTS-CLÉS : élément rectangulaire, élasticité à deux dimensions, approche en déformation, membrane, condensation statique.

1. Introduction

In a series of papers (Belarbi *et al.*, 2002, 1999, 1998, Belarbi, 2000) we have attempted to highlight the strain approach. This approach was originally developed by Sabir and Ashwell (Sabir *et al.*, 1971a), their search was begun on curved structures, they concluded that we can have better results with a reduced number of elements, compared with results given by the displacement model (Sabir *et al.*, 1971b). A new class of simple and effective finite elements for the problem of general plane elasticity (Sabir, 1983) was developed. Strains are independent and verify the criterion of completeness. This approach was later extended to three-dimensional elasticity (Belarbi *et al.*, 1999), shell problems (Assan, 1999, Djoudi *et al.*, 2003, 2004a, 2004b, Sabir *et al.*, 1996, 1997) and plate bending problems (Belounar *et al.*, 2005).

Several models such as rectangular plane elasticity elements were developed, among them the elements of Sabir (Sabir *et al.*, 1996, 1995) SBRIE and SBRIE1, each of them have two degrees of freedom (d.o.f) at each corner node. The first element is based on linear variation of direct strains (linear in *y* for ε_x and linear in *x* for ε_y) and constant shearing strain. The second element in addition to the corner nodes an internal node is also used, which is statically condensed. This element is based on linear variation of all three strain components (linear in *y* for $\varepsilon_{x,}$ linear in *x* for ε_y and linear in *x*,*y* for γ_{xy}).

In this paper, an improved rectangular strain based element will be presented; it is based on linear variation (linear in *x* and *y*) of direct strains and constant shearing strain. It has two d.o.f at each corner node and an internal node. Through the introduction of additional internal d.o.f, we managed to develop an element which proved to be more accurate, however it requires static condensation (Bathe *et al.*, 1976). This element is used to obtain solutions to general plane linear elasticity problems.

2. Analytical considerations

Consider the rectangular element shown in figure 1, the three components of the strain at any point in the Cartesian coordinate system are given in terms of the displacements U and V:

$$\varepsilon_{xx} = U,_x \qquad [1a]$$

$$\varepsilon_{yy} = V,_y \qquad [1b]$$

$$\gamma_{xy} = U,_y + V,_x \qquad [1c]$$

If the strains given by equations [1] are equal to zero, the integration of these equations allows obtaining the following expressions:

$$U = a_1 - a_3 y \quad [2a]$$

$$V = a_2 + a_3 x \quad [2b]$$

Equations [2] represent the displacement field in terms of its three rigid body displacements.

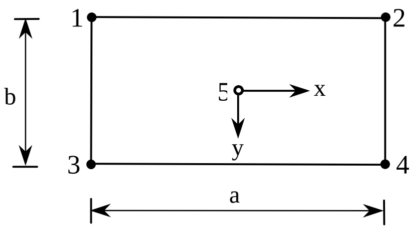


Figure 1. *Co-ordinates and nodal points for the rectangular R4BM element*

The present element is rectangular with four corner nodes and a central node, each node has two degrees of freedom. Thus, the displacement field should contain ten independent constants. Having used three (a_1, a_2, a_3) for the representation of the rigid body components, thus it is left seven constants (a_4, a_5,….. ,a_{10}) for expressing the displacement due to straining of the element. These seven independent constants are apportioned among the three strains as follow:

$$\varepsilon_{xx} = a_4 + a_5 y + a_9 x \quad [3a]$$

$$\varepsilon_{yy} = a_6 + a_7 x + a_{10} y \quad [3b]$$

$$\gamma_{xy} = a_8 \quad [3c]$$

By integrating equations [3], the displacement functions are obtained as follow:

$$U = a_4 x + a_5 xy - 0.5 a_7 y^2 + 0.5 a_8 y + 0.5 a_9 x^2 \quad [4a]$$

$$V = - 0.5 a_5 x^2 + a_6 y + a_7 xy + 0.5 a_8 x + 0.5 a_{10} y^2 \quad [4b]$$

The final displacement functions are obtained by adding equations [2] and [4] to obtain the following:

$$U = a_1 - a_3 y + a_4 x + a_5 xy - 0.5 a_7 y^2 + 0.5 a_8 y + 0.5 a_9 x^2 \quad [5a]$$

$$V = a_2 + a_3\,x - 0.5\,a_5\,x^2 + a_6\,y + a_7\,xy + 0.5\,a_8\,x + 0.5\,a_{10}\,y^2 \qquad [5b]$$

Another version of this element having the same strain assumptions as above, with a rearrangement of the different coefficients, the strain field will be:

$$\varepsilon_{xx} = a_4 + a_5\,x + a_6\,y \qquad [6a]$$

$$\varepsilon_{yy} = a_7 + a_8\,x + a_9\,y \qquad [6b]$$

$$\gamma_{xy} = a_{10} \qquad [6c]$$

This version produces rapid convergence of deflection and has the following displacement field:

$$U = a_1 - a_3\,y + a_4\,x + a_6\,xy + 0.5\,a_5\,x^2 - 0.5\,a_8\,y^2 + 0.5\,a_{10}\,y \qquad [7a]$$

$$V = a_2 + a_3\,x - 0.5\,a_6\,x^2 + a_7\,y + a_8\,xy + 0.5\,a_9\,y^2 + 0.5\,a_{10}\,x \qquad [7b]$$

The stiffness matrix is derived without using any tricks, which implies that it is obtained using exact and not reduced integration.

$$[K_e] = [A^{-1}]^T\,[K_0]\,[A^{-1}] \qquad [8a]$$

$$[K_0] = \iint_S [Q]^T [D][Q]\,dx.dy \qquad [8b]$$

With

$$[Q] = \begin{bmatrix} 0 & 0 & 0 & 1 & y & 0 & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & x & 0 & 0 & y \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \text{ and } [D] = \begin{bmatrix} D11 & D12 & 0 \\ D12 & D22 & 0 \\ 0 & 0 & D33 \end{bmatrix}$$

the usual constitutive matrix, where:

$$D11 = D22 = \frac{E}{(1-\nu^2)} \quad ; \quad D12 = \frac{\nu.E}{(1-\nu^2)} \quad ; \quad D33 = \frac{E}{2(1+\nu)}$$

For [A] and $[K_0]$ see the appendix

3. Numerical experiments

The numerical results of several quadrilateral plane elements are used and compared with those obtained from the present R4BM element and they are listed as follows:

– SBRIE: the strain based rectangular in-plane element (Sabir *et al.,* 1986);

– SBRIE1: the strain based rectangular in-plane element with an internal node (Sabir *et al.,* 1995);

– Q4: the standard four-node isoparametric element.

Most of the examples dealt with have been proposed at various stages in open literature to validate element performance. It will be seen that the SBRIE and the SBRIE1 versions show the same results for all cases.

3.1. *An elongated thin cantilever beam subjected to end shear*

An elongated thin cantilever beam subjected to end shear is a standard problem to test finite element accuracy. Young's modulus and Poisson's ratio are denoted by E and ν. These parameters and the mesh division are shown in figure 2, while the results are presented in Table 1, it should be noted that the R4BM element gives the most accurate results.

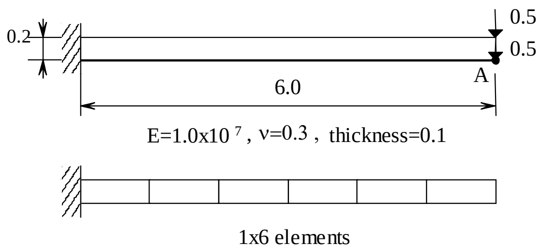


Figure 2. *An elongated thin cantilever beam subjected to end shear*

Table 1. *Normalized deflection at point A, of a thin cantilever beam under shear*

Normalized tip deflection	
Mesh	1x 6
SBRIE	0.903
Q4	0.093
R4BM	**0.992**
SBRIE1	0.903
Analyt.	1.000 (0.1081)

3.2. *An elongated thin cantilever beam subjected to end pure bending*

The tip deflection of an elongated thin cantilever beam under pure bending is compared using the present R4BM. The geometry, parameters and mesh discretization of the beam are shown in figure 3. Using four different mesh divisions, the normalised tip deflections of the R4BM are computed and compared with those obtained by other elements in Table 2. A pertinent point to note is that exact solution can be obtained for the R4BM element. The accuracy of the SBRIE and SBRIE1is quite high.

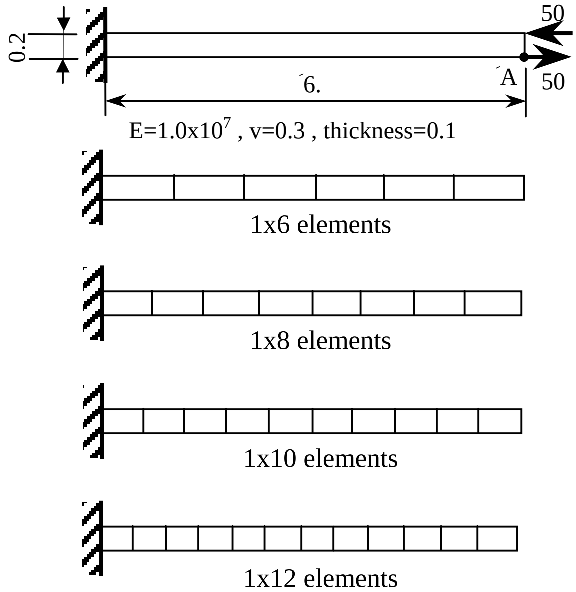


Figure 3. *An elongated thin cantilever beam subjected to end pure bending*

Table 2. *Normalised deflection at point A, of a thin cantilever beam under pure bending*

Normalised tip deflection				
Mesh	1x 6	1x 8	1x 10	1x 12
SBRIE	0.91	0.91	0.91	0.91
Q4	0.093	0.153	0.219	0.285
R4BM	**1.000**	**1.000**	**1.000**	**1.000**
SBRIE1	0.91	0.91	0.91	0.91
Analyt.	1.000 (0.270)			

3.3. *Aspect ratio tests for cantilever*

Two other tests for the cantilever problem can be conduct here, in the first case we keep the dimensions of the cantilever constant and we vary the meshes, in the second we vary the depth and keep the same mesh variation.

Case 1

In addition to the above examples, an extra test is included here to study the sensitivity of the present element to the variation in aspect ratio. We consider the response of a cantilever beam to a parabolic distributed shear applied as shown in figure 4. From the results presented in Table 3, we find that the displacement model gives bad results, and require more refinements in order to be able to approach the correct solution, whereas the R4BM performs very well.

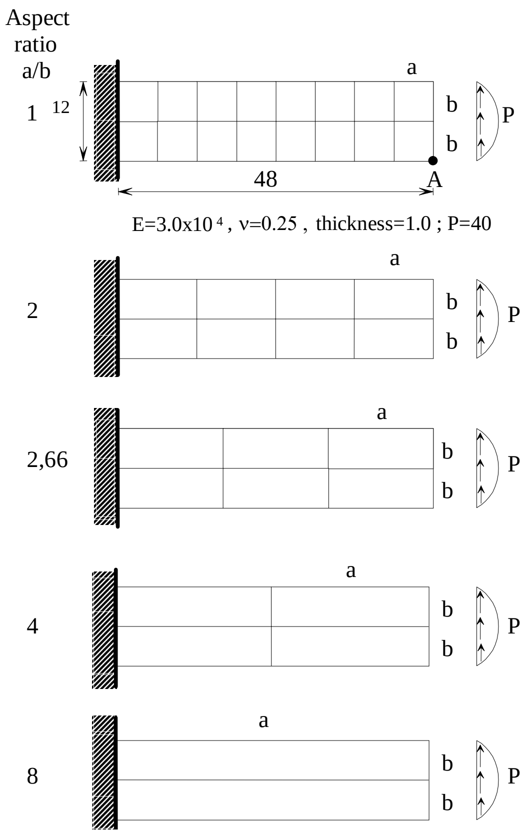


Figure 4. *Cantilever beam subjected to parabolically distributed shear, aspect ratio tests*

Table 3. *Normalized deflection at point A, of a cantilever beam*

Normalised tip deflection					
Element aspect ratio a/b	1.0	2.0	2.66	4.0	8.0
Mesh	2 x 8	2 x 4	2 x 3	2 x 2	2 x 1
SBRIE	0.972	0.957	0.943	0.905	0.738
Q4	0.888	0.699	0.573	0.378	0.134
R4BM	**0.988**	**0.972**	**0.958**	**0.919**	**0.750**
SBRIE1	0.972	0.957	0.943	0.905	0.738
Analyt.	1.000 (0.3558)				

Case 2

Sabir proposes tests which are shown in figure 5 involving deep, moderately deep and thin cantilever beams respectively. The cantilevers under consideration have a width b = 0.0625 m and length l = 10m, the material properties are taken to be 100 000 N/mm^2 and 0.2 for Young's modulus and Poisson's ratio, respectively. We calculate vertical deflection at the tip of the beam (Table 4), R4BM predicts displacements which are very close to the theoretical values for all the cases considered.

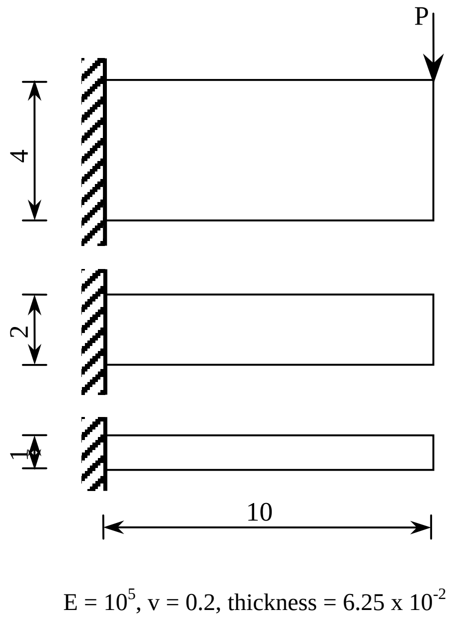


Figure 5. *Deep, moderately deep and thin cantilever beam*

Table 4a. *Normalized vertical deflection for deep cantilever*

Deep cantilever						
Mesh	2x2	2x4	6x6	8x8	12x12	10x16
SBRIE	0.913	0.929	0.988	0.992	0.995	0.995
Q4	0.591	0.597	0.926	0.956	0.979	0.971
R4BM	**0.922**	**0.932**	**0.989**	**0.993**	**0.996**	**0.995**
SBRIE1	0.913	0.929	0.988	0.992	0.995	0.995
Analyt.	1.000 (1.105)					

Table 4b. *Normalized vertical deflection for moderately deep*

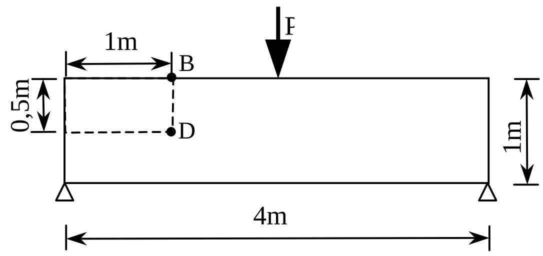
Moderately deep cantilever						
Mesh	2x2	2x4	6x6	8x8	12x12	10x16
SBRIE	0.916	0.923	0.988	0.992	0.996	0.995
Q4	0.275	0.275	0.771	0.856	0.930	0.903
R4BM	**0.925**	**0.926**	**0.989**	**0.993**	**0.996**	**0.995**
SBRIE1	0.916	0.923	0.988	0.992	0.996	0.995
Analyt.	1.000 (8.21)					

Table 4c. *Normalized vertical deflection for thin cantilever*

Thin cantilever						
Mesh	2x2	2x4	6x6	8x8	12x12	10x16
SBRIE	0.915	0.920	0.985	0.990	0.994	0.992
Q4	0.087	0.087	0.461	0.602	0.772	0.703
R4BM	**0.924**	**0.922**	**0.986**	**0.990**	**0.994**	**0.993**
SBRIE1	0.915	0.920	0.985	0.990	0.994	0.992
Analyt.	1.000 (64.52)					

3.4. *Simply supported beam loaded at mid-span*

The deep simply supported beam whose details are given in Figure 6 has been used in the finite element literature. It is also used here to test the performance of the present R4BM, and a comparison is made with the existing results given by the use of elements sited above.



P = 4.2 KN , E = 20x10^{6} KN/m² , v = 0.2
thickness = 0.5 m

Figure 6. *Simply supported beam loaded at mid-span*

Tables 5 and 6 show the results obtained for the bending stress at B and the shearing stress at D respectively. These tables show that the R4BM gives better results than all the other elements. Even for the coarse mesh this element produces results which are acceptable within practical engineering accuracy.

Table 5. *Bending stress at point B*

Bending stress at B				
Mesh	4x12	6x12	8x16	10x20
SBRIE	31.673	25.329	24.914	25.072
Q4	30.654	23.830	24.363	24.586
R4BM	**31.734**	**25.378**	**24.921**	**25.090**
SBRIE1	31.673	25.329	24.914	24.072
Exact	25.2			

Table 6. *Shearing stress at point D*

Bending stress at D				
Mesh	4x12	6x12	8x16	10x20
SBRIE	6.023	6.080	6.164	6.217
Q4	5.258	5.454	5.817	5.992
R4BM	**6.034**	**6.081**	**6.164**	**6.217**
SBRIE1	6.023	6.080	6.164	6.217
Exact	6.3			

4. Conclusion

The existing rectangular plane elements based on the strain model are reviewed. A new strain based element is developed for the analysis of general plane linear elasticity problems. This element can model beam bending action more closely than the other strain based elements and the standard displacement based element. It has only the customary two displacements d.o.f. Numerical examples demonstrate that the proposed element is both accurate and versatile. Its accuracy seem to be unaffected by the aspect ratio.

5. References

Ashwell D.G., Sabir A.B. and Roberts T.M., "Further studies in the application of curved finite elements to circular arches", *IJMS*, Vol. 13, 1971, pp. 507-517.

Assan A.E., "Analysis of multiple stiffened barrel shell structures by strain-based finite elements", *Thin-walled structures*, Vol. 35, 1999, pp. 233-253.

Bathe K.J. and Wilson E.L., *Numerical Methods in finite element analysis*, Printice Hall, New Jersey, 1976.

Belarbi M.T., Sedira L., "Contribution de l'intégration analytique dans l'amélioration géométriques et matérielle de l'élément SBRIE", *2ème colloque Maghrébin en génie civil CMGC'02*, Université de Biskra, Algérie, 2002, pp. 3-12.

Belarbi M.T., "Nouvel élément Triangulaire "SBT3" avec Drilling rotation", *Conférence Internationale sur les Mathématiques Appliquées et les Sciences de l'Ingénieur CIMASI'2000* les 23-24 et 25 octobre, Casablanca, Maroc, 2000.

Belarbi M.T., Charif A., "Développement d'un nouvel élément hexaédrique simple basé sur le modèle en déformation pour l'étude des plaques minces et épaisses", *Revue Européenne des éléments finis*, Vol. 8, No. 2, 1999, pp. 135-157.

Belarbi M.T., Charif A., "Nouvel élément secteur basé sur le modèle de déformation avec rotation dans le plan", *Revue Européenne des Eléments Finis*, Vol.7, No. 4, Juin 1998, pp. 439-458.

Belounar L., Guenfoud M., "A new rectangular finite element based on the strain approach for plate bending", *Thin-walled structures*, Vol. 43, 2005, pp. 47-63.

Djoudi M.S., Bahai H., "A cylindrical strain-based shell element for vibration analysis of shell structures", *Finite Elements in Analysis and Design*, Vol. 40, 2004, pp. 1947-1961.

Djoudi M.S., Bahai H., "Strain-based finite element for the vibration of cylindrical panels with openings", *Thin-walled structures*, Vol. 42, 2004, pp. 575-588.

Djoudi M.S., Bahai H., "A shallow shell finite element for the linear and nonlinear analysis of cylindrical shells", *Engineering structures*, Vol. 25, 2003, pp. 769-778.

Sabir A.B., Moussa A.I., "Analysis of fluted conical shell roofs using the finite element method", *Comput. Struct.*, Vol. 64, No. 1-4, 1997, pp. 239-251.

Sabir A.B., Moussa A.I., “Finite element analysis of cylindrical-conical storage tanks using strain-based elements”, *Structural Engineering Review,* (8), 4, 1996, pp. 367-374.

Sabir A.B., Sfendji A., “Triangular and Rectangular plane elasticity finite elements”, *Thin-walled Structures*, Vol. 21, 1995, pp. 225-232.

Sabir A.B., Salhi H.Y., “A strain based finite element for general plane elasticity in polar coordinates”, *Res. Mechanica*, Vol. 19, 1986, pp. 1-16.

Sabir A.B., “A new class of Finite Elements for plane elasticity problems”, *CAFEM 7th, Int. Conf. Struct. Mech. In Reactor Technology*, Chicago, 1983.

Sabir A.B. and Ashwell D.G., “A comparison of curved beam finite elements when used in vibration problems”, *Journal of Sound and Vibration*, Vol. 18, No. 11, 1971, pp. 555-563.

Appendix

$$[\mathrm{Ko}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & H1 & H2 & H3 & H4 & 0 & H5 & H6 \\ & & & & H7 & H8 & H9 & 0 & H10 & H11 \\ & & & & & H12 & H13 & 0 & H14 & H15 \\ & & \mathrm{Sym} & & & & H16 & 0 & H17 & H18 \\ & & & & & & & H19 & 0 & 0 \\ & & & & & & & & H20 & H21 \\ & & & & & & & & & H22 \end{bmatrix}$$

Where:

$H1 = b\,D11\,a$ $\quad$ $H6 = \frac{a\,D12\,b^2}{2}$ $\quad$ $H11 = \frac{a\,D12\,b^3}{3}$ $\quad$ $H16 = \frac{D22\,a^3\,b}{3}$ $\quad$ $H21 = \frac{a^2\,D12\,b^2}{4}$

$H2 = \frac{a\,D11\,b^2}{2}$ $\quad$ $H7 = \frac{a\,D11\,b^3}{3}$ $\quad$ $H12 = a\,D22\,b$ $\quad$ $H17 = \frac{D12\,a^3\,b}{3}$ $\quad$ $H22 = \frac{a\,D22\,b^3}{3}$

$H3 = b\,D12\,a$ $\quad$ $H8 = \frac{a\,D12\,b^2}{2}$ $\quad$ $H13 = \frac{a^2\,D22\,b}{2}$ $\quad$ $H18 = \frac{a^2\,D22\,b^2}{4}$

$H4 = \frac{a^2\,D12\,b}{2}$ $\quad$ $H9 = \frac{a^2\,D12\,b^2}{4}$ $\quad$ $H14 = \frac{a^2\,D12\,b}{2}$ $\quad$ $H19 = D33\,a\,b$

$H5 = \frac{a^2\,D11\,b}{2}$ $\quad$ $H10 = \frac{a^2\,D11\,b^2}{4}$ $\quad$ $H15 = \frac{a\,D22\,b^2}{2}$ $\quad$ $H20 = \frac{D11\,a^3\,b}{3}$

$$[A] = \begin{bmatrix}
1 & 0 & \frac{b}{2} & -\frac{a}{2} & \frac{a\,b}{4} & 0 & -\frac{b^2}{8} & -\frac{b}{4} & \frac{a^2}{8} & 0 \\
0 & 1 & -\frac{a}{2} & 0 & -\frac{a^2}{8} & -\frac{b}{2} & \frac{a\,b}{4} & -\frac{a}{4} & 0 & \frac{b^2}{8} \\
1 & 0 & \frac{b}{2} & \frac{a}{2} & -\frac{a\,b}{4} & 0 & -\frac{b^2}{8} & -\frac{b}{4} & \frac{a^2}{8} & 0 \\
0 & 1 & \frac{a}{2} & 0 & -\frac{a^2}{8} & -\frac{b}{2} & -\frac{a\,b}{4} & \frac{a}{4} & 0 & \frac{b^2}{8} \\
1 & 0 & -\frac{b}{2} & -\frac{a}{2} & -\frac{a\,b}{4} & 0 & -\frac{b^2}{8} & \frac{b}{4} & \frac{a^2}{8} & 0 \\
0 & 1 & -\frac{a}{2} & 0 & -\frac{a^2}{8} & \frac{b}{2} & -\frac{a\,b}{4} & -\frac{a}{4} & 0 & \frac{b^2}{8} \\
1 & 0 & -\frac{b}{2} & \frac{a}{2} & \frac{a\,b}{4} & 0 & -\frac{b^2}{8} & \frac{b}{4} & \frac{a^2}{8} & 0 \\
0 & 1 & \frac{a}{2} & 0 & -\frac{a^2}{8} & \frac{b}{2} & \frac{a\,b}{4} & \frac{a}{4} & 0 & \frac{b^2}{8} \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$