Finite element solution of the energy equation in lubricated contacts

Application to mechanical face seals

Noël Brunetière — Bernard Tournerie

Laboratoire de Mécanique des Solides – CNRS UMR 6610
SP2MI – Téléport 2 – Avenue Marie et Pierre Curie – BP 30179
F-86962 Futuroscope Chasseneuil cedex

noel.brunetiere@lms.univ-poitiers.fr

ABSTRACT. *In lubricated contacts, moving solids are separated by a strongly sheared thin fluid film. The resulting temperature rise due to viscous dissipation can greatly affect the behaviour of the contact. Therefore, it is essential to determine the temperature field in such contacts. Temperature is obtained by solving the energy equation (convection diffusion equation), which is modified to take into account the particular shape of the fluid film. Upwind schemes for the finite element method are presented for both the one- and two-dimensional steady configurations. They are then applied to simple lubrication problems and their results are compared. In some cases numerical oscillations occur. Modifications of the initial schemes are proposed to avoid those numerical problems. The influence of the boundary conditions and the effect of the orientation of the flow are analysed in more detail. Finally, the resolution of the three dimensional energy equation in a mechanical face seal is presented. There is a good correlation between the numerical results and the experimental data and this confirms the accuracy of the upwind scheme.*

RÉSUMÉ. *Dans les contacts lubrifiés, les solides en mouvement sont séparés par un film fluide fortement cisaillé. L'élévation de température due à la dissipation visqueuse peut nettement modifier le comportement du contact. Il est donc nécessaire de déterminer la température dans le film fluide. Elle est obtenue en résolvant l'équation de l'énergie qui prend une forme particulière du fait de la très faible épaisseur du film. Les schémas « upwind » pour les éléments finis sont présentés pour les problèmes stationnaires uni et bidimensionnel. Ils sont ensuite appliqués à des problèmes de lubrification simples et leurs résultats sont comparés. Dans certains cas, des oscillations apparaissent. Pour éviter cela des modifications sont proposées. L'influence des conditions aux limites et de la direction de l'écoulement est plus particulièrement analysée. Pour finir, la résolution numérique de l'équation de l'énergie tridimensionnelle dans une garniture mécanique est présentée. Les résultats numériques sont en bonne corrélation avec les données expérimentales, confirmant ainsi la précision du schéma « upwind ».*

KEYWORDS: *energy, temperature, convection, upwinding, lubrication, mechanical face seal.*

MOTS-CLÉS : *énergie, température, convection, lubrification, garniture mécanique.*

REEF – 14/2005. Finite elements in tribology, pages 213 to 235

1. Introduction

The efficiency of the finite element method in solving thermo elastic problems of complex-shape structures has been since several decades. It was natural to extend the method to fluid mechanics and, more particularly, to fluid film lubrication. The Reynolds equation that gives the pressure field in a lubricated contact has been solved by the finite element method since the mid-60's (Huebner, 1975). In fact, this equation is elliptic and can be easily solved using the well known Bubnov-Galerkin method, or by minimising a functional.

Thermal effects play an important role in the behaviour of a lubricated contact (Frêne, 1990), the key parameter of fluid viscosity being greatly dependent on temperature. Moreover, thermal distortions can be of the same order of magnitude as the fluid film thickness separating solids. It is therefore essential in fluid film lubrication to solve the energy equation in order to determine temperature..

However, the available methods (such as that of Bubnov-Galerkin) gave a poor fit in the solving of transport equations for fluids. Indeed, the convection terms in equations lead to numerical oscillations (Zienkiewicz, 2000). At the end of the 70's and in the early 80's many authors developed new schemes for tackling this problem. Several authors used them to solve the energy equation for hydrodynamic bearings (Kim, 1994, Kucinschi, 2000). But this energy transport equation is quite different from those relating to classical fluid flow. Consequently, upwind finite element schemes were modified to operate accurately for fluid film lubrication. However, it appears that what is lacking is a "state of the art" account relating to upwind schemes and the modifications needed to solve the energy equation in thin fluid film flows. Hence, this is the aim of the present paper.

In the first part of this paper, upwind schemes for the finite element method are presented for one- and two-dimensional steady state configurations. The second section concerns their application to simple lubrication problems. In some cases numerical oscillations occur and modifications to the initial schemes are proposed to avoid these calculation problems. The influence of the boundary conditions and the effect of the flow direction are analysed in more detail. In the third part of this paper, the resolution of the three-dimensional energy equation for a mechanical face seal is presented. The good correlation of the numerical results with experimental data confirms the accuracy of the upwind scheme.

2. Finite element schemes for the energy equation

For an incompressible fluid, the equation relating to the conservation of energy can be written as follows:

$$\rho C\left(\frac{\partial T}{\partial t}+V_i\frac{\partial T}{\partial x_i}\right)-\frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right)=\phi \qquad [1]$$

where ρ, C and k are respectively the fluid density, the fluid specific heat and the fluid thermal conductivity. The temperature of the fluid is T and V_i is the fluid velocity in the x_i direction. ϕ is the dissipation function. The first term on the left hand side represents energy convected by the fluid and consists of first order derivatives of T. This leads to problems during resolution and oscillations can occur in the calculated temperature field. In order to obtain a more accurate solution, several authors developed upwind schemes for the one-dimensional energy equation.

2.1. *The one-dimensional problem*

The stationary one-dimensional energy equation is expressed as:

$$\rho CV\frac{dT}{dx}-k\frac{d^2T}{dx}=\phi \qquad [2]$$

The temperature T can be discretized in the following way:

$$T\approx\sum N_i T_i \qquad [3]$$

where N_i are shape functions and T_i the nodal temperature values. By using a general weighting procedure, a linear system can be obtained:

$$K_{ij}T_j=f_i \qquad [4]$$

The terms are calculated by integration on the domain of the problem ($0\leq x\leq L$):

$$K_{ij}=\rho CV\int_0^L W_i\frac{dN_j}{dx}dx+k\int_0^L\frac{dW_i}{dx}\frac{dN_j}{dx}dx \text{ and } f_i=\int_0^L W_i\phi dx \qquad [5]$$

W_i is a weighting function.

2.1.1. *Bubnov – Galerkin method*

In the Bubnov-Galerkin method, weighting functions are equal to shape functions:

$$W_i = N_i \qquad [6]$$

For the linear shape function and equal elements size L_e, a typical assembled equation between nearby nodes is obtained (Zienkiewicz, 2000):

$$(-\mathrm{Pe}-1)T_{i-1} + 2T_i + (\mathrm{Pe}-1)T_{i+1} + \frac{\phi L_e}{k} = 0 \qquad [7]$$

where Pe is the mesh Peclet number:

$$\mathrm{Pe} = \frac{\rho C V L_e}{2k} \qquad [8]$$

If the source term ϕ is null, the temperature Ti must match the inequality:

$$\min(T_{i-1}, T_{i+1}) \le T_i \le \max(T_{i-1}, T_{i+1}) \qquad [9]$$

If |Pe|>1, the inequality [9] is violated by the equation [7]. In fact, oscillations appear in the temperature field given by the Bubnov-Galerkin method. A solution is to reduce Pe by reducing the element size L_e. But, in a convection-dominated problem, the number of nodes could become prohibitive. Another solution is to use an upwind scheme.

2.1.2. *Upwind schemes*

The general solution of the differential equation [2] is:

$$T(x) = A\exp\left(\frac{\rho C V}{k}x\right) + B \qquad [10]$$

where A and B are integration constants. This exact solution could be expressed in an assembled form:

$$[-\mathrm{Pe}(\alpha+1)-1]T_{i-1} + 2[1+\alpha\mathrm{Pe}]T_i + [-\mathrm{Pe}(\alpha-1)-1]T_{i+1} = 0 \qquad [11]$$

where the factor α is a function of the mesh Peclet number:

$$\alpha = \coth \mathrm{Pe} - \frac{1}{\mathrm{Pe}} \qquad [12]$$

The various upwind schemes lead to the equation [11] between nearby nodes and thus to the exact solution of the differential equation [2] without numerical oscillations.

The first possibility is to use a Petrov-Galerkin type of weighting in which $W_i \neq N_i$. The weighting functions are constructed in the following way (Zienkiewicz, 2000):

$$W_i = N_i + \alpha \tilde{W}_i \text{ with } \int_{\Omega_e} \tilde{W}_i dx = \pm \frac{L_e}{2} \text{ and } \int_{\Omega_e} \frac{d\tilde{W}_i}{dx} dx = 0 \quad [13]$$

Ω_e is the domain of one element. α is the factor defined in equation [12] and initially introduced by Christie *et al.* (Christie, 1976). According to the authors, one can find several expressions for the function $\tilde{W}_i$. Heinrich *et al.* (Heinrich, 1977) used a polynomial of the second degree. In 1979, Griffiths and Mitchell (Griffiths, 1979) proposed a simpler function:

$$\tilde{W}_i = \frac{L_e}{2} \frac{V}{|V|} \frac{dN_i}{dx} \quad [14]$$

Finally, the terms of the linear system are written in the following way:

$$K_{ij} = \begin{cases} \rho CV \int_0^L N_i \frac{dN_j}{dx} dx + \rho CV \frac{L_e}{2} \frac{V}{|V|} \alpha \int_0^L \frac{dN_i}{dx} \frac{dN_j}{dx} dx \\ + k \int_0^L \frac{dN_i}{dx} \frac{dN_j}{dx} dx \end{cases} \quad [15]$$

$$f_i = \int_0^L N_i \phi dx + \frac{L_e}{2} \frac{V}{|V|} \alpha \int_0^L \frac{dN_i}{dx} \phi dx$$

If the source term ϕ is null, the equation [15] can be obtained by applying the Bubnov Galerkin method to the equation:

$$\rho CV \frac{dT}{dx} - (k + k') \frac{d^2T}{dx^2} = 0 \text{ with } k' = \alpha \frac{\rho C |V| L_e}{2} \quad [16]$$

This method called balancing diffusion was simultaneously introduced by Hughes and Brookes (Hughes, 1979) and Kelly *et al.* (Kelly, 1980). However, if the source term is different from zero, it is necessary to use a weighting function defined

by equation [13] for the source term, in order to obtain an accurate solution (Brooks, 1980). Another way for upwinding was proposed by Hughes (Hughes, 1978). His solution operates if the integrals defined by equation [15] are calculated with a one-point Gauss method. It is based on the following property:

$$W_i(0) = N_i(0) + \alpha\tilde{W}_i(0) = N_i(\alpha) \qquad [17]$$

Thus, Hughes applied the Bubnov-Galerkin method to equation [2] and used a Gauss point located at the abscissa α instead of 0. It should be noted that there exist various other schemes for transport equations not presented here. Donea (Donea, 1991) used a Galerkin least square method. Faria and San Andrès (Faria, 2000) proposed a Galerkin method with high-order weighting and shape functions. An original method based on the Laplace transform was presented by Bou-Saïd and Colin (Bou-Saïd, 2000). In the present study, only the simplest and most useful upwind schemes have been described.

2.2. *Multidimensional schemes*

On the basis of the previously presented "one dimensional" schemes, it is possible to define two methods to solve the energy equation in two or three dimensions: the product method and the Streamline Upwind Petrov Galerkin (SUPG) method.

2.2.1. *The product method*

This method initiated by (Heinrich *et al.*, 1977) and (Hughes, 1978) is based on the following property: shape functions of a quadrilateral bilinear element or a hexahedral trilinear element are obtained by a product of the shape function of the bar linear element. The same procedure is applied to the weighting functions. Thus, this method is limited to quadrilateral and hexahedral elements.

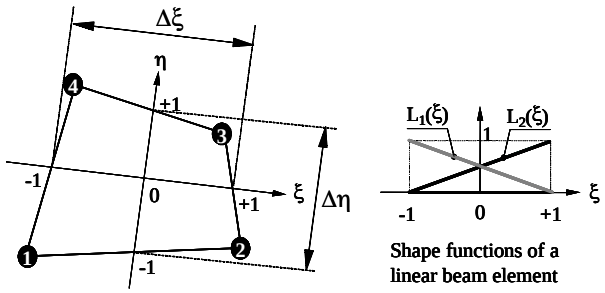

Figure 1. *Description of the product method*

On the basis of Figure 1, it is possible to define the shape function of the node 2 of the quadrilateral element:

$$N_2(\xi,\eta) = L_2(\xi).L_1(\eta) \quad [18]$$

In the same way, a weight function can be defined:

$$W_2(\xi,\eta) = \left[L_2(\xi) + \alpha_\xi \frac{V_\xi}{|V_\xi|} \frac{dL_2(\xi)}{d\xi} \right] \left[L_1(\eta) + \alpha_\eta \frac{V_\eta}{|V_\eta|} \frac{dL_1(\eta)}{d\eta} \right] \quad [19]$$

This method requires that the optimal value of the factor α and the fluid velocity are determined for each direction of the element. These factors α are based on the lengths defined on Figure 1.

2.2.2. *The SUPG method*

In this method the weighting functions are orientated by the fluid flow because the convection is only active in the direction of the resultant fluid velocity. Therefore, the weighting function is:

$$W_i = N_i + \alpha \frac{L_e}{2\|\vec{V}\|} \left(V_x \frac{\partial N_i}{\partial x} + V_y \frac{\partial N_i}{\partial y} \right) \text{ with } \vec{V} = V_x \vec{x} + V_y \vec{y} \quad [20]$$

The factor α and the Peclet number Pe are defined in the following way:

$$\alpha = \coth Pe - \frac{1}{Pe} \text{ and } Pe = \frac{\rho C \|\vec{V}\| L_e}{2k} \quad [21]$$

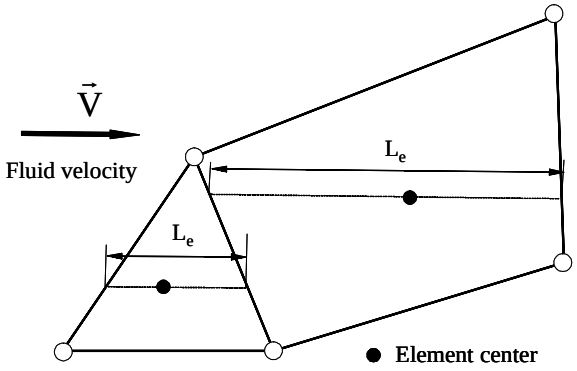

Figure 2. *Definition of the element length*

The element length L_e is calculated in the flow direction as shown in Figure 2. Contrary to the product method, this scheme works for both quadrilaterals and triangles. This method has been proposed by Brooks and Hughes (Brooks, 1980) and (Kelly *et al.*, 1980), using an anisotropic balancing diffusion.

3. Application to fluid film lubrication

Fluid film lubrication concerns fluid flow between two solid surfaces in relative motion, as shown in Figure 3. The fluid film thickness h is within 3 or 4 orders of magnitude less than the contact length L. Accordingly, using a dimensional analysis, it is possible to simplify the energy equation:

$$\rho C\left(V_x \frac{\partial T}{\partial x} + V_y \frac{\partial T}{\partial x} + V_z \frac{\partial T}{\partial x} \right) - \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z} \right) = \phi$$
$$= \mu \left[\left(\frac{\partial V_x}{\partial z} \right)^2 + \left(\frac{\partial V_y}{\partial z} \right)^2 \right] \qquad [22]$$

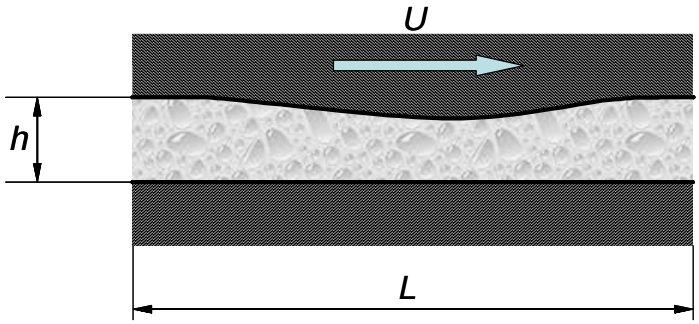

Figure 3. *A lubricated contact*

The z axis is normal to the surface of the solids. As shown by equation [22], the heat conduction in the x and y directions is negligible. Thus, if V_z is null, the problem is that of a pure convective flow in the x and y direction. Bou-Saïd and Colin (Bou-Said, 2000) showed that the temperature is independent of the downstream flow. Consequently, the factor α is always equal to 1 whatever the value of the mesh Peclet number. Moreover, no boundary condition is necessary for the exit area of the flow.

3.1. *Plane problems*

Let us consider various simple problems with analytical solutions to validate the schemes. In this section, the problem is two-dimensional. The film thickness h, the velocity V of the fluid and the source function are supposed to be constant. The energy equation [22] can be written in this way:

$$\rho CV\frac{\partial T}{\partial x}-\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)=\phi \qquad [23]$$

This equation is similar to the one-dimensional transient heat conduction equation where x is analogous to time t. There are several analytical solutions to this problem, depending on the boundary conditions (Myers, 1971).

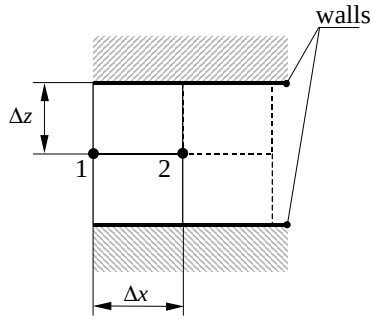

Figure 4. *Description of the problem and the mesh*

The problem and the mesh used are presented in Figure 4. It is composed of two elements in cross-film direction. The Peclet number of the mesh is defined by the following expression:

$$Pe=\frac{\rho CV\Delta z^2}{2k\Delta x} \qquad [24]$$

By using the weighting functions defined in equations [13] and [14], the terms of the linear system can be written in the following way:

$$K_{ij}=\rho CV\int_\Omega W_i\frac{dN_j}{dx}d\Omega+k\int_\Omega\frac{dW_i}{dz}\frac{dN_j}{dz}d\Omega \text{ and } f_i=\int_\Omega W_i\phi d\Omega \qquad [25]$$

If the source term ϕ is null, the numerical value of the temperature at node 2 given by equation [25] is:

$$\frac{T_2}{T_1} = \frac{Pe-1}{Pe+1} \qquad [26]$$

This equation clearly shows that the temperature T_2 can become negative if the Peclet number [24] is lower than 1. This is physically impossible and our solution is to modify the diffusion term to preserve the symetry between i and j. Thus N_j is replaced by W_j and equation [25] becomes:

$$K_{ij} = \rho CV \int_{\Omega} W_i \frac{dN_j}{dx} d\Omega + k \int_{\Omega} \frac{dW_i}{dz} \frac{dW_j}{dz} d\Omega \text{ and } f_i = \int_{\Omega} W_i \phi d\Omega \quad [27]$$

With this new finite element scheme, the nodal equation [27] is:

$$\frac{T_2}{T_1} = \frac{Pe}{Pe+2} \qquad [28]$$

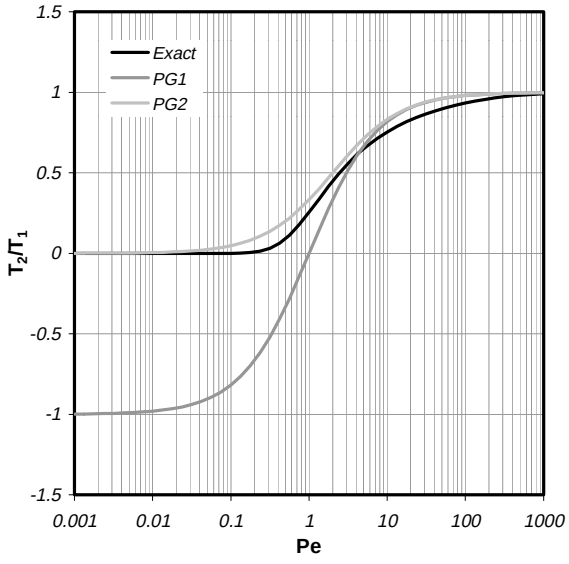

Figure 5. *Comparison of the exact solution with numerical methods (ϕ=0, $T_1 \neq 0$)*

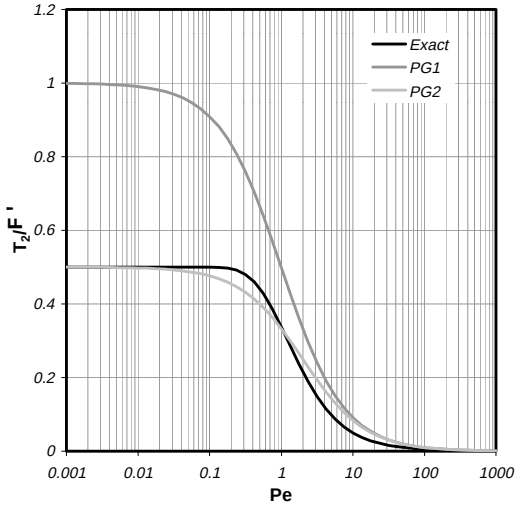

Figure 6. *Comparison of exact solution with numerical methods ($\phi \neq 0$, $T_1 = 0$)*

If the inlet temperature is assumed to vary linearly from the wall to node 1 in the problem described in Figure 4, it is possible to obtain an analytical solution. This is presented in Figure 5 as a function of Pe. The numerical solution given by scheme 1 (equation [25], labelled PG1) and scheme 2 (equation [27], labelled PG2) are also shown in Figure 5. Figure 6 presents the same type of comparison for a second problem with $T_1 = 0$ and $\phi' = \phi \Delta z^2 / k \neq 0$.

These two examples illustrate that a direct application of the Petrov Galerkin method (PG1) is not accurate. However, good results can be obtained using a modified version of this method (PG2). Note that a better correlation between PG2 and the analytical solution is observed with a higher number of elements in the cross-film direction z.

3.2. *Three-dimensional problems*

To compare the SUPG with the product methods, a three-dimensional problem is analyzed. The data from the studied case, along with the boundary conditions used, are presented in Figure 7. It shows the flow within very close two disks, one of which is rotating. A pressure gradient in the radial direction leads to a leakage flow. This fluid flow is similar to that occurring in mechanical face seals. An important parameter in the numerical solution is the ratio of the maximal leakage velocity V_L (velocity in the radial direction) and the maximal rotor velocity (velocity in the circumferential direction) V_ω. In mechanical seals this ratio can vary from 0 to 1. To

be consistent with the previous paragraphs, two mesh Peclet numbers, based on V_L and V_ω, are introduced:

$$Pe_L = \frac{\rho C V_L \Delta z^2}{2k\Delta R} \quad [29]$$

$$Pe_\omega = \frac{\rho C V_\omega \Delta z^2}{2kR\Delta\theta} \quad [30]$$

ΔR and $\Delta\theta$ are respectively the length in the radial direction and the angular extent of an element of the mesh.

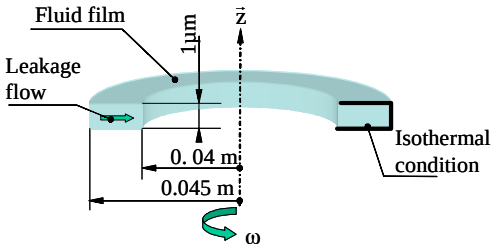

Figure 7. *Description of the three-dimensional problem*

The energy equation [22] was solved using both the SUPG and the product methods, while the new scheme for the conduction term was employed as in expression [27]. The mesh was the same for all cases (4x72x4 elements in the R, θ, z directions). The number of nodes is arbitrary but does not affect the following results. 8-node hexahedral elements are used. In terms of the configuration, the solution is independent of the angular position. Thus, only the temperature in the mid-plane of the film is presented as a function of the radius. Results obtained with both methods are presented in Figure 8 for values of Pe_L/Pe_ω equal to 0.01 and 1. Temperatures are given in relation to a local reference value T_{ref} defined in the following way:

$$T_{ref} = \mu \frac{\omega^2 R^2}{k} \quad [31]$$

In both cases, the SUPG method gives a solution containing oscillations. Moreover, by refining the mesh it is not possible to remove oscillations. This is not acceptable. By contrast, solutions obtained with the product method are very satisfactory.

In fact the SUPG method distributes the upwinding action according to the fluid velocity direction. However, in order to obtain an oscillation-free solution, it is necessary to apply a complete upwinding in the radial direction. This is not possible with the SUPG method and this example clearly demonstrates that the use of the SUPG method is not a good way for solving the energy equation for a lubricated contact. The product method, on the other hand, works very well.

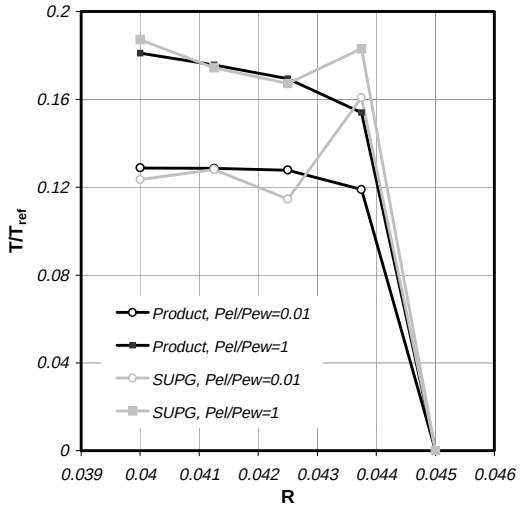

Figure 8. *Comparison of SUPG and product methods on a three-dimensional problem*

3.3. *Problems with reversed flow*

In an extremely loaded bearing, a reversed flow can occur at the entrance of the pad. In such situations, numerical oscillations may appear. To study this phenomenon, the resolution of the energy equation in an infinitely wide pad bearing is analysed. The configuration is presented in Figure 9. The ratio between h_1 and h_2 and the reference pressure are:

$$a = \frac{h_1}{h_2} = 8 \qquad [32]$$

$$P_{ref} = \frac{6\mu UB}{h_2^2(a-1)} = 1.071MPa \qquad [33]$$

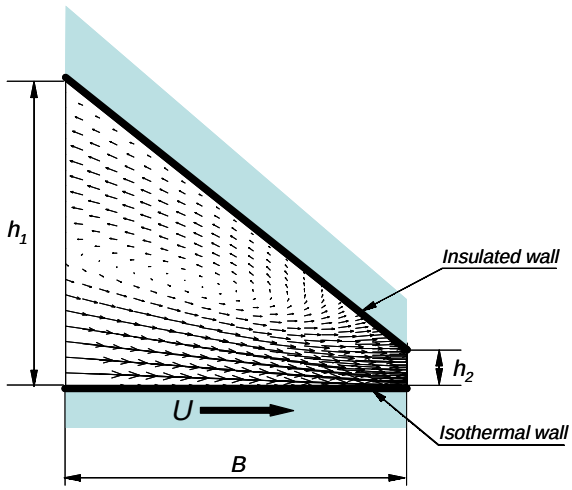

Figure 9. *Description of the three-dimensional problem*

In this case, a reversed flow occurs at the entrance of the pad, as shown in Figure 9. The energy equation [22] has been solved for a 20x20 elements mesh, with the boundary conditions likewise presented in Figure 9. In accordance with the previous studies, an upwinding factor is calculated for each element of the mesh:

$$\alpha \frac{V_x}{|V_x|} = \frac{V_x}{|V_x|} \qquad [34]$$

where V_x is the fluid velocity at the center of the element. The calculated temperature field is displayed in Figure 10. Temperatures are given in relation to a reference value T_{ref}, defined in the following way:

$$T_{ref} = \mu \frac{U^2}{k} \qquad [35]$$

Oscillations are observed at the boundary of the reversed flow zone. In fact, a 2-Gauss point integration method is used here to avoid singularity in the linear system. Thus there are Gauss points where the upwinding factor [34], calculated for the element, and the local velocity are acting in opposite directions, leading to numerical oscillations. The solution in this case is to calculate the upwinding factor for each Gauss point. Figure 11 shows the dimensionless temperature calculated with this method. No oscillations are observed.

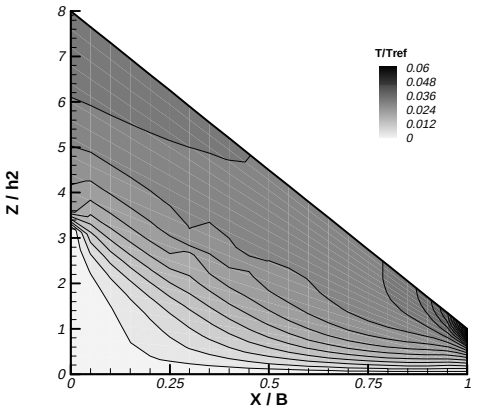

Figure 10. *Temperature field obtained using an upwinding factor computed at the element*

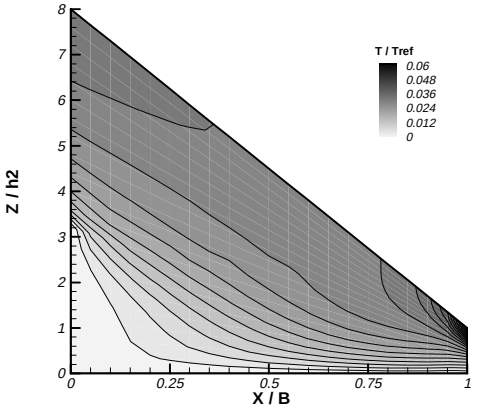

Figure 11. *Temperature field obtained using an upwinding factor computed at each Gauss point*

3.4. *Summary*

In this section, the finite element solution of the energy equation [22] for thin fluid film flows has been presented using upwind schemes. Details of the method are presented in the appendix. The question of upwinding in the cross-film direction has

not been addressed here. In fact, it has no significant effect on the calculated temperature and can be ignored. However, if the Peclet number in the z direction is close to or larger than 1, it could be necessary to use an upwind procedure in this direction. In this case the problem is not a pure convective one. Thus, the upwinding factor should be calculated in terms of the optimal expression, defined by equation [21].

4. Study of a mechanical face seal

Many authors have shown that the performance of non-contacting mechanical face seals is greatly influenced by thermal effects and, in particular, by thermal distortions that can be of the same order of magnitude as the film thickness (Lebeck, 1991). Within this framework a mechanical face seal test rig was built. The experimental device is presented in Figure 12 and in other work of the present authors (Brunetière, 2003). It consists of a carbon rotor and a stator, made of a calcium fluoride (CaF2) disk fixed on an annular floating piston. This ensures a perfect alignment of the surfaces.

Pressurized oil at constant temperature is supplied by hydraulic equipment. Pressurized air, acting on the back of the piston, balances the opening force resulting from the pressurized oil in the seal chamber. An infrared camera is placed below the seal and the rotor face temperature is determined from radiation transmitted through the semi-transparent stator. A mirror reflects radiation from the outer surface of the rotor, thereby providing axial temperature distribution.

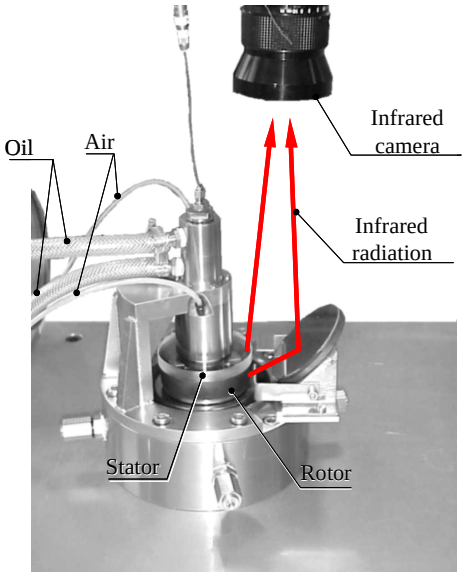

Figure 12. *Experimental mechanical face seal and measurement technique*

Brunetière *et al.* (2003) have developed a numerical model of non-contacting mechanical face seals. The geometric configuration of the fluid film separating the seal faces is presented in figure 7. Governing equations have been established as follows. Fluid velocities are obtained from simplified Navier-Stokes equations

$$\begin{aligned} V_x &= \frac{\partial p}{\partial x}\left(I - \frac{I_1}{J_1} J \right) + V_{X_2}\left(1 - \frac{J}{J_1} \right) \\ V_y &= \frac{\partial p}{\partial y}\left(I - \frac{I_1}{J_1} J \right) + V_{Y_2}\left(I - \frac{I_1}{J_1} J \right) \end{aligned} \qquad [36]$$

where I and J integrate the variations of the terms of viscosity μ through the film thickness:

$$\begin{aligned} I(z) &= \int_{H_2}^{z} \frac{\xi d\xi}{\mu} \text{ and } I_1 = I(H_1) \\ J(z) &= \int_{H_2}^{z} \frac{d\xi}{\mu} \text{ and } J_1 = J(H_1) \end{aligned} \qquad [37]$$

V_{X2} and V_{Y2} are the components of the rotor velocity.

Replacing relations [36] within the mass conservation equation and integrating through the film thickness, one can obtain a generalized form of the Reynolds equation

$$\frac{\partial}{\partial x}\left(G_1 \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y}\left(G_1 \frac{\partial p}{\partial y} \right) = \frac{\partial}{\partial x}\left(V_{X_2} \frac{I_1}{J_1} \right) + \frac{\partial}{\partial y}\left(V_{Y_2} \frac{I_1}{J_1} \right) + V_{Z_1} \quad [38]$$

where :

$$G_1 = \int_{H_2}^{H_1} \left(z - \frac{I_1}{J_1} \right) \frac{z dz}{\mu\left(1 + \delta_E \frac{\varepsilon_M}{\nu} \right)} \qquad [39]$$

The energy equation may be written as equation [22].

The pressure field is determined by solving the Reynolds equation [38] with the Bubnov-Galerkin method:

$$K_{ij} = \int_S \frac{\partial N_i}{\partial x} G_1 \frac{\partial N_j}{\partial x} dS + \int_\Omega \frac{\partial N_i}{\partial y} G_1 \frac{\partial N_j}{\partial y} dS$$

$$f_i = \int_S \frac{\partial N_i}{\partial x} V_{X_2} \frac{I_1}{J_1} dS + \int_S \frac{\partial N_i}{\partial y} V_{Y_2} \frac{I_1}{J_1} dS - \int_S N_i V_{Z_1} dS \qquad [40]$$

$$K_{ij} P_j = f_i$$

S is the annular plane domain limited by the inner and outer radius of the seal. Four-node quadrilateral elements are used. The mesh has 11 nodes in the radial direction and 72 in the circumferential direction. The finite element solution of the energy equation is obtained using the upwinding scheme presented here:

$$K_{ij} = \begin{cases} \int_\Omega W_i \rho C \left(V_x \frac{\partial N_j}{\partial x} + V_y \frac{\partial N_j}{\partial y} + V_z \frac{\partial N_j}{\partial z} \right) d\Omega + \\ \int_\Omega \frac{\partial W_i}{\partial z} k \frac{\partial W_j}{\partial z} d\Omega \end{cases}$$

$$f_i = \int_\Omega W_i \mu \left[\left(\frac{\partial V_x}{\partial z} \right)^2 + \left(\frac{\partial V_y}{\partial z} \right)^2 \right] d\Omega \qquad [41]$$

$$K_{ij} T_j = f_i$$

Ω is the fluid domain presented in figure 9. 8-node hexahedral elements are used. The mesh has 11 nodes in the radial direction, 72 in the circumferential direction and 9 in the z direction. Both linear systems [40] and [41] are solved using multi frontal algorithms developed by Davis and Duff (Davis, 1994 and Davis, 1997). All the data and boundary conditions used in this calculation are presented in (Brunetière, 2003).

The steps in deriving a solution are as follows. From an initial fluid film temperature field, dynamic viscosity is calculated. The Newton-Raphson method is used to determine the geometric parameters of the stator position, assuring that the pressure field balances external forces. After calculating the velocity and temperature of the fluid particles, the temperatures of the faces are determined by means of influence coefficient matrices. Thus heat flux conservation is ensured at the fluid / solid interface. The distortions of the faces are calculated from the heat fluxes at the seal faces and the influence coefficient matrices of displacement. The entire procedure is repeated until the fluid film temperature remains unchanged from one iteration to another. The calculation presented in figure 13 needs about 300 iterations, that is to say 30 minutes of CPU time on a PC.

Figure 13 shows the temperature of the fluid film for a rotation speed of 600 rpm. There are no numerical oscillations. The temperature is at a maximum near the exit of the sealing dam, where the viscous dissipation is maximal. On the right hand side of Figure 13, the temperature of the solids is represented. As expected, the highest values are observed near the contact. Since no significant temperature gradients are observed, no mesh refinement was carried out in this case.

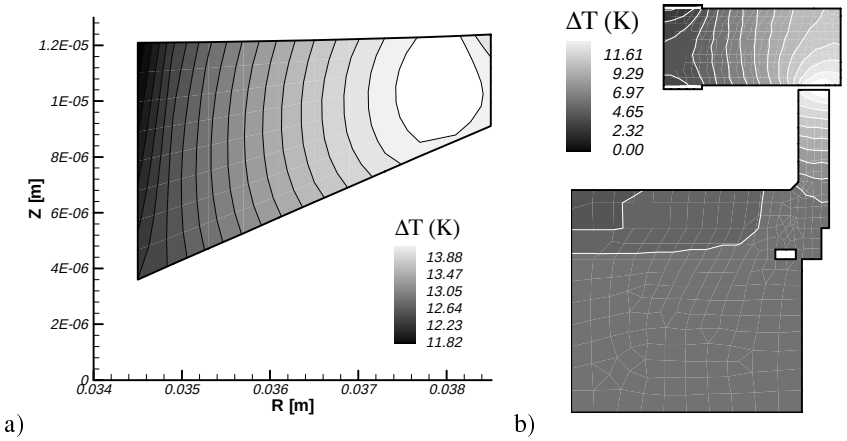

Figure 13. *Temperature of a) the fluid film separating the stator and the rotor and b) of the stator and the rotor of the experimental seal*

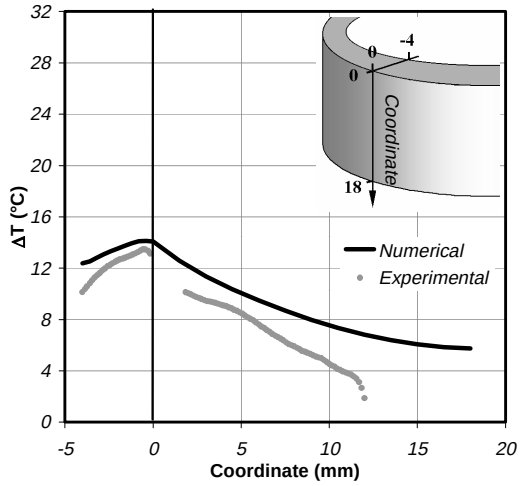

Figure 14. *Comparison between numerical and experimental results*

The experimental and numerical curves presented in Figure 14 are very similar. These are the temperature profiles on the seal rotor. The average deviation between experimental results and computational predictions in the contact area is about 9%. Considering the uncertainty of temperature measurements (about 5%) and the uncertainty surrounding the operating conditions (pressure of the sealed fluid: 2%, load supported by the contact: 10%), this result is satisfying. The finite element scheme for solving the energy equation works very well.

5. Conclusions

The energy equation [22] for thin fluid film flows cannot be solved with a classical finite element scheme (such as the Bubnov-Galerkin method). Indeed, convective terms lead to numerical oscillations, even for low mesh Peclet numbers [24]. A way of resolving this problem is to use upwind schemes and the simplest and most useful of such schemes developed for fluid mechanics have been presented here. The energy equation in lubrication is purely a convective problem for the directions parallel to the surfaces of the solids. Accordingly, initial schemes have been modified to take into account these circumstances.

Finally, an efficient method to solve the energy equation for a fluid film lubricated contact has been obtained. It is based on the Petrov Galerkin method. Weighting functions depend on the fluid flow direction and are obtained by a product method. (Full details are presented in the appendix).

Some of the problems encountered in lubrication, such as reversed flow and the ratio between the leakage flow and the Couette flow, have been investigated and resolved with the scheme presented here. Moreover, a particular application for mechanical face seals has been proposed. The good correlation of the numerical results with experimental data confirms the accuracy of the upwind scheme.

6. References

Bou-Saïd B., Colin F., “Laplace transform to figth the effect of Peclet number in thermal problems”, *International Tribology Conference*, JST, Nagasaki, Paper 3P11-7, 2000.

Brooks A., Hughes T.J.R., “Streamline upwind/Petrov Galerkin methods for advection dominated flows”, *Proc. of the 3rd International Conference on FEM in Fluid Flows*, Blanff, Canada, 1980, Vol. 2, pp. 283-292.

Brunetière N., Tournerie B., Frêne J., “TEHD lubrication of mechanical face seals in stable tracking mode – Part 1 – Numerical model and experiments”, ASME *Journal of Tribology*, Vol. 125, No. 3, 2003, pp. 608-616.

Christie I., Griffiths D.F., Mitchell A.R., Zienkiewicz O.C., “Finite element methods for second order differential equations with significant first derivatives”, *International Journal of Numerical Methods in Engineering*, Vol. 10, 1976, pp. 1389-1396.

Davis T.A., Duff I.S., An unsymmetric-patern multifrontal method for sparse LU factorization, TR94-038, http://www.cise.ufl.edu, 1994.

Davis T.A., Duff I.S., A combined unifrontal/multifrontal method for unsymmetric sparse matrices, TR97-016, http://www.cise.ufl.edu, 1997.

Donea J., "Generalized Galerkin methods for convection dominated transport phenomena", *ASME Applied Mechanical Review*, Vol. 44, No. 5, 1991.

Faria M.T.C, San Andrès L., "On the numerical modelling of high-speed hydrodynamic gas bearings", *ASME Journal of tribology*, Vol. 122, No. 1, 2000, pp. 124-130.

Frêne J., Nicolas D., Degueurce B., Berthe D., Godet M., *Lubrification hydrodynamique – paliers et butées*, Collection de la direction des Etudes et Recherche d'EDF, Eyrolles, 1990.

Griffiths D.F., Mitchell A.R., "On generating upwind finite element methods", *Finite Element Methods for Convection Dominated Flows*, ed. T.J.R. Hughes, AMD 34, ASME, New York, 1979, pp. 91-104.

Heinrich J.C., Huyakorn P.S., Zienkiewicz O.C., Mitchell A.R., "An upwind finite element scheme for two dimensional convective transport equation", *International Journal of Numerical Methods in Engineering*, Vol. 11, pp. 131-143, 1977.

Huebner K.H., "Analysis of fluid film lubrication – A survey", *Finite elements in fluids*, John Wiley and Sons, Vol. 2, 1975, pp. 225-254.

Hughes T.J.R., "A simple scheme for developing upwind finite elements", *International Journal of Numerical Methods in Engineering*, Vol. 12, 1978, pp. 1359-1365.

Hughes T.J.R., Brooks A., "A multidimensional upwind scheme with no crosswind diffusion", *Finite Element Methods for Convection Dominated Flows*, ed. T.J.R. Hughes, AMD, 34, ASME, New York, 1979, pp. 19-35.

Kelly D.W., Nakasawa S., Zienkiewicz O.C., Heinrich J.C.. "A note on upwinding and anisotropic balancing dissipation in finite element approximations to convective diffusion problems", *International Journal of Numerical Methods in Engineering*, Vol. 15, pp. 1705-1711, 1980.

Kim J., Palazzolo A.B., Gadangi R.K., "TEHD analysis for tilting-pad journal bearings using upwind finite element method", *Tribology Transactions*, Vol. 37, No. 4, 1994, pp. 771-783.

Kucinschi R.B., Fillon M., Frêne J., Pascovici M.D., "A transient thermoelastohydrodynamic study of steadily loaded plain journal bearings using Finite Element Method analysis", *ASME Journal of Tribology*, Vol. 122, No. 1, 2000, pp. 219-226.

Lebeck A.O., *Principles and design of mechanical face seals*, Wiley-Interscience Publication, 1991.

Myers G.E., *Analytical methods in conduction heat transfer*, Mc Graw & Hill, 1971.

Zienkiewicz O.C., Taylor R.L., *The finite element method*, Vol. 3, *Fluid dynamics*, Fifth edition, Butterworth Heineman, 2000.

7. Appendix

The energy equation is defined by equation [22]. The weighting functions will be defined for an 8-node trilinear hexahedral element as showed on Figure 15. It is assumed that the cross film direction z and the ζ direction of the element are perfectly aligned.

In this method, it is necessary to determine the components of the fluid velocity (V_ξ, V_η, V_ζ) at each Gauss points in the element natural coordinates (ξ, η, ζ). In the case of upwinding in the cross-film direction, the thickness $\Delta\zeta$ of the element in the ζ direction should be determined at each Gauss Point. Then an optimal α factor can be calculated for the ζ direction by using equation [12]

$$\alpha_\zeta = \coth \mathrm{Pe}_\zeta - \frac{1}{\mathrm{Pe}_\zeta} \text{ with } Pe_\xi = \frac{\rho C |V_\zeta| \Delta\zeta}{2k} \qquad [42]$$

As shown previously, the α factor in the ξ and η directions is equal to 1, owing to the pure convective transport conditions.

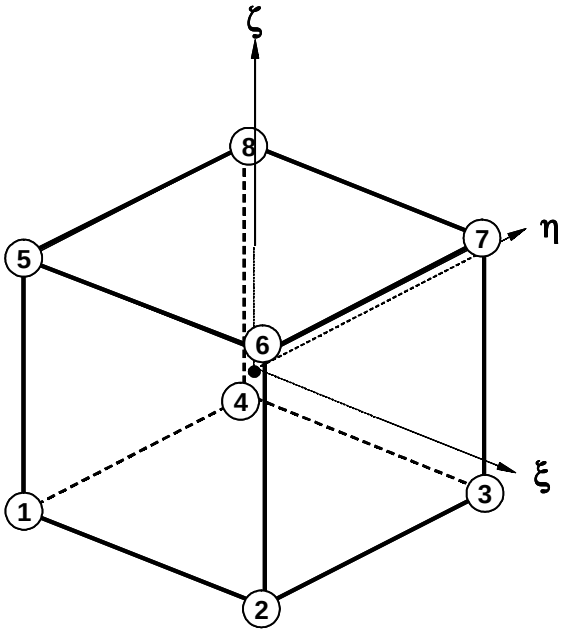

Figure 15. *An 8-node trilinear hexahedral element*

The 8 weighting functions are then defined in the following way:

$$
\begin{aligned}
W_1 &= \left(1-\xi-\frac{V_\xi}{|V_\xi|}\right)\left(1-\eta-\frac{V_\eta}{|V_\eta|}\right)\left(1-\zeta-\alpha_\zeta\frac{V_\zeta}{|V_\zeta|}\right)\\
W_2 &= \left(1+\xi+\frac{V_\xi}{|V_\xi|}\right)\left(1-\eta-\frac{V_\eta}{|V_\eta|}\right)\left(1-\zeta-\alpha_\zeta\frac{V_\zeta}{|V_\zeta|}\right)\\
W_3 &= \left(1+\xi+\frac{V_\xi}{|V_\xi|}\right)\left(1+\eta+\frac{V_\eta}{|V_\eta|}\right)\left(1-\zeta-\alpha_\zeta\frac{V_\zeta}{|V_\zeta|}\right)\\
W_4 &= \left(1-\xi-\frac{V_\xi}{|V_\xi|}\right)\left(1+\eta+\frac{V_\eta}{|V_\eta|}\right)\left(1-\zeta-\alpha_\zeta\frac{V_\zeta}{|V_\zeta|}\right)\\
W_5 &= \left(1-\xi-\frac{V_\xi}{|V_\xi|}\right)\left(1-\eta-\frac{V_\eta}{|V_\eta|}\right)\left(1+\zeta+\alpha_\zeta\frac{V_\zeta}{|V_\zeta|}\right)\\
W_6 &= \left(1+\xi+\frac{V_\xi}{|V_\xi|}\right)\left(1-\eta-\frac{V_\eta}{|V_\eta|}\right)\left(1+\zeta+\alpha_\zeta\frac{V_\zeta}{|V_\zeta|}\right)\\
W_7 &= \left(1+\xi+\frac{V_\xi}{|V_\xi|}\right)\left(1+\eta+\frac{V_\eta}{|V_\eta|}\right)\left(1+\zeta+\alpha_\zeta\frac{V_\zeta}{|V_\zeta|}\right)\\
W_8 &= \left(1-\xi-\frac{V_\xi}{|V_\xi|}\right)\left(1+\eta+\frac{V_\eta}{|V_\eta|}\right)\left(1+\zeta+\alpha_\zeta\frac{V_\zeta}{|V_\zeta|}\right)
\end{aligned}
\qquad [43]
$$

Finally, the terms of the linear system are written:

$$
K_{ij} = \begin{cases} \rho C \int_\Omega W_i\left(V_x\frac{dN_j}{dx}+V_y\frac{dN_j}{dy}+V_z\frac{dN_j}{dz}\right)d\Omega \\ +\int_\Omega \frac{dW_i}{dz}k\frac{dW_j}{dz}d\Omega \end{cases} \quad \text{and} \quad f_i = \int_0^L W_i\phi dx \qquad [44]
$$