Mixed mode fracture of a higher order beam model in gradient-dependent plasticity

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ABSTRACT. *A higher-order shear-deformable beam model with gradient-dependent plasticity regularisation is presented in this work. The model takes into account a realistic non-linear variation of stresses and strains through the beam thickness. This enables mixed mode fracture analysis. Numerical results are presented for a multilayered beam. The discrete problem size is significantly reduced through this semi-local approach.*

RÉSUMÉ. *Un modèle de poutre avec gauchissement des sections, dans le cadre de la plasticité au gradient, est présenté. Il prend en compte une variation réaliste des contraintes et déformations de cisaillement sur la section. La confrontation avec les expériences montre l'aptitude de ce modèle multicouche à décrire le mode mixte de ruine, tout en réduisant considérablement la taille du problème discret.*

KEYWORDS: *Higher-order shear theory, gradient-dependent plasticity, beam model.*

MOTS-CLÉS : *gauchissement, plasticité gradient, modèle de poutre.*

1. Introduction

Strain localisation of quasi-brittle materials can be considered as an instability in the macroscopic constitutive description of inelastic deformation. In classical continuum mechanics, the appearance of localisation is associated with a change of type of the governing equations, which lose ellipticity in statics. This is an indicator of the initiation of a material surface discontinuity. Since the localisation zone can be infinitely thin, a displacement discontinuity may develop, leading to numerical difficulties when conventional finite element methods are used. The calculated energy dissipation tends to zero upon mesh refinement. As a consequence, in softening regime, the global response becomes infinitely brittle; this is known as mesh dependency.

Higher-order continuum theories can be used to remedy the mesh dependency. Aifantis (Aifantis, 1992) and Vardoulakis (Vardoulakis, 1991) demonstrated the regularising role of higher order strain gradients in localisation phenomena. The governing equations possess a mathematical structure, which is amenable to non-linear stability analysis.

A 2D shear-deformable beam model with cross section warping incorporating gradient-dependent plasticity regularisation is presented in this work. An appropriate form of the warping function was chosen in order to provide more accurate solutions, thus eliminates the use of shear correction coefficients, as it is the case in Timoshenko's and Reissner's theory for beams. The chosen kinematics enables a better description of localised failure under shear, tension and mixed-mode conditions for thick to thin beams.

2. Finite element formulation

Gradient dependent plasticity models bear significant advantages of mesh independency and preservation of ellipticity. However, the increment of the plastic strain can not be obtained at a local level, since the consistency condition, which governs the plastic flow, becomes a second order partial differential equation thereof. Using a finite difference method as proposed by Belytschko and Lasry (Belytschko, 1989), the algorithm is then a sequence of separate approximate solutions of the equilibrium problem. de Borst and Mühlhaus (de Borst *et al.*, 1992) developed the following approach, which uses only finite elements. It solves the problems of the functional dependence of the yield function on the plastic strain and its Laplacian.

We consider the following set of field equations:

$$\mathbf{L}^T \dot{\boldsymbol{\sigma}} = 0 \qquad [1.a]$$

$$\dot{\boldsymbol{\varepsilon}} = \mathbf{L}\dot{\mathbf{u}} \qquad [1.b]$$

$$f\left(\boldsymbol{\sigma}, \kappa, \nabla^2 \kappa\right) = 0 \qquad [1.c]$$

$$\dot{\boldsymbol{\sigma}} = \mathbf{D}\left(\dot{\boldsymbol{\varepsilon}} - \dot{\lambda} \frac{\partial f}{\partial \boldsymbol{\sigma}} \right) \quad [1.d]$$

which defines the elasto-plastic rate boundary problem during associated plastic flow. In the equations mentioned above, $\mathbf{L}$ is a differential operator matrix, $\dot{\boldsymbol{\sigma}}$ and $\dot{\boldsymbol{\varepsilon}}$ are the stress and strain rate tensors (in vector form), respectively, $\dot{\mathbf{u}}$ is the displacement rate vector, $\mathbf{D}$ is the elastic stiffness matrix, λ is the plastic multiplier being a measure of the plastic flow intensity, f is the gradient-dependent yield function and κ is the hardening parameter which corresponds to the cumulated plastic strain.

The cumulated plastic strain Laplacian, appearing in yield function, introduces an internal length scale in the continuum description. The localisation zones have therefore finite widths, controlled by this internal length, and mesh dependency is avoided (de Borst *et al*., 1992).

An incremental-iterative algorithm presented in (de Borst *et al*., 1992) has been derived for gradient plasticity. Unlike classical plasticity, this algorithm requires a weak satisfaction of the equilibrium equation [1.a] and the yield condition [1.c] at the end of iteration j+1 of current loading step, leading to a variational formulation of the gradient dependent plasticity problem. For the detailed derivation of the weak formulation, the reader is referred to (de Borst *et al*., 1992), (Meftah, 1998).

The weak satisfaction of the yield condition besides the equilibrium equation requires the discretisation of the plastic multiplier field together with the displacement one (de Borst *et al*., 1992):

$$\mathbf{u} = \mathbf{Na}\ ,\ \lambda = \mathbf{H}^{\mathrm{T}}\boldsymbol{\Lambda} \text{ and } \nabla^2\lambda = \mathbf{P}^{\mathrm{T}}\boldsymbol{\Lambda} \quad [2]$$

where $\mathbf{a}$ are the nodal displacements and $\boldsymbol{\Lambda}$ the nodal degrees of freedom related to the plastic multiplier λ. Further, $\mathbf{N}$ and $\mathbf{H}$ are the respective shape functions and $\mathbf{P}$ the matrix containing second derivatives of $\mathbf{H}$ elements.

By introducing the discretisation form of the two unknown fields into the weak form of equations [1.a] and [1.c], we obtain the set of algebraic equations that will allow for solving numerically the initial problem [1]:

$$\begin{bmatrix} \mathbf{K}_{aa} & \mathbf{K}_{\lambda a}^{T} \\ \mathbf{K}_{\lambda a} & \mathbf{K}_{\lambda\lambda} \end{bmatrix} \begin{bmatrix} \mathrm{d}\mathbf{a} \\ \mathrm{d}\Lambda \end{bmatrix} = \begin{bmatrix} \mathbf{f}_e - \mathbf{f}_i \\ \mathbf{f}_\lambda \end{bmatrix} \quad [3]$$

with the elastic stiffness matrix K_{aa}, the off diagonal matrix $K_{a\lambda}$ introducing coupling between kinematics and the plastic flow, the gradient dependent matrix $K_{\lambda\lambda}$, the external force vector f_e, the internal force vector f_i, and the vector f_λ of the residual forces resulting from the inexact fulfilment of the yield condition.

3. The multilayered finite element model

Meftah (Meftah, 1997), (Meftah, 1998) has originally developed a multi-layered finite element model in gradient plasticity. It is a model allowing for finite element analyses of beam failure with reduced degrees of freedom. In this model, the displacement field reduces to $\mathbf{a}=(u,v,\beta)^T$, where u and v are respectively the axial and the transverse displacements of the beam middle plane, β is the rotation thereof. The plastic multiplier field is, however, interpolated through the beam depth which is divided into superposed layers, giving its variation by the mean of nodal parameter $\Lambda^k = \left(\Lambda_i^k, \Lambda_j^k\right)$ at each layer k. C^1 continuous interpolation polynomials are considered for the plastic multiplier field. Therefore, the equation of the equilibrium process for n layers becomes:

$$\begin{bmatrix} [K_{aa}] & [K_{\lambda a}^1]^T & \cdots & [K_{\lambda a}^k]^T & \cdots & [K_{\lambda a}^n]^T \\ [K_{\lambda a}^1] & [K_{\lambda\lambda}^1] & \cdots & [0] & \cdots & [0] \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ [K_{\lambda a}^k] & [0] & \cdots & [K_{\lambda\lambda}^k] & \cdots & [0] \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ [K_{\lambda a}^n] & [0] & \cdots & [0] & \cdots & [K_{\lambda\lambda}^n] \end{bmatrix} \begin{bmatrix} d\mathbf{a} \\ d\Lambda^1 \\ \vdots \\ d\Lambda^k \\ \vdots \\ d\Lambda^n \end{bmatrix} = \begin{bmatrix} f_e - f_i \\ f_\lambda^1 \\ \vdots \\ f_\lambda^k \\ \vdots \\ f_\lambda^n \end{bmatrix} \qquad [4]$$

The multilayered beam model was previously approximated by the Euler-Bernoulli theory of bending which leads to serious discrepancies in the case of beams with small aspect ratio (Meftah, 1997). Thus, this theory is not appropriate for shear failure and mixed-mode fracture.

4. Higher-order shear theory

In order to simulate mixed mode fracture, shear stress has to be taken into account. In the Timoshenko's theory for beams, the transverse shear strain remains constant through the thickness. The shear-free boundary is not satisfied. To avoid discrepancies in the shear constitutive equations, a shear correction coefficient is then introduced.

The higher-order deformation theory incorporates a realistic non-linear variation of the longitudinal displacements through the beam thickness thus, eliminates the use of shear correction coefficients. This theory allows the cross-section to rotate and to warp into a non-planar surface. The following kinematical assumption is made:

$$\begin{cases} u(x,y) = u(x) + y\beta(x) - \dfrac{4}{3h^2}\left(\beta(x) + \dfrac{dv}{dx}\right)y^3 \\ v(x,y) = v(x) \end{cases} \qquad [5]$$

where u and v are respectively the axial and the transverse displacements of the middle plane (x-axis), β is the rotation thereof and h is the beam depth (Levinson, 1981), (Kant *et al*., 1989).

This kinematical assumption was introduced in the displacement field of the multilayered beam previously developed with the Bernoulli theory (Salomon, 2000). This higher-order theory satisfies the shear free boundary conditions on the lateral surface of the beam. The gradient regularisation concerns only the axial normal stress. Shear and normal stresses are coupled in a Drucker-Prager yield criterion, since the model does not take into account the transverse normal stress.

5. Numerical examples

For mode I fracture, a comparison of the present beam model with the experimental data is in good agreement (Salomon, 2000). The ability of the proposed model to describe mixed mode fracture is now assessed.

5.1. *Single edge notched beam*

The four-point-shear tests (single and double edge notched beams) were initially proposed to study the yield strength of welded joints. The geometry was adapted to concrete materials in order to improve the behaviour of the shearing zone (Bažant *et al.*, 1986). Those beams do not follow Saint-Venant's hypothesis, since the displacement control point is next to the loading platen.

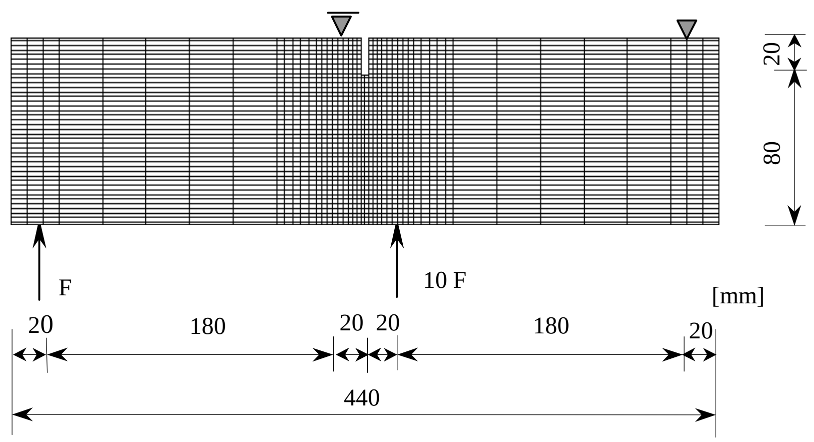

Figure 1. *Geometry of the single edge notched beam*

The geometry and the material data for the notched concrete beam are based on the experimental values of Schlangen (Schlangen, 1993): Young's modulus E = 35000 N/mm^2, tensile strength f_t = 3.0 N/mm^2, compressive strength f_c = 46.6 N/mm^2, fracture energy G_f = 0.10 N/mm and the internal length l = 3 mm. Arc length method is used.

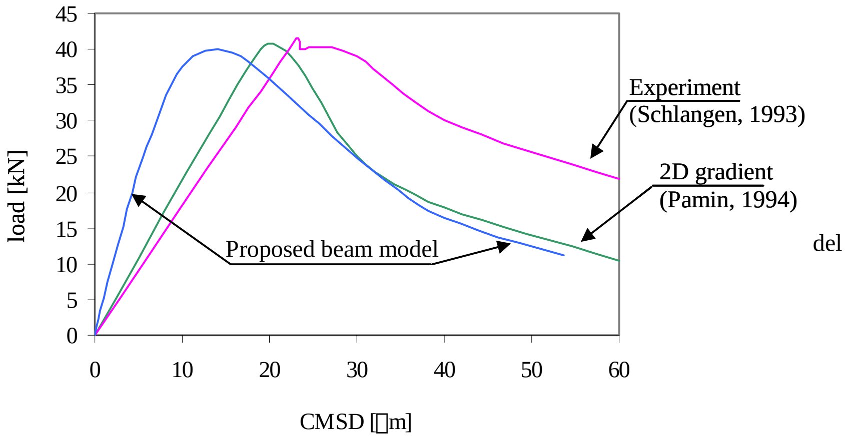

Figure 2. *Load versus Crack Mouth Sliding Displacement* (*CMSD*) *of the single edge notched beam*

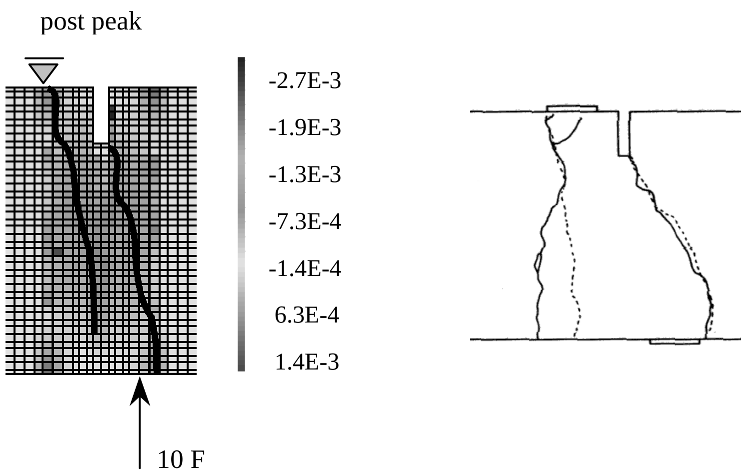

Figure 3. *Crack pattern of the shear test a) beam model, b) experiment* (Schlangen, 1993)

In the neutral axis of the beam, high shear stresses rise in the notched area, while the bending moment vanishes. According to the beam kinematics, this means no axial stress in the notched area (Salomon, 2000). This should give a better insight in the global response of the beam model (cf. figure 2).

Figure 2 gives the response of the notched beam in term of load versus the crack mouth sliding displacement (CMSD). The CMSD is defined as the relative vertical displacement of the notch faces at the top of the beam. The present model underestimates the peak load by 2.5%, the sliding displacement by 32%. At early load stage, the model could not simulate the normal stress concentration at the notched. As a consequence the response in the pre-peak regime is too stiff. However, the failure mode and the peak load have been properly simulated. In this test, mixed mode fracture occurs. Indeed, a crack is initiated at the right corner of the notch and opposite to the central loading platen. Then it continues to grow, following a curved path which ends to the right of the lower loading platen (cf. figure 3b). A closed form of the crack pattern, given by the iso-values of the cumulated plastic strain (cf. black curve in figure 3a), is obtained by the simulation. Note that The cracking of the notched zone is due to the shear stress that compensates for the zero normal stresses.

5.2. Double edge notched beam

This beam is put to the same loading condition as the single edge notched one. In this test, however, the shearing zone is smaller, while the shear stress is greater than the transverse normal stress (Bažant *et al.*, 1986). Therefore, the proposed beam theory based model should be able to predict properly the observed behaviour. The material data are the same as those of the previous test. The geometry is given in figure 4.

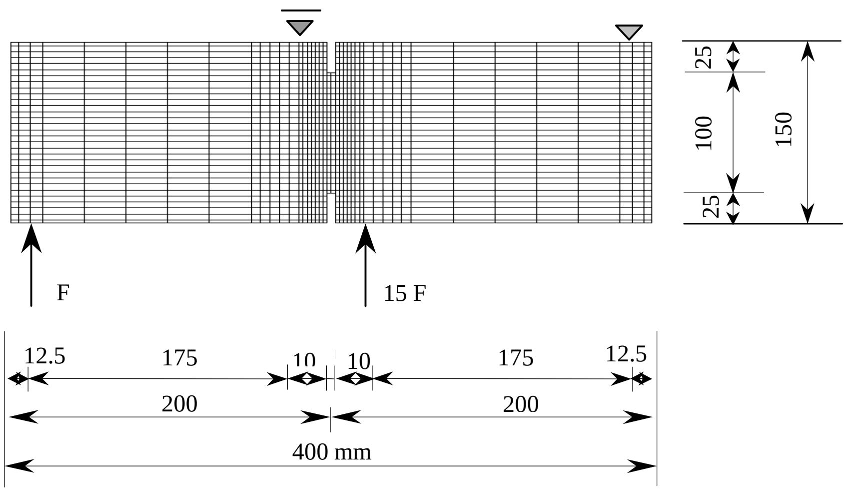

Figure 4. *Geometry of the double edge notched beam*

Figure 5 gives the response of the notched beam in term of load versus the average value of both the crack mouth opening and sliding displacements (CMOS and CMSD, respectively), as reported experimentally by Schlangen (Schlangen, 1993). The CMOD is defined as the relative horizontal displacement of the notch faces at the top of the beam.

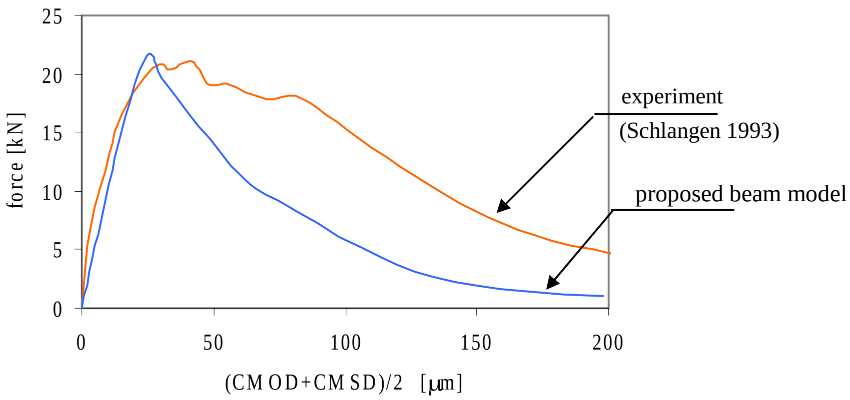

Figure 5. *Load versus Displacement of the double edge notched beam*

In the pre-peak regime, a good agreement between the model and the experiment is obtained, since in the shearing zone, shear stresses are greater than the transverse normal stresses (Bažant *et al.*, 1986). Therefore the beam model does not overestimate the stiffness as is the case in the single edge notched beam test. The post-peak regime predicted by the proposed model shows, however, a more brittle behaviour. Since the shear failure prevails in this case, the brittleness of the numerical response is mainly due to the fact that shear fracture energy introduced in the model underestimates the real dissipated energy.

6. Conclusion

In this paper, a beam element based on a higher-order shear deformation theory with gradient plasticity regularisation is developed and studied. For thin beams, the shear stress can often be neglected by using the Bernoulli beam theory. When high shear gradients are present in the beam, excluding warping due to transverse shear may not be justified. The chosen warping function provides accurate solutions and thus, eliminates the use of shear correction coefficients.

The higher order beam element accurately predicts the peak load for the shear test analysis. Mixed mode fracture was properly simulated, and the response of the model is almost similar to those given by a full bidimensional analysis. The numerical results indicate that the higher order beam theory coupled with gradient plasticity suffices for the examination of many beam problems and provides a beam element, which accounts for shear deformation effects. This beam model consists in a quasi two-dimensional method, allowing finite element analyses with reduced degrees of freedom. This means memory workload and calculation cost decrease.

7. References

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