Nonlinear analysis of reinforced concrete structures using a new constitutive model

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ABSTRACT. *An analytical model, which can simulate the biaxial description of the nonlinear behavior of reinforced concrete structures, is introduced. The behavior of concrete is assumed orthotropic inside the ultimate failure surface and a compressive softening law of concrete is presented. The behavior of cracked concrete is simulated using the smeared crack model, which the tension stiffening effect based on a cracking criterion derived from the fracture mechanics principles is considered. A computer program is developed for analyzing the over and under-reinforced concrete beams. Several parameters such as the non-linearity proprieties, the cut off and tension stiffening models and shear retention factor are studied. The correlation between analytical and experimental results shows the validity of the proposed models and the significance of various effects. The global responses are evaluated to verify simultaneously the reliability of the proposed model and the performance of the numerical program.*

RÉSUMÉ. *Un modèle analytique qui peut simuler la description biaxiale du comportement non linéaire des structures en béton armé est introduit. Le comportement du béton est supposé orthotropique à l'intérieur de la surface ultime de rupture et une loi d'adoucissement en compression du béton est présentée. Le comportement du béton fissuré est simulé en utilisant le modèle de fissuration diffus dont l'effet de "tension stiffening" basé sur un critère de rupture dérivé à partir des principes de mécanique de la rupture est considéré. Un programme de calcul a été développé à l'analyse des poutres sur et sous-armées. Plusieurs paramètres tels que les propriétés non linéaires, les modèles "cut-offv" et "tension stiffening" et le facteur de transfert de cisaillement ont été étudiés. Une corrélation entre les résultats analytiques et expérimentaux, montre la validité des modèles proposés et l'importance de divers effets. Les réponses globales sont évaluées pour vérifier simultanément la fiabilité du modèle proposé et la performance du programme numérique.*

KEYWORDS: *compressive softening behavior of concrete, non-linear analysis, reinforced concrete structures, modeling, cracked concrete, tension stiffening, cut off, beams.*

MOTS-CLÉS : *comportement adoucissement du béton, analyse non linéaire, structures en béton armé, modélisation, béton fissuré, tension stiffening, cut off, Poutres.*

1. Introduction

Reinforced concrete is one of the most commonly used applied materials into all kinds of structures. For many years ago, the structural analysis of reinforced concrete members was based on empirical approaches and elasticity equations. This analysis doesn’t reflect the accurate response of structures which oblige to go back to the nonlinear analysis that are often required especially for complex structures or under extreme load conditions.

To solve the differential equations governing the behavior of structures, it is necessary to resort to the numerical methods. Since Ngo and Scordelis (Ngo *et al.*, 1967) applied the first nonlinear finite element method to analyze the reinforced concrete beams. This numerical method became a significant tool for complex reinforced concrete structures analyses (Zienkiewicz *et al.*, 1991). This technique has played an important role in the composite structures field to allow the knowledge of the formation and propagation of cracks, their mechanism and process of failure.

Concrete is complex material with cracking phenomenon, tension-stiffening effect, shearing transfer factor, plasticity and non-linearity proprieties. These last years, a flood of modeling of plain, reinforced concrete and their interface has been presented. In this context, we can quote, principally, the elasto-plastic models (Rots, 1988, Bicanic *et al.*, 1994) and damage models (Mazars, 1984, Mazars *et al.*, 1992, La Borderie, 2003) knew a broad use.

Modeling of concrete cracking is further complicate by the interaction effect between cracked concrete and steel bars which tension stiffening effect is incorporated in the stress-strain relationship of concrete (Kupta *et al.*, 1990, Kwak *et al.*, 2001-2002). It is recognized that overall nonlinear response of reinforced concrete material is significantly affected by this effect.

In the other hand, cracking of concrete gives birth to the shear stress transfer. Shear can be transmitted across the cracked concrete faces. In this study, its effect is treated by reducing of the shear stiffness. Due to this simplicity, this approach is more useful for computational purpose and various values are granted to the reduced factor (Gilbert *et al.*, 1978, Lin *et al.*, 1975, Leibengood *et al.*, 1986). A deepened analysis allows recommending an average value of shear retention factor to reinforced concrete beams analysis.

This study presents a new analytical model of concrete, which describes the compressive softening behavior of concrete. The stress-strain relationships of both ascending and descending curves are presented. The ascending part is assumed to be elasto-plastic while the descending part is represented by an exponential function, which describes the stiffness degradation with a negative hardening. In this region, the effect of finite element size is discussed in connection with the smeared crack model. In this concern, a numerical procedure is established that taken into account the above parameter influencing structural responses and the stress-strain relationship is implemented into the finite element program.

The developed finite element model is validated by comparison the obtained results and experimental data (Pera, 1973). The correlation through the examples studied shows the reliability of the proposed model and the performance of the developed program (Khalfallah,.2003).

2. Material models

This section details the concrete and reinforcing bars models adopted in the present work.

2.1. *Reinforcing* bars

The reinforcing bar is modeled as a linear elastic, linear strain hardening material with yield stress f_y as shown in figure 1.The smeared stress-strain of mild steel bars embedded in concrete is expressed by two straight lines as:

$$f_s = E_s \varepsilon_s \qquad \varepsilon_s \leq \varepsilon_y \qquad [1]$$

where E_s is the modulus of elasticity. In the plastic range when the effective stress exceeds the yield stress f_y of reinforcing steel, the stress-strain law is expressed as:

$$f_s = (E_s - E_p)\,\varepsilon_y + E_p\,\varepsilon_s \qquad \varepsilon_y \leq \varepsilon_s \leq \varepsilon_u \qquad [2]$$

where ε_y and ε_u are the yield strain and the ultimate strain respectively and E_p is modulus of plasticity.

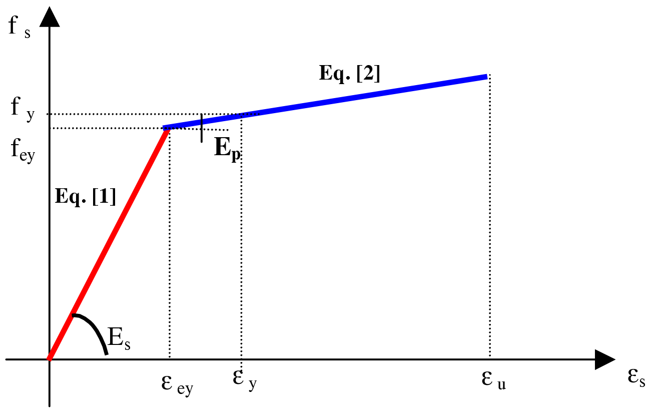


Figure 1. *Constitutive law of steel bars*

According to the experiment results established by (Belarbi *et al*., 1994 and Zhu *et al*., 2001), the smeared yielding stress f_{ey} of the steel bar embedded in concrete is lower then that of the bare steel bar f_y. Our present model was based on this idea by assuming first that the smeared yield stress of the steel bars (figure 1) is the same to the yield stress of the bare steel bars.

When a steel bar embedded in concrete starts to yield at level of the cracks, the stresses in the steel bars between two successive cracks will be less than the yield stress at the cracks, The use of the smeared steel stresses (équations, 1-2) in combination with smeared concrete stresses in tension (équation, 13) allows the tension stiffening of steel bar by concrete to be considered.

2.2. *Concrete*

2.2.1. *Concrete in compression*

To simulate the stress state of concrete under biaxial loading, the elasto-plastic theory is adopted. The behavior of the material depends on the actual stress state situation in the local axes. In the biaxial compression region, the model remains linear elastic before reaching the initial yield surface (Kupta *et al*., 1969). The incremental stress-strain relationship is expressed as:

$$\Delta\sigma_{ij} = C_{ijkl}\, \Delta\varepsilon_{kl} \qquad [3]$$

where C_{ijkl} is the elastic matrix of concrete, $\Delta\sigma_{ij}$ and $\Delta\varepsilon_{kl}$ are the increments of stress and strain respectively.

The ultimate failure surface is described by an affinity of the initial yield surface according to von-Mises criterion. Inside the ultimate failure envelope, the behavior of concrete is described by a non-linearity elasto-plastic theory and the corresponding stress-strain relationship is became as:

$$\Delta\sigma_{ij} = C_{ijkl}^{ep}\, \Delta\varepsilon_{kl} \qquad [4]$$

In the positive hardening region, the behavior of concrete is governed with the elasto-plastic matrix expressed by:

$$C_{ijkl}^{ep} = C_{ijkl} - \frac{C_{ijkl}\, \frac{\partial F}{\partial \sigma} (\frac{\partial F}{\partial \sigma})^T C_{ijkl}}{-A + (\frac{\partial F}{\partial \sigma})^T C_{ijkl}\, \frac{\partial F}{\partial \sigma}} \qquad [5]$$

where A is the flow rule parameter and F is the current plasticity function or (flow function), defined as:

$$F_p\ (\sigma_{ij}, \varepsilon^p_{kl}, k) = 0 \qquad [6]$$

with ε^p_{kl} is the plastic strain; null inside the initial loading surface and k is the parameter of work hardening.

In describing the uniaxial stress-strain behavior of concrete, the model of Hognestad (Hognestad, 1951) is adopted, largely used in recent researches (Kwak *et al.*, 2001) or for unsoftened curve (Pang *et al.*, 1995; Wang *et al.*, 2001; Zhu *et al.*, 2001), as shown in figure 2, can be expressed as:

$$f^1_c = f'_c\ \frac{\varepsilon_c}{\varepsilon_p}\left(2-\frac{\varepsilon_c}{\varepsilon_p}\right) \qquad \frac{\varepsilon_c}{\varepsilon_p} \le 1. \qquad [7]$$

where f'_c is the cylinder compressive strength of concrete, ε_p is the concrete strain at maximum compressive stress. The parabolic ascending branch as shown in figure 2 represents the behavior of concrete, which is assumed elasto-plastic with isotropic strain hardening.

When the biaxial stresses exceed the von-Mises failure surface, the behavior of concrete enters in the compressive softening range. In this region, failure occurs by crushing of concrete when the principal compressive strain exceeds a limit value ε_r. The descending branch, representing the compressive strain softening region, obeys to an exponential law given as:

$$f^2_c = \psi + \alpha\ e^{\vartheta \frac{\varepsilon_c}{\varepsilon_p}} \qquad [8]$$

where ψ, α and θ are constants; which can be determined by applying of the continuity and compatibility conditions and they are expressed as:

$$\psi = -f'_c, \qquad \alpha = 2\ f'_c\ e^{-\frac{1}{6}\ln\frac{6}{10}} \quad \text{and} \quad \theta = \frac{1}{6}\ln\frac{6}{10}$$

Substituting these expressions into Equation (8) lead to:

$$f^2_c = f'_c\ (-1 + 2.17773\ e^{-0.08513\frac{\varepsilon_c}{\varepsilon_p}}) \qquad 1. \le \frac{\varepsilon_c}{\varepsilon_p} \le \frac{\varepsilon_u}{\varepsilon_p} \qquad [9]$$

Still in descending portion of the concrete stress-strain curve, the lowest stress value was taken constant in region between the ultimate strain ε_u and the crushing strain ε_r of concrete (figure 2), this is expressed as:

$$f^3_c = 0.306\ f'_c \qquad \frac{\varepsilon_u}{\varepsilon_p} \le \frac{\varepsilon_c}{\varepsilon_p} \le \frac{\varepsilon_r}{\varepsilon_p} \qquad [10]$$

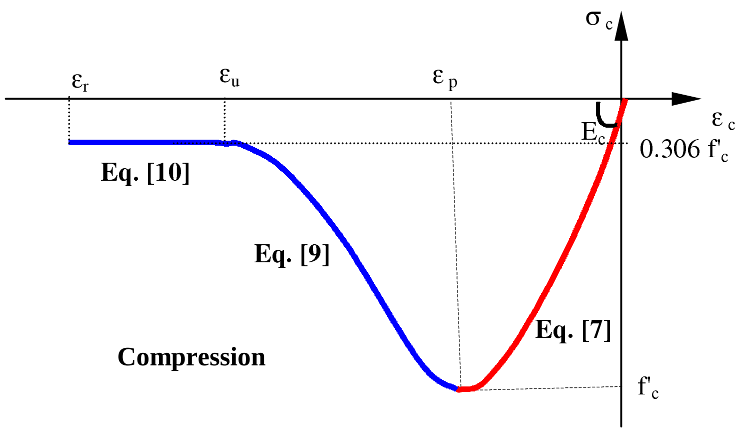


Figure 2. *Uniaxial constitutive law of concrete in compression*

The constitutive relationships described in equations (7-12) must be controlled by a failure criterion for concrete. Various proposals have been made to describe the failure strength characteristics of concrete (Kupfer *et al.*, 1969; Liu *et al.*, 1972; Walter, 1999), which provide the 2-D strength of concrete.

In this study, a von-Mises criterion in biaxial compression coupled with the Rankine criterion, which provides the biaxial strength of concrete under tension-tension and tension-compression are used (figure 3). This multiple failure surfaces criterion was implemented in the developed program. At failure its surface is given by:

$$F_r\,(\sigma^r_{ij},\, \varepsilon^{pr}_{kl},\, k^r) = 0 \qquad [11]$$

where σ^r_{ij}, ε^{pr}_{kl} and k^r are the stress, the permanent strain and the of work hardening parameter at failure respectively.

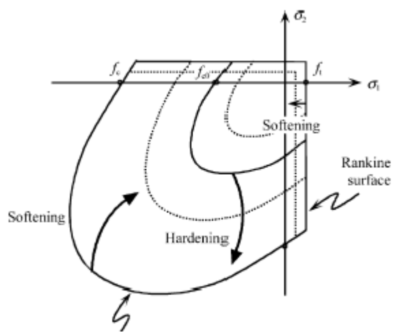


Figure 3. *Loading surface defined in the2-D principal axes*

2.2.2. *Concrete in tension*

The behavior of concrete before cracking is assumed as an elastic isotropic solid. The average stress-strain curve in tension is shown in figure 4 where the ascending and descending branches are given for both cut-off and tension- stiffening models; as:

$$\begin{cases} f_t = E_c \cdot \varepsilon_t & 0 \le \varepsilon_t \le \varepsilon_{cr} & \text{Cut-off model} \\ f_t = 0. & \varepsilon_t > \varepsilon_{cr} & \\ f_t = f_{cr}\left(1 - \dfrac{\varepsilon_t}{\varepsilon_m}\right) & \varepsilon_{cr} \le \varepsilon_t \le \varepsilon_m & \text{Tension-stiffening model} \\ f_t = 0. & \varepsilon_t > \varepsilon_m & \end{cases} \qquad [12]$$

where E_c is Young's modulus of concrete, f_{cr} and ε_{cr} are the cracking stress and strain of concrete respectively, ε_m is the strain corresponding to a null stress.

During the cracking phenomenon, the smeared crack model is used to represent the discontinuity of the macro-cracking behavior. It is recommended (Asce, 1982) to take into account the contribution of cracked concrete in the normal direction of the cracks named tension-stiffening effect. In this investigation, a limiting value of ε_m equal to 0.001 (Hibbit *et al.*, 1997) has been used (figure 4).

To simulate the cracked concrete behavior, a linear softening model was adopted in this study for its simplicity and computational efficiency. After the first crack has occurred elastic orthotropic model is considered with reduced elastic modulus in the direction normal to the crack plane. The stiffness matrix describing the cracked concrete behavior is expressed as:

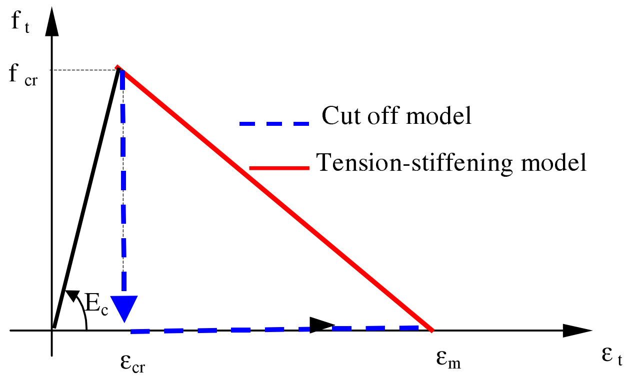


Figure 4. *Cracked concrete model*

$$[D_{cr}] = \frac{1}{1-\nu^2}\begin{bmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & (1-\nu)G_c^* \end{bmatrix} \quad [13]$$

where E_1 and E_2 are the secant moduli of elasticity in the directions of the orthotropic axes, ν and G_c^* are Poisson's ratio of concrete and the secant shear modulus of cracked concrete in the principal plan.

Beyond the tensile strength, the tensile stress decreases linearity with increasing of principal tensile strain. The ultimate failure occurs by cracking when the principal tensile strain exceeds the value ε_m (figure 4), which can be determined from the fracture mechanics concept. The experimental studies (Welch *et al.*, 1969) show that ε_m depends on the finite element mesh size. Through, previous numerical analyses were guaranteed the insensitivity of the analytical solution to the mesh size using the above approach defining the limited strain of softening ε_m.

Cracking of concrete will develop and propagate in the direction normal to the major principal strain starting from the section where a crack first occurs. In the post cracking stage, the cracked concrete can still transfer forces through shear friction, which is termed shear retention. Assuming that the stiffness modulus of intact concrete is cracked G_c, then the reduced shear modulus G_c^* of cracked concrete can be expressed as:

$$G_c^* = \beta.\, G_c \quad [14]$$

The shear retention factor is expressed as:

$$\beta = 1 - \frac{\varepsilon_n}{\varepsilon^*} \quad [15]$$

where ε_n and ε^* are the strain normal to the cracked direction and the strain at which the parameter β reduces to zero (figure 5).

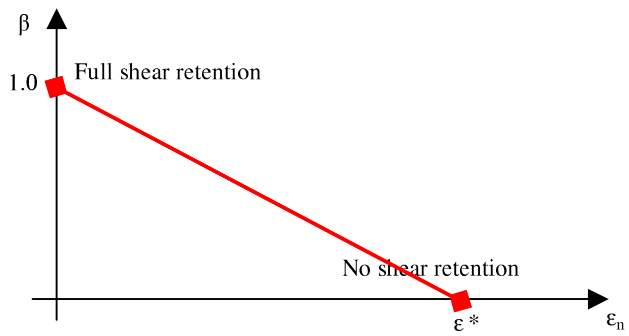


Figure 5. *Shear retention factor*

Several values of β are selected in the range of 0 to 1. A value of β equal to zero corresponds to no shear retention and the shear modulus of cracked concrete is assumed to be 0. For full shear retention the parameter β is selected to be 1 (figure 5), the shear modulus of the cracked concrete is assumed to be the same as that of the intact concrete. It is recommended from the obtained results that an average value of β = 0.40 is to be taken into account of reinforced concrete beams analysis.

3. Nonlinear solution technique

Every nonlinear analysis algorithm of the resolution is composed of 4 steps: (1) the formulation of the tangent stiffness matrix, (2) the resolution of the equilibrium equations, (3) the formulation of the internal force vector and (4) the test of convergence. Since, the stiffness matrix of structure depends on the displacement increments, the nonlinear resolution of equilibrium equations required the utilization of an iterative method and the internal force vector is established as the difference between the exterior and interior forces. The selected nonlinear solution is that the Newton Raphson method. The criterion of the convergence is based on the accuracy of satisfying the global equilibrium equations, which is guided of the unbalanced nodal forces.

The load is divided into small increments. For each iteration, the nodal force vector is calculated as:

$$\{F_{\text{int}}\} = \sum_{e} \int_{v} [B]^{T} \{\sigma_{\text{int}}\} dv \qquad [16]$$

where [B] is the element geometric matrix and $\{\sigma_{\text{int}}\}$ is the total internal stresses vector. The residual force vector is expressed as:

$$\{F_{res}\} = \{F_{ext}\} - \{F_{\text{int}}\} \qquad [17]$$

where $\{F_{ext}\}$ and $\{F_{\text{int}}\}$ are the external and internal force vectors respectively. For each iteration the convergence criterion has to be achieved and this process is repeated for all load increments until the structural failure is attained.

Finally, the effect of finite element mesh size was studied for the shear dominant structures on the basis of the used tension-stiffening model. When the tension stiffening effect was not included, there was a marked dependence of the analytical results on the finite element mesh size, while the proposed model exhibited satisfactory behavior and led to response predictions which are essentially independent from the finite element mesh size when the mesh size is less than seven times the maximum aggregate size that’s about 17.8 cm (Kim, 1999).

4. Applications

Two simply supported reinforced concrete beams are investigated. They have been tested by Pera (Pera, 1973). The Serindipity membrane plane stress elements with 3×3 of Gauss integration points are selected to modeling the concrete while truss linear elements with 3 nodes are used for modeling the steel bars. A perfect bond between membranous elements and truss steel bars is assumed and the materials proprieties of beams are summarized in table 1.

Table 1. *Materials proprieties used in applications (in MPa)*

Model	Concrete				Reinforcing bars				
	E_c	ν	F''_c	f_{cr}	f_y	f_r	E_s	ρ_s(%)	ρ'_s(%)
P1	37600	0.22	41	3.90	447.50	722.50	2.210^5	6.35	0.11
P2	37600	0.22	41	3.90	368	488	2.210^5	1.79	0.11

The tested beams are subjected to a concentrated load at mid-span. The geometry and the cross section of over and under-reinforced concrete beams are shown in figure 6a. The finite element model in figure 6b represents only half of the structure (symmetry in geometry and loading).

The predicted ultimate load of the over-reinforced beam (P1) is equal to 400 KN, that is 2.50% lower than the experimental value evaluated at 410 KN. For the under-reinforced beam (P2), the predicted value of 240 KN is 1.69% higher than the experimental load, 236 KN.

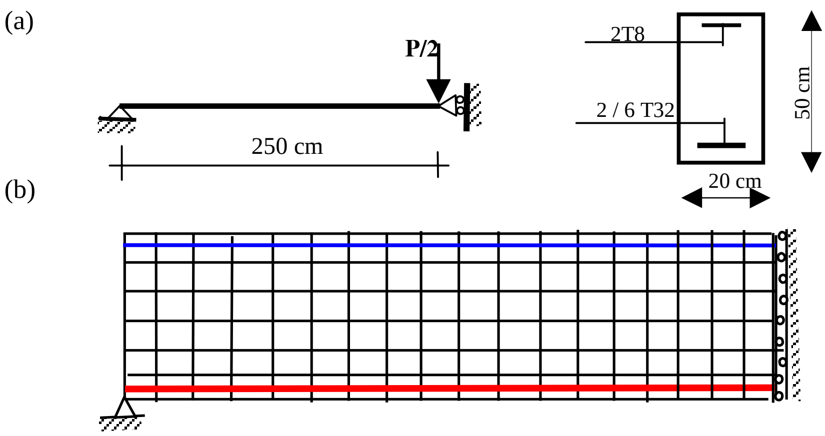


Figure 6. *Description of the beams: (a) configuration, (b) discretization*

Various computed load-deflection curves are compared with the experimental results (figures 7-8) and it can be observed that their shapes are very similar. The diagrams show that the nonlinear behavior accurately reproduces the experimental behavior but the linear analysis does not reflect the structural behavior of RC beams. The contribution of the cracked concrete has an importance in the nonlinear analysis and is in good agreement with the experimental data for over and under-reinforced concrete beams. The predicted responses show that the tension-stiffening branch adopted in this study exhibits satisfactory behavior and reasonable results. It is recorded that the under-reinforced concrete beam (P2) sustains the post-yielding behavior but the over-reinforced beam (P1) present a sudden failure after the yield of steel. It is clear that the global behavior is very well simulated until the beam failure.

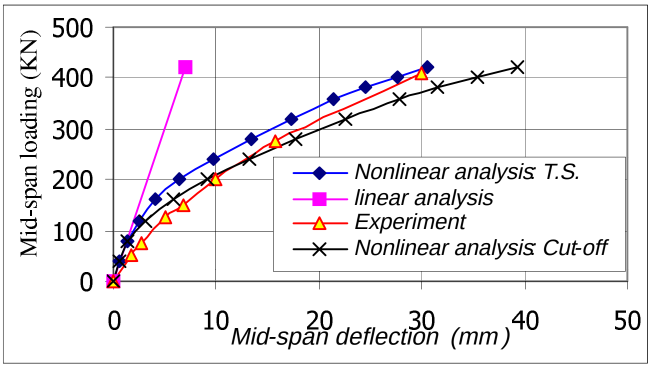


Figure 7. *Load- deflection curve at mid-span of over-reinforced beam*

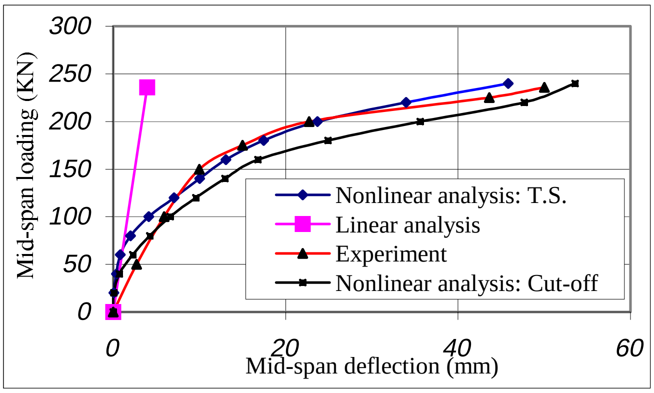


Figure 8. *Load- deflection curve at mid-span of under-reinforced beam*

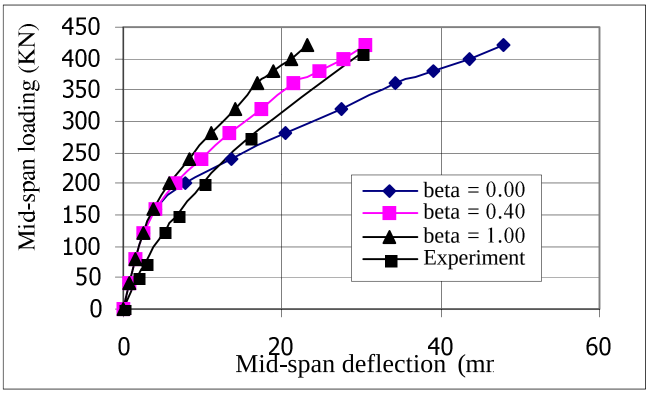


Figure 9. *Shear retention factor effect of over-reinforced concrete beam with tension stiffening model*

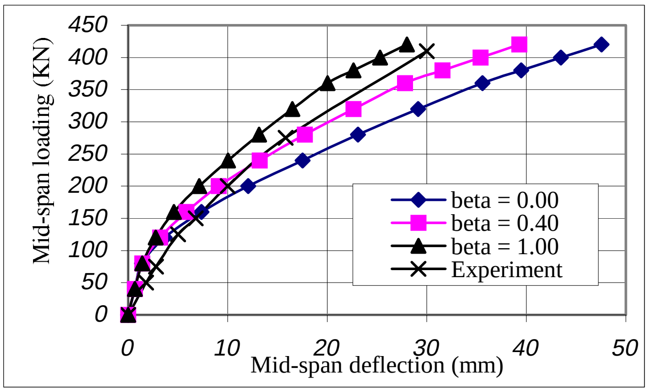


Figure 10. *Shear retention factor effect of over-reinforced beam with cut off model*

The over-reinforced beam

The reinforced concrete beam is not correctly simulated when the shear retention factor is neglected which presented independency of the tension stiffening effect. The P-δ curves obtained with values of $\beta = 0$ and $\beta = 1.0$ show the clear difference between the test results and the numerical results of the proposed model (figures 9-10). This difference becomes more important for the case of cracked concrete with tension stiffening model compared with that the cut off model.

The increase of stiffness due to the contribution of the un-cracked concrete, the ultimate deflection is well simulated with the tension stiffening model (2%) and it is over-estimated without the tension stiffening (31%).

For two cases of modeling, an average value of β = 0.4, the proposed model reproduces the experimental behavior more accurately. It is thus advisable to take it as a reference value in the nonlinear finite element analysis of reinforced concrete beams.

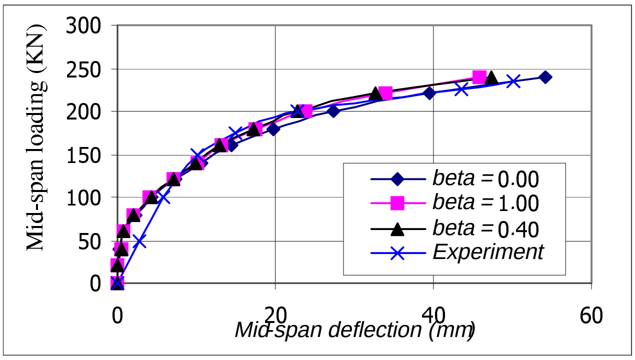


Figure 11. *Shear retention factor effect of under-reinforced beam with tension stiffening model*

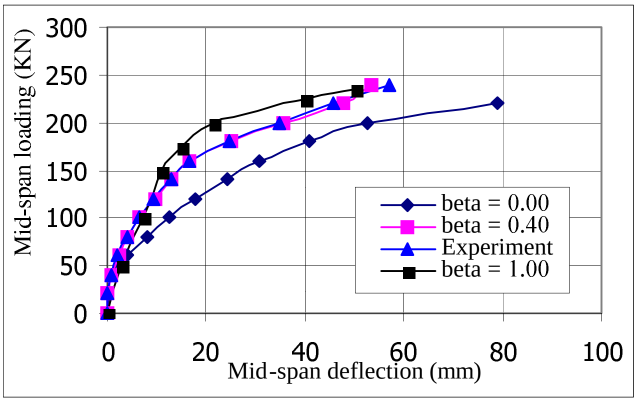
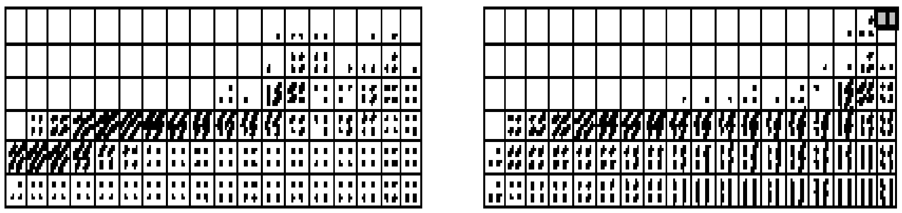


Figure 12. *Shear retention factor effect of under-reinforced beam with cut off model*

The under-reinforced beam

In this case, the shear retention factor does not have an appreciable influence with the tension-stiffening model (figure 11). The factor β seems to have an influence in the total response of the reinforced concrete beams (figure 12) without the tension stiffening effect. These obtained results are in contradiction with those presented by Hu and Liang (Hu *et al.*, 2000).

Finally, it is seen that by using the proposed model, the 2-D finite element analysis can predict not only the ultimate load and the deflection satisfactory, but also the failure process of the beams (figure 13), the ductility and the effect of the percentage of reinforcements on the response of the reinforced concrete beams.



Under-reinforced section Over-reinforced section

Figure 13. *Analytical failure modes*

5. Conclusion

An analytical model for the material nonlinear analyses of reinforced concrete structures has been proposed. The model includes a new softened stress-strain relationship of concrete in compression. The proposed model, which takes in account the following parameters, has been verified by comparison between numerical and experimental results.

The studies RC beams were analyzed with the purpose of investigating the relative effects of the physical proprieties including the proposed softening behavior, the tension stiffening and the shear retention factor. The correlation between analytical and experimental results led to the following conclusions: (1) the compressive softening behavior of concrete and the smeared stress-strain relation of steel embedded in concrete can predict more accurately the response and the ultimate strength of RC beams, (2) the tension stiffening effect has not an appreciable effect but its negligence has an influence especially on under-reinforced concrete beams. The tension-stiffening model has a notable effect at the cracked stage of concrete of the under-reinforced concrete beams because the contribution of the cracked concrete increases as the structure is more under-reinforced. However, the predicted responses show that the adopted tension-stiffening model exhibits reasonable results (3) an average value of the shear retention factor ($\beta \approx 0.4$) is recommended for this analysis. The previous figures (7-13) show that the finite element analysis based on the proposed model can provide accurate prediction of the nonlinear behavior, including the ultimate strength, deflection and failure mechanism of RC structures.

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