A monolithic strategy based on an hybrid domain decomposition method for multiphysic problems

Application to poroelasticity

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ABSTRACT. *A monolithic strategy based on an hybrid domain decomposition method for the numerical simulation of multiphysic problems is presented. It relies on a "physical" choice of primal interface unknowns. First numerical assessments are described for poroelasticity problems.*

RÉSUMÉ. *Une stratégie monolithique associée à une nouvelle approche de décomposition de domaine dite hybride pour la simulation numérique de problèmes multi-physiques est proposée. Celle-ci repose sur un choix "physique" des champs primaux d'interface. Une première application à un problème de compactage de sol (poroélasticité) est présentée.*

KEYWORDS: *multiphysic, domain decomposition, multifield problems, porous media.*

MOTS-CLÉS : *multiphysique, décomposition de domaine, problèmes multichamps, milieux poreux.*

REEF – 13/2004. Giens 2003, pages 523 à 534

1. Introduction

Multiphysic problems form a wide range of problems occurring in mechanics. They arise when modelling complex systems involving different phenomena with couplings. A classical solution is then to associate to each phenomenon various unknown physical fields; for instance in poroelasticy, studying the fluid flow inside a deformable body leads to the research of both skeleton displacement and fluid pressure. However, after finite element discretization, the presence of several unknown fields leads to large and massive systems of equations.

Different approaches have been explored to solve such problems with computational efficiency: partitioned methods and staggered algorithms [FEL 88, LEW 91, FEL 01] are often preferred to monolithic approaches as they make it possible to carry out iterative resolutions uncoupling large systems; among others let us cite Successive OverRelaxation (SOR) based methods (like ISPP [MAT 96]) and Large Time INcremental method [DUR 03]. All these methods are based on an uncoupling of the various physical phenomena, with the possibility of using different solvers for each uncoupled problem. These methods may lack efficiency when couplings become too strong because many iterations are then required so that computational cost considerably increases, or when huge 3D problems are considered because the dimension of the pure mechanical problem is then far greater than the fluid part dimension leading to unbalanced subsystems. In such cases a monolithic approach can be privileged, even if such classical method does not easily take into account the multiphysic nature of the problem.

An interesting computational strategy, which is perfectly suited to modern computing hardware, is to use non-overlapping domain decomposition methods. Basically these methods consist in substructuring the reference domain, condensing the problem on the interface between substructures to ensure the continuity of primal unknowns and the equilibrium of fluxes, and then solving this interface problem using a Krylov iterative solver. Best known non-overlapping domain decomposition methods are the primal approach (BDD [MAN 93]) and the dual approach (FETI [FAR 91]). For a classical elastic problem, the first consists in searching the continuous interface displacement which ensures the action-reaction principle between substructures, while the second consists in searching the equilibrated interface efforts which ensure the continuity of the displacement between substructures. Unfortunately the use of classical domain decomposition methods to solve a coupled problem may lead to poorly conditioned systems because of the heterogeneousness of the physical dimension of the unknown fields, even when using an adimensionalization preprocessing.

Hence we propose to use the hybrid domain decomposition method [GOS 04], which makes it possible to have a physics-friendly approach of multifield problems. The main idea is to enable a degree-of-freedom-specific treatment (basically primal or dual) of the interface, so that the interface unknown is physically homogeneous, which should lead to better performance results when solving strongly coupled problems.

The paper is organized as follows: in section 2 a general modelling and basic simulation strategy of poroelastic problems is presented, in section 3 hybrid domain decomposition approach is described in the context of poromechanics. This method is applied in section 4 to the simulation of the hardening of a porous media. Finally section 5 provides conclusions and prospects.

2. Reference poroelasticity problem

2.1. *Description*

Let us consider the isotherm evolution of a structure Ω made up of a saturated porous medium, underlying small perturbations around a reference position during interval of time $[0, t_f]$ [BIO 41, COU 90]. The structure is disturbed by a boundary loading made up of displacement $\vec{u}_d$ on $\partial_u\Omega$, and of effort $\vec{F}_d$ on complementary part $\partial_F\Omega$, of pressure p_d on $\partial_p\Omega$ and of flow of fluid mass $\vec{M}_d$ on complementary part $\partial_M\Omega$. No body force is considered in the medium.

$\vec{u}$ represents skeleton displacement, ε the skeleton linearised strain tensor, σ the Cauchy stress tensor, p the pore pressure, m the rate of fluid mass accumulation since time of reference. $\vec{M}$ is the fluid mass flow.

2.2. *Linear poroelasticity constitutive equations*

Two balance equations control a porous medium during isotherm evolution: momentum balance [1] and fluid mass balance [2]:

$$\vec{\text{div}}\,\sigma = \vec{0} \quad [1]$$

$$\dot{m} = -\text{div}\,\vec{M} \quad [2]$$

The structure behavior is assumed to be linear elastic. Within this framework, a thermodynamic study leads to Hooke's law [3] and fluid state equation [4]:

$$\sigma = \mathbb{A} : \varepsilon - B\,p \quad [3]$$

$$\frac{m}{\rho} = N\,p + B : \varepsilon \quad [4]$$

where $\mathbb{A}$ is the fourth order Hooke's tensor of the drained skeleton, B is Biot's tensor and N is the inverse of Biot modulus. It is necessary to add the complementary relation which rules the fluid transport phenomenon, expressed in Darcy's law [5]:

$$\frac{\vec{M}}{\rho} = -K\,\vec{\text{grad}}\,p \quad [5]$$

where K is the fluid permeability tensor.

2.3. *Problem formulation*

The resolution of the previous problem leads to determine various fields $\vec{u}$, σ, p, m and $\vec{M}$ in their various respective admissible space $\mathcal{U}^{ad}$, σ^{ad}, $\mathcal{P}^{ad}$, $\mathfrak{M}^{ad}$ and $\mathcal{M}^{ad}$, at any time $t \in [0, t_f]$ ensuring equations [1] to [5] of the model. The weak formulation reads:

$$\int_\Omega \sigma : \varepsilon^* \, d\Omega = \int_{\partial_F \Omega} \vec{F}_d . \vec{u^*} dS, \forall \vec{u^*} \in \mathcal{U}_0^{ad} \quad [6]$$

$$\int_\Omega \vec{M} . \vec{\text{grad}}\, p^* d\Omega - \int_\Omega \dot{m}\, p^* d\Omega = \int_{\partial_M \Omega} M_d \, p^* dS, \ \forall p^* \in \mathcal{P}_0^{ad} \quad [7]$$

A finite element space discretization is adopted which leads to a time first order linear differential equations system:

$$\begin{bmatrix} 0 & 0 \\ -B^T & -C \end{bmatrix} \begin{pmatrix} \dot{u} \\ \dot{p} \end{pmatrix} + \begin{bmatrix} K_1 & -B \\ 0 & -K_2 \end{bmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f_u \\ f_p \end{pmatrix} \quad [8]$$

where K_1, K_2 are the rigidity and permeability matrices, B is the poromechanical coupling matrix and C the fluid compressibility matrix. f_u is the vector of generalized forces and f_p the vector of generalized flows. An incremental implicit Euler scheme is then used to solve [8]:

$$\begin{bmatrix} K_1 & -B \\ -B^T & -(C + K_2 \, \Delta t) \end{bmatrix} \begin{pmatrix} u_{n+1} \\ p_{n+1} \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ -B^T & -C \end{bmatrix} \begin{pmatrix} u_n \\ p_n \end{pmatrix} + \begin{pmatrix} f_u^{n+1} \\ \Delta t \, f_p^{n+1} \end{pmatrix} \quad [9]$$

or using standard notation:

$$K x_{n+1} = f_{n+1} \quad [10]$$

3. An hybrid domain decomposition method

3.1. *Principle and choice of primal variables*

Let us first consider a two-substructure partitioning of a porous medium Ω, $\Omega = \Omega^{(1)} \bigcup \Omega^{(2)}$ and $\Upsilon = \partial\Omega^{(1)} \bigcap \partial\Omega^{(2)}$ is the interface between substructures (Figure 1).

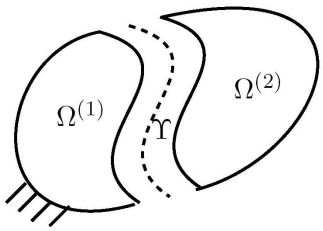

Figure 1. *Two-subdomain decomposition*

It is possible to completely rewrite equations [1] to [5] on the two substructures $\Omega^{(1)}$ and $\Omega^{(2)}$:

$$s = 1 \text{ or } 2 \left\{ \begin{array}{ll} \vec{\text{div}}\,\sigma^{(s)} = \vec{0} & \text{in } \Omega^{(s)} \\ \dot{m}^{(s)} = -\text{div}\,\vec{M}^{(s)} & \text{in } \Omega^{(s)} \\ \sigma^{(s)} = \mathbb{A} : \varepsilon^{(s)} - B\,p^{(s)} & \text{in } \Omega^{(s)} \\ \frac{m^{(s)}}{\rho} = N\,p^{(s)} + B : \varepsilon^{(s)} & \text{in } \Omega^{(s)} \\ \frac{\vec{M}^{(s)}}{\rho} = -K\,\vec{\text{grad}}\,p^{(s)} & \text{in } \Omega^{(s)} \\ \vec{u}^{(s)} = \vec{u}_d^{(s)} & \text{on } \partial_u\Omega^{(s)} \bigcap \partial_u\Omega \\ \sigma^{(s)}\vec{n}^{(s)} = \vec{F}_d & \text{on } \partial_F\Omega^{(s)} \bigcap \partial_F\Omega \\ p^{(s)} = p_d & \text{on } \partial_p\Omega^{(s)} \bigcap \partial_p\Omega \\ \vec{M}^{(s)}\vec{n}^{(s)} = M_d & \text{on } \partial_M\Omega^{(s)} \bigcap \partial_M\Omega \end{array} \right. \quad [11]$$

the two systems are linked by interface boundary conditions:

$$\left\{ \begin{array}{l} \vec{u}^{(1)} = \vec{u}^{(2)} \\ p^{(1)} = p^{(2)} \end{array} \right. \text{ on } \Upsilon \qquad \text{unknowns continuity} \quad [12]$$

$$\left\{ \begin{array}{l} \sigma^{(1)}\vec{n}^{(1)} + \sigma^{(2)}\vec{n}^{(2)} = \vec{0} \\ \vec{M}^{(1)}\vec{n}^{(1)} + \vec{M}^{(2)}\vec{n}^{(2)} = 0 \end{array} \right. \text{ on } \Upsilon \qquad \text{fluxes balance} \quad [13]$$

The domain decomposition method in primal form (BDD [MAN 93]) consists in looking for the continuous interface displacement and pressure fields which ensure the action-reaction principle and flow balance [13] between substructures, while the dual approach consists in searching the equilibrated interface effort and flow which ensure the continuity of the displacement and pressure between substructures [12]. For both methods, the interface problem is physically heterogeneous as it mixes pressure and displacement unknowns or effort and fluid flow unknowns.

In order to get an interface unknown homogeneous to an effort we propose to treat in a primal way the pressure and in a dual way the displacement. In other words we

search the interface fields of balanced intereffort λ_u and continuous pressure which guarantee the continuity of displacements and the conservation of flow:

$$\begin{array}{cc} \text{seeking for } (\vec{\lambda}_u, p) & \left\{ \begin{array}{ll} \vec{\lambda}_u = \sigma^{(1)}\vec{n}^{(1)} = -\sigma^{(2)}\vec{n}^{(2)} & \\ p = p^{(1)} = p^{(2)} & \end{array} \right. \quad \text{on } \Upsilon \\ \text{with respect of} & \left\{ \begin{array}{ll} \vec{u}^{(1)} = \vec{u}^{(2)} & \\ \vec{M}^{(1)}\vec{n}^{(1)} + \vec{M}^{(2)}\vec{n}^{(2)} = 0 & \end{array} \right. \quad \text{on } \Upsilon \end{array}$$

This step lies within the more general scope of the hybrid domain decomposition method as described in [GOS 04].

3.2. *A dual/primal condensed hybrid system*

Let us consider one linear displacement/pressure system ([10]) (we omit subscript n). We denote by i the internal degrees of freedom, p and u interface pressure and displacement degrees of freedom. Considering a two-subdomain decomposition, the system reads:

$$\left\{ \begin{array}{l} \begin{pmatrix} K_{ii}^{(1)} & K_{iu}^{(1)} & 0 & 0 & K_{ip}^{(1)} A_p^{(1)^T} \\ & K_{uu}^{(1)} & 0 & 0 & K_{up}^{(1)} A_p^{(1)^T} \\ & & K_{ii}^{(2)} & K_{iu}^{(2)} & K_{ip}^{(2)} A_p^{(2)^T} \\ & & & K_{uu}^{(2)} & K_{up}^{(2)} A_p^{(2)^T} \\ & & & & \sum A_p^{(s)} K_{pp}^{(s)} A_p^{(s)^T} \end{pmatrix} \begin{pmatrix} x_i^{(1)} \\ x_u^{(1)} \\ x_i^{(2)} \\ x_u^{(2)} \\ x_p \end{pmatrix} = \\ \qquad\qquad \begin{pmatrix} f_i^{(1)} \\ f_u^{(1)} \\ f_i^{(2)} \\ f_u^{(2)} \sum A_p^{(s)} f_p^{(s)} \end{pmatrix} + \begin{pmatrix} 0 \\ \underline{A}_u^{(1)^T} \\ 0 \\ \underline{A}_u^{(2)^T} \\ 0 \end{pmatrix} \underline{\lambda}_u \\ \\ \sum \underline{A}_u^{(s)} x_u^{(s)} = 0 \end{array} \right. \qquad [14]$$

Matrices $A_p^{(s)}$ are boolean assembly operators from local interface to global geometric interface, matrices $\underline{A}_u^{(s)}$ are signed boolean assembly operators from local interface to global connectivity interface so that the last line of the previous system expresses the continuity of displacement field (see figure 2).

In order to simplify writings and extend to several subdomains, we adopt the following notations (b represents either u or p and $\bar{b}$ its complementary) :

$$K_b = \begin{pmatrix} K_{ii} & K_{i\bar{b}} \\ K_{\bar{b}i} & K_{\bar{b}\bar{b}} \end{pmatrix} \quad \bar{K}_b = \begin{pmatrix} K_{ib} \\ K_{\bar{b}b} \end{pmatrix}$$

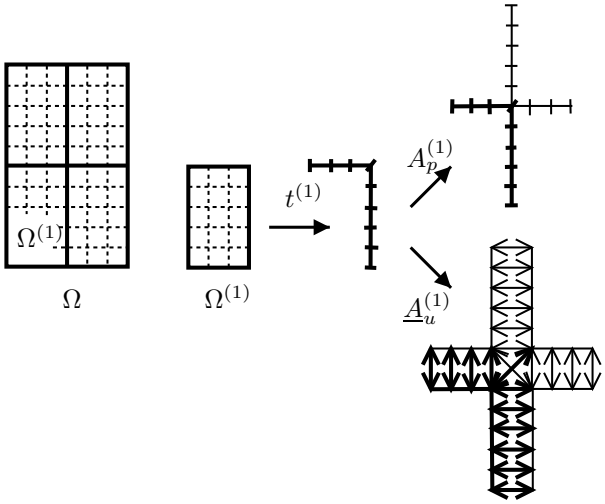

Figure 2. *Trace and assembly operators*

$$\text{if } v = \begin{pmatrix} v_i \\ v_1 \\ v_2 \end{pmatrix} \text{ then } \bar{v}_b = \begin{pmatrix} v_i \\ v_{\bar{b}} \end{pmatrix} \qquad \text{Trace operator: } t_b^T v_b = \begin{pmatrix} 0_i \\ v_b \end{pmatrix}$$

System ([14]) then reads:

$$\left\{ \begin{array}{l} \begin{pmatrix} K_p^{(1)} & 0 & \bar{K}_p^{(1)} A_p^{(1)^T} \\ & K_p^{(2)} & \bar{K}_p^{(2)} A_p^{(2)^T} \\ & & \sum A_p^{(s)} K_{pp}^{(s)} A_p^{(s)^T} \end{pmatrix} \begin{pmatrix} \bar{x}_p^{(1)} \\ \bar{x}_p^{(2)} \\ x_p \end{pmatrix} = \\ \qquad\qquad\qquad\qquad \begin{pmatrix} \bar{f}_p^{(1)} \\ \bar{f}_p^{(2)} \\ \sum A_p^{(s)} f_p^{(s)} \end{pmatrix} + \begin{pmatrix} t_u^{(1)^T} \underline{A}_u^{(1)^T} \\ t_u^{(2)^T} \underline{A}_u^{(2)^T} \\ 0 \end{pmatrix} \underline{\lambda}_u \\ \sum \underline{A}_u^{(s)} t_u^{(s)} \bar{x}_p^{(s)} = 0 \end{array} \right. \qquad [15]$$

Solving on each subdomain a system where pressure and effort boundary conditions are imposed on the interface we eliminate internal degrees of freedom and unknown interface displacements, and then we only keep pressure and intereffort interface unknowns. If the subdomain lacks Dirichlet boundary conditions ("floating substructure") we have to introduce admissible solid body motions; we denote by $R_p^{(s)}$ a basis of $\ker(K_p^{(s)})$ and $\alpha^{(s)}$ the magnitude of these motions.

$$\left\{ \begin{array}{l} \bar{x}_p^{(s)} = K_p^{(s)^+} \left(-\bar{K}_p^{(s)} {A_p^{(s)}}^T x_p + \bar{f}_p^{(s)} + {t_u^{(s)}}^T {\underline{A}_u^{(s)}}^T \underline{\lambda}_u \right) + R_p^{(s)} \alpha_p^{(s)} \\ {R_p^{(s)}}^T \left(-\bar{K}_p^{(s)} {A_p^{(s)}}^T x_p + \bar{f}_p^{(s)} + {t_u^{(s)}}^T {\underline{A}_u^{(s)}}^T \underline{\lambda}_u \right) = 0 \end{array} \right. \quad [16]$$

The condensed problem with unknowns x_p and $\underline{\lambda}_u$ then reads:

$$\begin{pmatrix} S_{p\underline{u}} & \begin{pmatrix} G_p \\ \underline{G}_u \end{pmatrix} \\ \begin{pmatrix} -G_p^T & \underline{G}_u^T \end{pmatrix} & 0 \end{pmatrix} \begin{pmatrix} \begin{pmatrix} x_p \\ \underline{\lambda}_u \end{pmatrix} \\ \alpha \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} b_p \\ -\underline{b}_u \end{pmatrix} \\ -e \end{pmatrix} \quad [17]$$

with

$$\left\{ \begin{array}{l} G_p = \left(\dots A_p^{(s)} (\bar{K}_p^{(s)})^T R_p^{(s)} \dots \right) \\ \underline{G}_u = \left(\dots \underline{A}_u^{(s)} t_u^{(s)} R_p^{(s)} \dots \right) \end{array} \right. \quad \left\{ \begin{array}{l} b_p = \sum A_p^{(s)} \left(f_p^{(s)} - (\bar{K}_p^{(s)})^T K_p^{+(s)} \bar{f}_p^{(s)} \right) \\ \underline{b}_u = \sum \underline{A}_u^{(s)} t_u^{(s)} {K_p^{(s)}}^+ \bar{f}_p^{(s)} \end{array} \right.$$

$$S_{p\underline{u}} = \sum \begin{pmatrix} A_p^{(s)} & 0 \\ 0 & \underline{A}_u^{(s)} \end{pmatrix} S_{pu}^{(s)} \begin{pmatrix} A_p^{(s)} & 0 \\ 0 & \underline{A}_u^{(s)} \end{pmatrix}^T, \; \alpha = \begin{pmatrix} \vdots \\ \alpha^{(s)} \\ \vdots \end{pmatrix}, \; e = \begin{pmatrix} \vdots \\ {R_p^{(s)}}^T \bar{f}_p^{(s)} \\ \vdots \end{pmatrix}$$

$$S_{pu}^{(s)} = \begin{pmatrix} \left(K_{pp}^{(s)} - \bar{K}_p^{(s)T} {K_p^{(s)}}^+ \bar{K}_p^{(s)} \right) & \left(\bar{K}_p^{(s)T} {K_p^{(s)}}^+ {t_u^{(s)}}^T \right) \\ -\left(\bar{K}_p^{(s)T} {K_p^{(s)}}^+ {t_u^{(s)}}^T \right)^T & \left(t_u^{(s)} {K_p^{(s)}}^+ {t_u^{(s)}}^T \right) \end{pmatrix}$$

From a local point of view, we can notice that the top left block of matrix $S_{pu}^{(s)}$ is the primal schur complement related to pressure degrees of freedom and that the bottom right block is the dual Schur complement related to displacement degrees of freedom, the extradiagonal blocks are skew-symmetric coupling terms.

3.3. *Preconditioning/coarse problem*

The condensed linear system is a constraint system. Matrix $S_{p\underline{u}}$ being a sum of local contributions, an efficient numerical strategy is then to use a projected Krylov algorithm (a GMRes solver is preferred since the linear system is non-symmetric or non-positive when symmetrized). The definition of a smart preconditioner is an important issue and we propose to use the classical strategy consisting in approximating the inverse of a sum of local contributions by a scaled sum of local inverses.

Noticing that $S_{pu}^{(s)}$ is the local operator which associates resulting interface flux and displacement to given interface pressure and effort, its inverse is $S_{up}^{(s)}$ [18] which associates resulting interface pressure and effort to given flux and displacement.

$$S_{up}^{(s)} = \begin{pmatrix} \left(t_p^{(s)} {K_u^{(s)}}^{-1} {t_p^{(s)}}^T \right) & -\left(\bar{K}_u^{(s)T} {K_u^{(s)}}^{-1} {t_p^{(s)}}^T \right)^T \\ \left(\bar{K}_u^{(s)T} {K_u^{(s)}}^{-1} {t_p^{(s)}}^T \right) & \left(K_{uu}^{(s)} - \bar{K}_u^{(s)T} {K_u^{(s)}}^{-1} \bar{K}_u^{(s)} \right) \end{pmatrix} \qquad [18]$$

Note that in such a situation, matrix $K_u^{(s)}$ is always inversible. Preconditioning operator $\hat{S}_{p\underline{u}}^{-1}$ writes

$$\hat{S}_{p\underline{u}}^{-1} = \sum \begin{pmatrix} W_p^{(s)} A_p^{(s)} & 0 \\ 0 & \underline{W}_u^{(s)} \underline{A}_u^{(s)} \end{pmatrix} S_{up}^{(s)} \begin{pmatrix} W_p^{(s)} A_p^{(s)} & 0 \\ 0 & \underline{W}_u^{(s)} \underline{A}_u^{(s)} \end{pmatrix}^T \qquad [19]$$

where W_p and $\underline{W}_u$ are diagonal scaling matrices. These can be defined according to the multiplicity of degrees of freedom or according to the difference of behavior between substructures [RIX 99, GOS 02].

Let $G = \begin{pmatrix} -G_p \\ G_d \end{pmatrix}$, constraint $G^T \begin{pmatrix} x_p \\ \underline{\lambda}_u \end{pmatrix} = -e$ is taken into account by initialization $\begin{pmatrix} x_p \\ \lambda_u \end{pmatrix}_0$ and projector P_Q where Q is an approximation of the preconditioner.

$$\begin{pmatrix} x_p \\ \lambda_u \end{pmatrix}_0 = QG \left(G^T Q G \right)^{-1} e, \qquad P_Q = I_d - QG \left(G^T Q G \right)^{-1} G^T \qquad [20]$$

3.4. *On a primal/dual approach*

Previous notations make it possible to give the formulation where displacement is treated as a primal unknown and pressure as a dual unknown. The resulting interface unknown is displacement/flux. Such an approach is not based on any physical consideration.

$$S_{\underline{p}u} \begin{pmatrix} \underline{\lambda}_p \\ x_u \end{pmatrix} = \begin{pmatrix} -\underline{b}_p \\ b_u \end{pmatrix} = \begin{pmatrix} -\sum \underline{A}_p^{(s)} t_p^{(s)} {K_u^{(s)}}^+ \bar{f}_u^{(s)} \\ \sum A_u^{(s)} \left(f_u^{(s)} - (\bar{K}_u^{(s)})^T K_u^{+(s)} \bar{f}_u^{(s)} \right) \end{pmatrix} \qquad [21]$$

$$S_{\underline{p}u} = \sum \begin{pmatrix} \underline{A}_p^{(s)} & 0 \\ 0 & A_u^{(s)} \end{pmatrix} S_{pu}^{(s)} \begin{pmatrix} \underline{A}_p^{(s)} & 0 \\ 0 & A_u^{(s)} \end{pmatrix}^T \qquad [22]$$

$$\hat{S}_{\underline{p}u}^{-}1 = \sum \begin{pmatrix} \underline{W}_p^{(s)} \underline{A}_p^{(s)} & 0 \\ 0 & W_u^{(s)} A_u^{(s)} \end{pmatrix} S_{pu}^{(s)} \begin{pmatrix} \underline{W}_p^{(s)} \underline{A}_p^{(s)} & 0 \\ 0 & W_u^{(s)} A_u^{(s)} \end{pmatrix}^T \qquad [23]$$

$$H = \begin{pmatrix} \dots \underline{W}_p^{(s)} \underline{A}_p^{(s)} (\bar{K}_p^{(s)})^T R_p^{(s)} \dots \\ \dots - W_u^{(s)} A_u^{(s)} t_p^{(s)} R_p^{(s)} \dots \end{pmatrix} \left\{ \begin{array}{l} P = I_d - H \left(H^T S_{\underline{p}u} H \right)^{-1} H^T S_{\underline{p}u} \\ \begin{pmatrix} \underline{\lambda}_p \\ x_u \end{pmatrix}_0 = H \left(H^T S_{\underline{p}u} H \right)^{-1} H^T \begin{pmatrix} -\underline{b}_p \\ b_u \end{pmatrix} \end{array} \right.$$

4. Numerical assessments

Let us consider the problem of soil compaction described in figure 3. The structure is made up of a portion of soil of 5m length, 4m of width and 3m of depth. A compaction effort (effort growing linearly till a threshold) is imposed on a 2m $\times$ 2m zone. The top part is submitted to the atmospheric pressure, elsewhere the walls are assumed to be impermeable and only transverse displacements are possible.

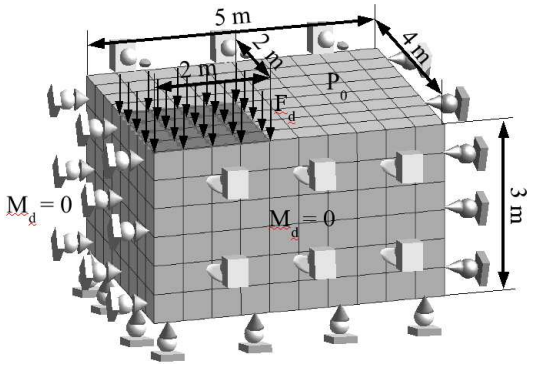

Figure 3. *Porous media compaction*

Only isotropic and linear phenomena are considered and the set of coefficients for an argilite medium, according to [GOR 99] reads: mechanical characteristics of the skeleton $\lambda = 662$ MPa, $\mu = 1885$ MPa, module of Biot $\frac{1}{N} = 6800$ MPa, coefficient of Biot $B = 0.6$, coefficient of permeability $K = 10.\,10^{-16}$m^2 Pa $^{-1}$ s^{-1}. Because of the significant difference of magnitude between coefficient values, an adimensionalization process was performed.

The structure is meshed with hexaedral elements. A $Q_2 - Q_1$ approximation (Q_2 for displacement, Q_1 for pressure) is used. The considered grid contains 480 elements and 8300 degrees of freedom. The decomposition into 3 subdomains is automatically carried out using the ZeBuLoN [FOE 96] code mesher. The time discretization leads to the resolution of 20 linear systems.

The performance results of the primal, dual and hybrid domain decomposition approaches based on the average number of iterations to converge, are presented in Tab. 1. Those results comply with a same relative GMRes error ratio, lower than 10^{-10}.

Approach	without preconditioning	with preconditioning
Primal/dual	417	26
Primal	286	26
Dual	265	28
Dual/primal	214	27

Table 1. *Average iteration number for linear system convergence*

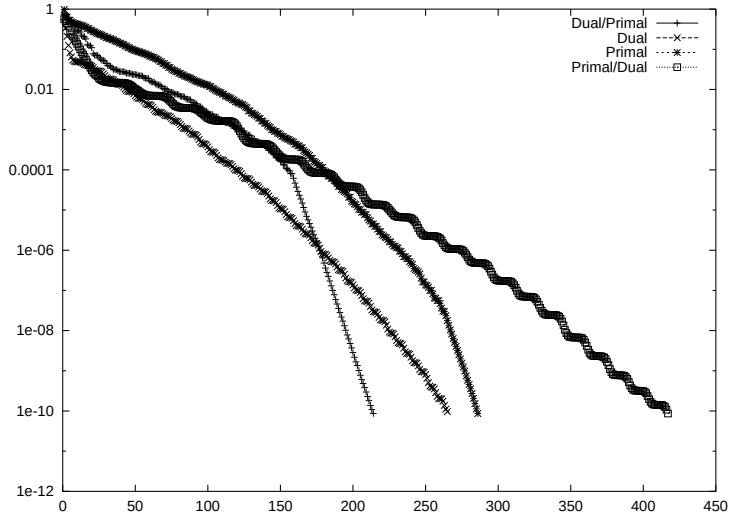

Figure 4. *Convergence of residual norm for non-preconditioned problems*

These results highlight the better performance levels of dual/primal approach when no preconditioning was used. The primal/dual approach which does not have physical sense leads to even poorer efficiency.

It is shown that the preconditioning gives the same kind of performance results as usual methods (FETI or BDD) with their optimal preconditioners, what proves the good behavior of the hybrid preconditioner.

Figure 4 presents the evolution of the residual norm for the first linear system which is mostly linked to the spectrum of the operator. It seems that being more respectful of the physic nature of unknowns, the hybrid method benefits from the good behaviors of the dual and primal approaches: first iterations look like the dual approach with a quick ratio-decrease, after a slowdown a superconvergent behavior similar to the one of the primal approach is quickly achieved.

5. Conclusion and prospect

A monolithic strategy based on an hybrid non-overlapping decomposition method was presented. First results obtained on a simple test look promising. Another sig-

nificant validation process should be undergone for ambitious porous problems with mechanical and fluid non-linearities (plasticity, damage, variation of permeability). Since the optimal preconditioner exists, a very interesting prospect is to study "simplified" preconditioners based on a partial physical or spacial scale uncoupling (like *lumped* preconditioner for FETI), benefitting from the advantageous spectrum of hybrid operator. Work on uncoupling according to physical time scales will also be carried out.

6. References

[BIO 41] BIOT M. A., "General theory of three dimensional consolidation", *Journal of Applied Physics*, vol. 12, num. 2, 1941, p. 188-164.

[COU 90] COUSSY O., *Mécanique des Milieux Poreux*, Technip, 1990.

[DUR 03] DUREISSEIX D., LADEVÈZE P., SCHREFLER B., "A computational strategy for multiphysics problems – application to poroelasticity", *International Journal for Numerical Methods in Engineering*, vol. 56, num. 10, 2003, p. 1489-1510.

[FAR 91] FARHAT C., ROUX F.-X., "A Method of Finite Element Tearing and Interconnecting and its Parallel Solution Algorithm", *Int. J. Numer. Meth. Engrg.*, vol. 32, 1991, p. 1205-1227.

[FEL 88] FELIPPA C. A., GEERS T., "Partionned analysis of coupled mechanical systems", *Engineering Computation*, vol. 5, 1988, p. 123-133.

[FEL 01] FELIPPA C. A., PARK K., FARHAT C., "Partionned analysis of coupled mechanical systems: Formulation", *Computer Methods in Applied Mechanics and Engineering*, vol. 190, 2001, p. 3247-3270.

[FOE 96] FOERCH R., *Un environement orienté objet pour la modélisation en mécanique des matériaux*, Thèse de doctorat, Ecole Nationale Supérieure des Mines de Paris, 1996.

[GOR 99] GOREYCHI M., *Comportement Thermo-hydro mécanique des argilites de l'Est et son influence sur les ourages souterains*, EDP Sciences Ed., 1999.

[GOS 02] GOSSELET P., REY C., DASSET P., LENE F., "A domain decomposition method for quasi incompressible formulations with discontinuous pressure field", *Revue Européenne des Elements Finis*, vol. 11, 2002, p. 363-377.

[GOS 04] GOSSELET P., REY C., *An hybrid domain decomposition method*, Rapport interne, to be submitted, 2004.

[LEW 91] LEWIS R., SCHREFLER B., SIMONI L., "Coupling versus uncoupling in soil consolidation", *International Journal for Numerical and Analytical Methods in Geomechanics*, vol. 15, 1991, p. 533-548.

[MAN 93] MANDEL J., "Balancing Domain Decomposition", *Comm. Appl. Num. Meth. Engrg.*, vol. 9, 1993, p. 233-241.

[MAT 96] MATTEAZZI R., SCHREFLER B., VITALIANI R., *Comparisons of partioned solution procedures for transient coupled problems in sequential and parallel processing*, Advances in Comput. Struc. Tech. Civil-Comp Ltd, Edinburgh, Scotland, 1996.

[RIX 99] RIXEN D., FARHAT C., "A Simple and Efficient Extension of a Class of Substructure Based Preconditioners to Heterogeneous Structural Mechanics Problems", *Int. J. Num. Meth. Eng.*, vol. 44, num. 4, 1999, p. 489-516.