Numerical coupling between shakedown and periodic homogenization for heterogeneous elastic plastic media

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ABSTRACT. *This paper presents a direct method to numerically study the strength, in the sense of shakedown, of elastic perfectly plastic media with a periodic microstructure, submitted to variable loads. The macroscopic admissible strength domains are obtained by solving constrained nonlinear optimization problems on a three-dimensional unit cell. These problems represent the shakedown analysis problems. Static and kinematic approaches of shakedown are tested by applying the developed method to a layered material and to a periodically perforated sheet.*

RÉSUMÉ. *On propose une méthode directe permettant d'étudier numériquement la résistance, au sens de l'adaptation, de matériaux élastoplastiques parfaits à microstructure périodique, lorsqu'ils sont soumis à des sollicitations variables. Les domaines de chargements macroscopiques admissibles sont obtenus en résolvant, sur une cellule de base tridimensionnelle, des problèmes d'optimisation non linéaire sous contraintes traduisant les problèmes d'adaptation. Les approches statique et cinématique de l'adaptation sont testées par application de la méthode aux cas d'un matériau stratifié et d'une plaque périodiquement trouée.*

KEYWORDS: *elastic plastic shakedown, static approach, kinematic approach, periodic homogenization, numerical modeling, 3D unit cell, convex optimization.*

MOTS-CLÉS : *adaptation élastoplastique, approche statique, approche cinématique, homogénéisation périodique, modélisation numérique, cellule de base 3D, optimisation convexe.*

1. Introduction

The emergence of new materials with specific properties requires latest researches in many fields such as aircraft, car or biomechanical industries. In this sense, this study concerns the strength capacity of elastic perfectly plastic materials which exhibit a periodic microstructure and which are submitted to variable mechanical loadings. This consists in coupling two theories:

– firstly the periodic homogenization theory, which allows to take into account the influence of the heterogeneities of the studied material on its macroscopic behavior,

– secondly the shakedown theory, which allows to take into account the variable aspect of the loadings by requiring very little information. Indeed, to know if a structure submitted to variable loads shakes down, that is to know if the plastically dissipated energy is bounded, there is no need to describe the time evolution of the loadings: bounds are sufficient to carry out direct shakedown analysis.

The methodology is the same as the one proposed by [SUQ 83] for the coupling between periodic homogenization and limit analysis. The shakedown analysis (either the static approach of Melan [MEL 36] or the kinematic approach of Koiter [KOI 60]) is carried out on a unit cell –three-dimensional in our study– representative of the heterogeneities of the structure and the results are expressed in terms of admissible strength domains of external loads: the macroscopic strains or stresses. By using the finite element method and the von-Mises yield criterion, direct shakedown analysis becomes a mathematical programming problem that is solved by a nonlinear constrained optimization software.

After some theoretical definitions concerning periodic homogenization (section 2), section 3 presents the static approach of the coupling by introducing a specific formulation which allows to take rigorously into account the periodicity and average relations lying the microscopic and the macroscopic quantities. This formulation allows to solve the cellular problems and to express the optimization conditions. The admissible domains are eventually obtained by implementing this coupling in the finite element software SIC ([AUN 90]) interfaced with the optimization software LANCELOT ([CON 92]). Section 4 deals then with the dual problem [DEB 76] –the kinematic approach of the coupling– in order to compare the performances of the two approaches. Finally, section 5 presents two application examples: the study of the strength capacity of a layered material and of a perforated sheet submitted to macroscopic loadings in the plane of the layers and of the sheet, respectively.

2. Periodic homogenization and localization problems

As mentioned in Introduction, the studied materials exhibit a periodic microstructure obtained by periodic translation, in the three spatial directions, of a pattern called *unit cell* and denoted by V. For instance, figure 1 shows a three-dimensional unit cell associated with an elastic plastic layered material. To take into account the influence

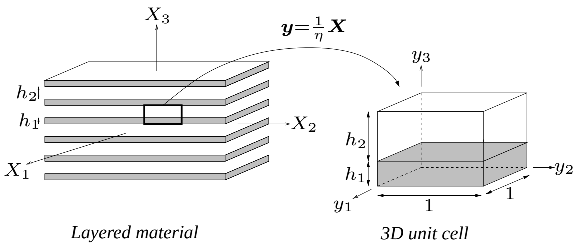

Figure 1. *Heterogeneous material and associated 3D unit cell*

of the heterogeneities on the macroscopic behavior of the layered material, it is thus necessary to introduce two scales: the macroscopic scale of the structure, denoted by $\boldsymbol{X}$, and the microscopic scale of the heterogeneities, denoted by $\boldsymbol{y}$. In the following all the macroscopic quantities are expressed at the point $\boldsymbol{X}$ and represented by capital letters, and all the microscopic quantities are expressed at the point $\boldsymbol{y}$ and represented by small letters. When there are many heterogeneities, the state variables fields fluctuate highly in the heterogeneous material. Therefore, when the size of the unit cell becomes much smaller than the one of the structure, the macroscopic strain $\boldsymbol{E}$ and stress $\boldsymbol{\Sigma}$ are linked ([SUQ 83]) by the average relations [1] where V^* is the volume occupied by the material part of V.

$$\boldsymbol{\Sigma} =< \boldsymbol{\sigma} > \triangleq \frac{1}{|V|} \int_{V^*} \boldsymbol{\sigma} \, dV \quad \text{and} \quad \boldsymbol{E} =< \boldsymbol{\varepsilon}(\boldsymbol{u}) > \triangleq \frac{1}{|V|} \int_{\partial V} \boldsymbol{u} \otimes_s \boldsymbol{n} \, dS \qquad [1]$$

As explained in [BOR 01], the periodicity properties of the media also imply that the microscopic strain and stress fields, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$, are V-periodic. As a consequence, the local strain field $\boldsymbol{\varepsilon}$ is splitted into two parts: a constant one, $\boldsymbol{E}$, deriving from $\boldsymbol{E}.\boldsymbol{y}$ and a fluctuating one, $\boldsymbol{\varepsilon}^{per}$, deriving from a periodic displacement on the unit cell, $\boldsymbol{u}^{per}$, such that:

$$\boldsymbol{u} = \boldsymbol{E}.\boldsymbol{y} + \boldsymbol{u}^{per} \quad \Leftrightarrow \quad \boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{E} + \boldsymbol{\varepsilon}(\boldsymbol{u}^{per}) \qquad [2]$$

The local elastic problems appearing in the shakedown analysis (*cf.* section 3) can then be expressed either in terms of given macroscopic strain or stress. This leads to consider the following localization problems:

Strains approach: P_{strain}	Stresses approach: P_{stress}
$\boldsymbol{E}$ given, $\boldsymbol{E} =< \boldsymbol{\varepsilon}(\boldsymbol{u}) >$	$\boldsymbol{\Sigma}$ given, $\boldsymbol{\Sigma} =< \boldsymbol{\sigma} >$
Find $\boldsymbol{u}^{per}$ and $\boldsymbol{\sigma}$ s.t.:	Find $\boldsymbol{u}^{per}$ and $\boldsymbol{\sigma}$ s.t.:
$\begin{cases} \textbf{div}\,\boldsymbol{\sigma} = \boldsymbol{0} \text{ in } V^* \\ \boldsymbol{\sigma} = \boldsymbol{d} : \boldsymbol{\varepsilon}(\boldsymbol{u}) \text{ in } V^* \\ \boldsymbol{u}^{per} = \boldsymbol{u} - \boldsymbol{E}.\boldsymbol{y} \text{ per. on } \partial V \\ \boldsymbol{\sigma}.\boldsymbol{n} \text{ antiperiodic on } \partial V \\ \boldsymbol{\sigma}.\boldsymbol{n} = \boldsymbol{0} \text{ on } \partial H \end{cases}$	$\begin{cases} \textbf{div}\,\boldsymbol{\sigma} = \boldsymbol{0} \text{ in } V^* \\ \boldsymbol{\sigma} = \boldsymbol{d} : \boldsymbol{\varepsilon}(\boldsymbol{u}) \text{ in } V^* \\ \boldsymbol{u}^{per} = \boldsymbol{u} - < \boldsymbol{\varepsilon}(\boldsymbol{u}) > .\boldsymbol{y} \text{ per. on } \partial V \\ \boldsymbol{\sigma}.\boldsymbol{n} \text{ antiperiodic on } \partial V \\ \boldsymbol{\sigma}.\boldsymbol{n} = \boldsymbol{0} \text{ on } \partial H \end{cases}$

where H denotes the potential holes such that $\partial H \cap \partial V = \varnothing$, $\boldsymbol{d}$ is the elastic stiffness tensor, and $\boldsymbol{n}$ is the outer normal on the boundary ∂V of V or the inner normal on the boundary ∂H of H. It must be noted that these formulations do not take into account the case when there are rigid inclusions in the unit cell. As explained in section 3, the resolution of either P_{strain} or P_{stress} is essential to write the optimization problem.

3. The static approach of the coupling

3.1. *Some definitions and assumptions*

Let us first introduce some definitions and assumptions necessary to describe the framework of the study and also to write the shakedown problem.

The cellular analysis to carry out in order to obtain the macroscopic admissible domains can be summarized as follows: an initial domain of prescribed macroscopic loading paths is given and we search how much this domain can be amplified such that shakedown occurs in the unit cell. In our case, the loading paths are assumed to be included in a convex polyhedron $\boldsymbol{\mathcal{D}}$ defined as a linear combination of n independent loads $\boldsymbol{\mathcal{P}}_i^0, i = 1..n$:

$$\boldsymbol{\mathcal{D}} = \left\{ \boldsymbol{\mathcal{P}}(t) \mid \boldsymbol{\mathcal{P}}(t) = \sum_{i=1}^{n} \mu_i(t) \boldsymbol{\mathcal{P}}_i^0, \quad \mu_i(t) \ \in \ [{\mu_i}^-, {\mu_i}^+] \right\} \qquad [3]$$

In this definition, $\boldsymbol{\mathcal{P}}$ represents either a state of macroscopic strains or a state of macroscopic stresses, and t is a loading parameter denoting the time in the following.

The medium is supposed to exhibit at each point $\boldsymbol{y}$ of the unit cell a plasticity domain $\boldsymbol{F}(\boldsymbol{y})$ in which the microscopic stress $\boldsymbol{\sigma}$ must stay to be plastically admissible. In our study we assume furthermore that each component, for instance each layer of the layered material, is elastic-perfectly plastic and that the local plasticity domain is defined by the way of the von-Mises yield function ($\boldsymbol{\sigma}^d$ denotes the deviatoric part of $\boldsymbol{\sigma}$ and σ^0 the yield stress):

$$\boldsymbol{F}(\boldsymbol{y}) = \left\{ \boldsymbol{\sigma}(\boldsymbol{y}) \mid \mathcal{F}(\boldsymbol{\sigma}(\boldsymbol{y}), \sigma^0(\boldsymbol{y})) = \sqrt{\frac{3}{2} \boldsymbol{\sigma}^d(\boldsymbol{y}) : \boldsymbol{\sigma}^d(\boldsymbol{y})} - \sigma^0(\boldsymbol{y}) \leq 0 \right\} \qquad [4]$$

It is also necessary to introduce a so called reference unit cell, noted $V^{(e)}$. This fictive unit cell has the same geometrical characteristics as the real one, but each component is supposed to exhibit a purely elastic behavior, defined by the Hooke's law. It means that once submitted to the same loading as the real unit cell, the response of this fictive unit cell is purely elastic. In the following, all the purely elastic fields are noted by the superscript symbol $^{(e)}$.

3.2. *Formulation of the shakedown problem as a maximization problem*

On these bases, the coupling between the static theorem of Melan and the periodic homogenization theory is then enounced as follows: a periodic heterogeneous material submitted to any variable loading path $\boldsymbol{\mathcal{P}} : \ t \in [0, +\infty[\ \rightarrow \ \boldsymbol{\mathcal{P}}(t) \in \boldsymbol{\mathcal{D}}$ shakes down if there exist a safety factor $\alpha > 1$ and a time-independent self-equilibrated residual stress field $\boldsymbol{\rho}$ such that its superposition with the field of the purely elastic stress path $\alpha\boldsymbol{\sigma}^{(e)}$ constitutes a safe state of stresses:

$$\forall \boldsymbol{y} \in V^*, \ \ \forall t, \quad \boldsymbol{\sigma}(\boldsymbol{y}, t) = \alpha\, \boldsymbol{\sigma}^{(e)}(\boldsymbol{y}, t) + \boldsymbol{\rho}(\boldsymbol{y}) \quad \in \quad \boldsymbol{F}(\boldsymbol{y}) \qquad [5]$$

where:

– the stress state $\boldsymbol{\sigma}^{(e)}$ due to the loading path $\boldsymbol{\mathcal{P}}$ occurs in $V^{(e)}$ and satisfies the localization problem P_{strain} (resp. P_{stress}) if the macroscopic strain $\boldsymbol{E}$ (resp. stress $\boldsymbol{\Sigma}$) is prescribed,

– the residual stress field $\boldsymbol{\rho}$ satisfies the periodicity and average conditions:

Strains approach: P^{res}_{strain}	Stresses approach: P^{res}_{stress}
$\begin{cases} \mathbf{div}\boldsymbol{\rho} \ = \ \mathbf{0} \quad \text{in } V^* \\ \boldsymbol{\rho}.\boldsymbol{n} \quad \text{antiperiodic on } \partial V \\ \boldsymbol{\rho}.\boldsymbol{n} \ = \ \mathbf{0} \quad \text{on } \partial H \end{cases}$	$\begin{cases} \mathbf{div}\boldsymbol{\rho} \ = \ \mathbf{0} \quad \text{in } V^* \\ \boldsymbol{\rho}.\boldsymbol{n} \quad \text{antiperiodic on } \partial V \\ \boldsymbol{\rho}.\boldsymbol{n} \ = \ \mathbf{0} \quad \text{on } \partial H \\ \langle \boldsymbol{\rho} \rangle \ = \ \mathbf{0} \end{cases}$

The resolution of the static approach of the coupled problem shakedown / homogenization leads thus to search the maximum value of the coefficient α, such that for any given loading path taking its values in $\boldsymbol{\mathcal{D}}$ the nonlinear conditions (yield conditions) and the linear conditions (the one to be satisfied by $\boldsymbol{\rho}$) are satisfied. This becomes thus a constrained nonlinear optimization problem, P^{opt}, in which P^{res} represents either P^{res}_{strain} or P^{res}_{stress}:

Find α_{SD} such that: $\quad \alpha_{SD} = \max\limits_{\boldsymbol{\rho}}(\alpha) \quad$ where $\boldsymbol{\rho}$ satisfies:

$$\mathcal{F}\left(\alpha\, \boldsymbol{\sigma}^{(e)}(\boldsymbol{y}, \boldsymbol{\mathcal{P}}(t)) + \boldsymbol{\rho}(\boldsymbol{y}), \sigma^0(\boldsymbol{y})\right) \leq 0 \quad \forall\, \boldsymbol{y} \ \in \ V^*, \ \forall\, \boldsymbol{\mathcal{P}}(t) \subset \boldsymbol{\mathcal{D}}, \ \forall\, t \qquad [6a]$$

$$P^{res} \qquad [6b]$$

3.3. *Discretization of the coupled problem*

In order to consider unit cells as complex as possible, it is necessary to discretize and to implement this method.

3.3.1. *Discretization of the initial load domain*

The discretization of the initial load domain –the time discretization– is achieved thanks to the following property [KOE 78]: a media shakes down for any loading path included in the polyhedron $\boldsymbol{\mathcal{D}}$ if and only if it shakes down for all the vertices of $\boldsymbol{\mathcal{D}}$. The set $\mathfrak{D}$ of these vertices is defined as follows:

$$\mathfrak{D} = \left\{ P_k^0 \mid P_k^0 = \sum_{i=1}^{n} \mu_i \boldsymbol{\mathcal{P}}_i^0, \quad \text{with } \mu_i = {\mu_i}^- \text{ or } {\mu_i}^+, \;\; k = 1..2^n \right\} \qquad [7]$$

As a consequence, the yield conditions [6a] need to be expressed only for the 2^n vertices P_k^0, which are called *loading points*.

3.3.2. *A specific formulation*

The variational formulations of the localization problems $P_{strain/stress}$ and P^{res} are written by introducing the general formulation [8]:

$$P_{gen} \left\{ \begin{array}{l} \mathbf{div}\, \boldsymbol{s} \;=\; \mathbf{0} \quad \text{in } V^* \\ \boldsymbol{s}.\boldsymbol{n} \quad \text{antiperiodic on } \partial V \\ \boldsymbol{s}.\boldsymbol{n} \;=\; \mathbf{0} \quad \text{on } \partial H \\ \langle \boldsymbol{s} \rangle = \boldsymbol{S} \quad \text{only for} \quad P_{stress} \;\text{ and }\; P^{res}_{stress} \end{array} \right. \qquad [8]$$

in which $\boldsymbol{s}$ is either the microscopic stress $\boldsymbol{\sigma}$ such that $\boldsymbol{\sigma} = \boldsymbol{d} : (\boldsymbol{E} + \boldsymbol{\varepsilon}(\boldsymbol{u}^{per}))$ or the microscopic residual stress $\boldsymbol{\rho}$, and $\boldsymbol{S}$ is either the macroscopic stress $\boldsymbol{\Sigma}$ or the null tensor $\mathbf{0}$. The variational formulation of [8] is then written using the principle of virtual work with a virtual kinematics of type [2], the average relations [1], the periodicity conditions [8b], and the traction free conditions [8c]:

$$\forall\, \delta\boldsymbol{u}^{per},\; \forall\, \delta\boldsymbol{E}, \qquad \int_{V^*} (\boldsymbol{\varepsilon}(\delta\boldsymbol{u}^{per}) + \delta\boldsymbol{E}) : \boldsymbol{s}\; dV \;=\; |V|\, \delta\boldsymbol{E} : \boldsymbol{S} \qquad [9]$$

Using the finite element method and adding to all the elements of the mesh a fictive node [DEB 86], the degrees of freedom of which are the components of the macroscopic strain $\boldsymbol{E}$ and the nodal forces of which are the components of the macroscopic stress $\boldsymbol{\Sigma}$ multiplied by the volume of the unit cell $|V|$, we obtain [MAG 02] the following discrete form: $\forall\, \delta\boldsymbol{u}^{per,e},\; \forall\, \delta\boldsymbol{E}$,

$$\sum_{e=1}^{NELT} \langle \delta\boldsymbol{u}^{per,e}, \delta\boldsymbol{E} \rangle \sum_{i=1}^{NGE} \omega_i [{\boldsymbol{B}_i}^{hom,e}]^T \{\boldsymbol{s}_i\} \; \det \boldsymbol{J}_i = |V| \, \langle \delta\boldsymbol{E} \rangle \; \{\boldsymbol{S}\} \qquad [10]$$

where $NELT$ is the total number of elements of the mesh, e designs the current element, $\delta\boldsymbol{u}^{per,e}$ is the virtual displacements vector of nodes of the element e, NGE the number of Gauss points of the element e, and $[\boldsymbol{B}^{hom,e}]$ replaces the classical matrix of the derivated shape functions:

$$[\boldsymbol{B}^{hom,e}] = [\boldsymbol{B}^e, \boldsymbol{I}] \quad \text{s.t.} \quad \{\boldsymbol{\varepsilon}^e\} = [\boldsymbol{B}^{hom,e}]\, \{\boldsymbol{u}^e\} = [\boldsymbol{B}^e]\, \{\boldsymbol{u}^{per,e}\} + \{\boldsymbol{E}\} \qquad [11]$$

This formulation allows us to write the discrete forms of $P_{strain/stress}$ and P^{res}:

– *Elastic problem:* taking into account the chosen kinematics [2], [10] becomes the following linear system of equations:

$$[\boldsymbol{K}] \; \langle \boldsymbol{u}^{per}, \boldsymbol{E} \rangle^T = \langle \mathbf{0}, |V| \boldsymbol{\Sigma} \rangle^T \qquad [12]$$

where the matrix $[\boldsymbol{K}]$ results from the assembly of the elementary stiffness matrices $[\boldsymbol{K}^{hom,e}]$ such that:

$$[\boldsymbol{K}^{hom,e}] = \sum_{i=1}^{NGE} \omega_i [\boldsymbol{B}_i^{hom,e}]^T [\boldsymbol{d}^e][\boldsymbol{B}_i^{hom,e}] \det \boldsymbol{J}_i \qquad [13]$$

After implementation in the finite element code SIC, applying a macroscopic strain or stress becomes as easy as applying a displacement or a force in a classical finite element code. This problem [12] is thus solved by SIC in order to obtain the purely elastic stress fields $\boldsymbol{\sigma}_i^{(e)}(P_k^0)$, which are the stresses at each integration point i resulting from the loading point P_k^0 and necessary to express the inequality constraints [6a].

– *Residual stresses problem:* the discrete form of P^{res} also is a linear system of equations, which constitutes the equality constraints of P^{opt}:

$$[\boldsymbol{C}]\,\{\boldsymbol{\rho}\} = \boldsymbol{0} \qquad [14]$$

where $\{\boldsymbol{\rho}\}$ is the vector of the residual stresses at all integration points. The $[\boldsymbol{C}]$-matrix is called *equilibrium matrix* and results from the assembly of the elementary equilibrium matrices $[\boldsymbol{C}^e]$ such that:

$$[\boldsymbol{C}^e] = [\omega_{_1} [\boldsymbol{B}_{_1}{}^{hom,e}]^T \det \boldsymbol{J}_1 | \ldots | \omega_{_{NGE}} [\boldsymbol{B}_{_{NGE}}{}^{hom,e}]^T \det \boldsymbol{J}_{_{NGE}}] \qquad [15]$$

The method to build the $[\boldsymbol{C}]$-matrix has also been implemented in SIC, by using an optimized storage methodology in order to treat the maximum of equality constraints.

3.3.3. *Discretization of the optimization problem*

Thanks to these developments in the finite element software SIC, all the data necessary to write the discrete form of P^{opt} are now available. The following nonlinear constrained optimization problem is then solved with the optimization software LANCELOT, dedicated to large scale optimization problems and based on an augmented lagrangian method:

Find α_{SD} such that: $\quad \alpha_{SD} = \max\limits_{\boldsymbol{\rho}}(\alpha) \quad$ where $\boldsymbol{\rho}$ satisfies:

$$\mathcal{F}\left(\alpha\, \boldsymbol{\sigma}_i^{(e)}(P_k^0) + \boldsymbol{\rho}_i, \sigma_i^0\right) \leq 0 \qquad \forall\ k = 1..2^n,\ \ \forall\ i = 1..NGP \qquad [16a]$$

$$[\boldsymbol{C}]\,\{\boldsymbol{\rho}\} = \boldsymbol{0} \qquad [16b]$$

This static approach of the coupling leads thus to a maximization problem with: a linear objective function, $6 * NGP$ optimization variables (NGP is the total number of integration points), $2^n * NGP$ nonlinear inequality constraints (σ_i^0 is the yield stress of the integration point i), and $NDOF$ linear equality constraints where $NDOF$ is the total number of degrees of freedom.

4. The kinematic approach of the coupling

In order to compare the kinematic and the static approach, we formulate here the coupling between periodic homogenization and shakedown by using the kinematic theorem of Koiter [KOI 60]. The definitions of the initial load domain and of the reference unit cell, as well as the assumption on the local material behavior are the same as the ones presented in section 3.1.

4.1. *The coupled problem*

From the kinematic point of view, the unit cell shakes down only if its plastically dissipated energy is strictly superior to the one due to the loading $\boldsymbol{\mathcal{P}}$. The coupling states thus the following [CAR 99]: if there exists $\gamma > 1$ such that for any plastic strains rate $\dot{\boldsymbol{\varepsilon}}^p$ and for any loading path $\boldsymbol{\mathcal{P}}:\ t \in [0,T[\ \rightarrow\ \boldsymbol{\mathcal{P}}(t) \in \boldsymbol{\mathcal{D}}$ satisfying:

$$\int_0^T \dot{\boldsymbol{\varepsilon}}^p(\boldsymbol{y},t)dt = \mathbf{grad_s}\boldsymbol{u}(\boldsymbol{y}) \quad \forall\, \boldsymbol{y} \text{ in } V^* \qquad [17a]$$

$$\text{tr}\ \dot{\boldsymbol{\varepsilon}}^p(\boldsymbol{y},t) = 0 \quad \forall\, \boldsymbol{y} \text{ in } V^*,\ \forall\, t \qquad [17b]$$

$$\boldsymbol{u}(\boldsymbol{y}) = \boldsymbol{E}.\boldsymbol{y} + \boldsymbol{u}^{per}(\boldsymbol{y}) \quad \text{with } \boldsymbol{u}^{per} \text{ periodic on } \partial V \qquad [17c]$$

$$\int_0^T \int_{V^*} \boldsymbol{\sigma}^{(e)}(\boldsymbol{y},t) : \dot{\boldsymbol{\varepsilon}}^p(\boldsymbol{y},t)\ dV\ dt = 1 \qquad [17d]$$

the following relation holds: $$\int_0^T \int_{V^*} \boldsymbol{\sigma}(\boldsymbol{y},t) : \dot{\boldsymbol{\varepsilon}}^p(\boldsymbol{y},t)\ dV\ dt \geq \gamma \qquad [18]$$

then the unit cell shakes down for any loading path in $\boldsymbol{\mathcal{D}}$.

Note that this theorem is enounced for a prescribed macroscopic stress, but holds also for a prescribed macroscopic strain: in this case, the displacement $\boldsymbol{u}(\boldsymbol{y})$ must satisfy [17c] with $\boldsymbol{E} = \langle \boldsymbol{\varepsilon} \rangle = \mathbf{0}$. The purely elastic stress $\boldsymbol{\sigma}^{(e)}$ is likewise solution of either the localization problem P_{stress} or the localization problem P_{strain}.

4.2. *The associated minimization problem*

After having discretized the initial load domain in the same way as presented in section 3.3.1, a convex analysis reasoning based on the duality of both the static and kinematic approaches ([DEB 76]) allows us to deduct the discrete kinematic coupling from the discrete static one. Introducing appropriate vectorial spaces and bilinear forms, this coupling becomes the following constrained minimization problem:

$$\text{Find } \alpha_{SD} \text{ s. t.: } \alpha_{SD} = \min_{\dot{\boldsymbol{\varepsilon}}^p, \boldsymbol{u}^{per}, \boldsymbol{E}} \sum_{i=1}^{NGP} \sum_{k=1}^{2^n} \omega_i \det \boldsymbol{J}_i \sigma_i^0 \sqrt{\frac{2}{3} \dot{\boldsymbol{\varepsilon}}_i^p(P_k^0) : \dot{\boldsymbol{\varepsilon}}_i^p(P_k^0)}$$

where $\dot{\boldsymbol{\varepsilon}}^p$, $\boldsymbol{u}^{per}$, and $\boldsymbol{E}$ satisfy:

$$\sum_{k=1}^{2^n} \dot{\boldsymbol{\varepsilon}}_i^p(P_k^0) = [\boldsymbol{B}_i^{hom,e}] \langle \boldsymbol{u}^{per,e}, \boldsymbol{E} \rangle^T \quad \forall\, i = 1..NGP \qquad [19a]$$

$$\text{tr}\ \dot{\boldsymbol{\varepsilon}}_i^p(P_k^0) = 0 \quad \forall\, i = 1..NGP,\ k = 1..2^n \qquad [19b]$$

$$\sum_{i=1}^{NGP} \sum_{k=1}^{2^n} \omega_i \det \boldsymbol{J}_i\ \boldsymbol{\sigma}_i^{(e)}(P_k^0) : \dot{\boldsymbol{\varepsilon}}_i^p(P_k^0) = 1 \qquad [19c]$$

By comparison with the static approach, the kinematic approach leads to consider a minimization problem with: a more complex objective function (it is indeed nonlinear and non differentiable at the origin), more optimization variables ($6 * 2^n * NGP + NDOF$), no inequality constraints, and $(2^n + 6)NGP + 1$ linear equality constraints (in general $(2^n + 6)NGP + 1 > NDOF$) composed of $6 * NGP$ compatibility constraints, $2^n * NGP$ incompressibility constraints, and one normalization constraint. As emphasized by the formulation [19], the data necessary to express this kinematic coupling are the matrix $[\boldsymbol{B}_i^{hom,e}]$ and the stresses $\boldsymbol{\sigma}_i^{(e)}(P_k^0)$, which have already been computed by the finite element developments of the static approach.

5. Application examples

5.1. *The two studied materials*

	Layer 1	Layer 2
Young's modulus	$2.1 * 10^5$ MPa	$1.2 * 10^5$ MPa
Poisson's ratio	0.3	0.27
Yield stress	$\sigma_0^1 = 180$ MPa	$\sigma_0^2 = 80$ MPa
Thickness ratio	$h_1/h_2 = 1/3$	

Table 1. *Properties of each layer of the material shown figure 1*

In order to validate and apply the proposed methods, we consider two types of materials. The first one is the layered material presented figure 1, which is constituted by the superposition of two layers of different characteristics (*cf.* table 1). This layered material is first submitted to the macroscopic strain state $(E_{11},\ E_{22},\ 0,\ 0,\ 0,\ 0)$ and then to the macroscopic stress state $(\Sigma_{11},\ \Sigma_{22},\ 0,\ 0,\ 0,\ 0)$. For the second type of loading, the static and the kinematic approach are treated. Thanks to the invariance in the in-plane directions, each layer of the unit cell is meshed with a unique quadratic brick element with 20 nodes.

The second studied material is the periodically perforated aluminum sheet shown figure 2 which is submitted to the macroscopic stress state $(\Sigma_{11},\ \Sigma_{22},\ 0,\ 0,\ 0,\ 0)$ and whose mechanical and geometrical characteristics are: $E = 69550\ MPa, \nu = 0.337$,

$\sigma^0 = 159\ MPa$ and $2r/d = 0.3$. Some symmetry considerations allow to work on only one eighth of the unit cell which is meshed with the same quadratic brick elements as the layered material. These symmetries lead ([LEN 84]) to consider the following localization problems:

$$\left\{\begin{array}{ll} \textbf{div}\ \boldsymbol{s} = \mathbf{0} \quad \text{in } V^* & \text{with additional relations for } P_{stress} : \\ \boldsymbol{s}.\boldsymbol{n} = \mathbf{0} \quad \text{on } \partial V_{top} \cup \partial H & \boldsymbol{\sigma} = \boldsymbol{d} : \varepsilon(\boldsymbol{u}), \\ \boldsymbol{s}.\boldsymbol{n} - (\boldsymbol{n}.\boldsymbol{s}.\boldsymbol{n}).\boldsymbol{n} = \mathbf{0} \text{ on } \partial V_{bot} \cup \partial V_{lat} & \boldsymbol{u}^{per}.\boldsymbol{n} = \mathbf{0} \text{ on } \partial V_{lat} \cup \partial V_{bot}, \\ \langle \boldsymbol{s}_{11} \rangle = \boldsymbol{S}_{11},\ \langle \boldsymbol{s}_{22} \rangle = \boldsymbol{S}_{22} & E_{i3} = E_{3i} = E_{12} = E_{21} = 0 \end{array}\right. \qquad [20]$$

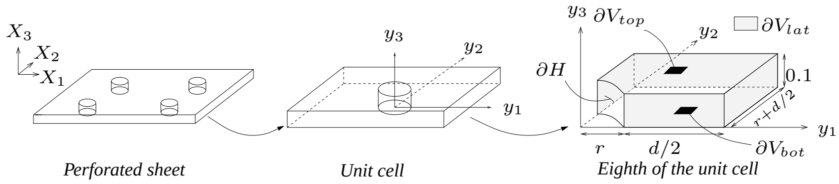

Figure 2. *3D unit cell of a perforated sheet*

5.2. *Numerical simulations*

Because of the symmetry of the von-Mises criterion with respect to the origin, all the results presented in this section are plotted in the half-plane $\mathcal{P}_2 \geq 0$ when the materials are submitted to bi-axial tensile tests in the plane $(\mathcal{P}_1, \mathcal{P}_2)$. For all the following results, the initial load domains are rectangles with one vertex at the origin. Thus, the unit cell shakes down for any variable loading path included in such a rectangle, with the vertex opposite from the origin inside the shakedown envelope.

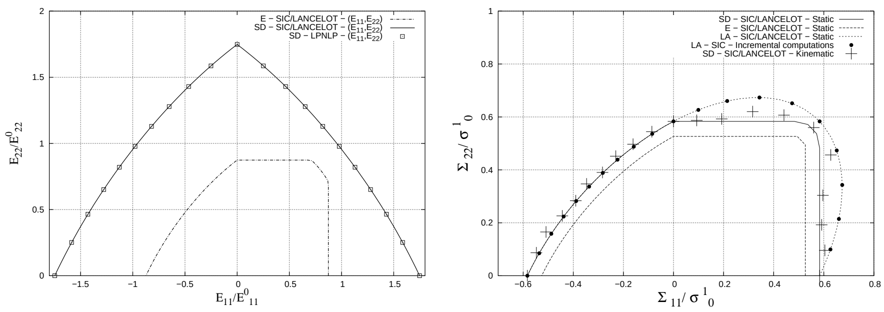

Figure 3. *Layered material computed in the planes (E_{11}, E_{22}) and $(\Sigma_{11}, \Sigma_{22})$ with $E^0_{11} = E^0_{22} = 10^{-3}$*

The results of the numerical simulations on the layered material are presented figure 3 and show the macroscopic strength domains with respect to elasticity (E), shakedown (SD), and limit analysis (LA). The computations in the plane (E_{11}, E_{22}) emphasize a good correlation between the results obtained by the presented static method and the one obtained by a semi-analytical method using LPNLP ([HAC 01], [PIE 75]). It also emphasizes that the collapse loads with respect to shakedown coincide with the ones with respect to alternating plasticity. The computations in the plane $(\Sigma_{11}, \Sigma_{22})$ imply two major comments: firstly, the limit analysis results (considering one loading point in [16a] are in good correlation with the ones issued from incremental computations in SIC ([BOU 98])[1]; and secondly, the shakedown results obtained by the static approach and the ones obtained by the kinematic approach are not totally identical however they should be. This probably comes from the non differentiability of the dissipation function at the origin, which is not treated in LANCELOT. This point should be further investigated following the works of [CAR 99] for example.

The results of the numerical simulations on the perforated sheet are presented figure 4. Concerning the limit analysis results, there is once more a good correlation between the proposed results and the ones obtained by the incremental method developed in SIC. The gap between our limit analysis domain and the one of [DEB 85] emphasizes the influence of the plane stress hypothesis that is assumed in [DEB 85].

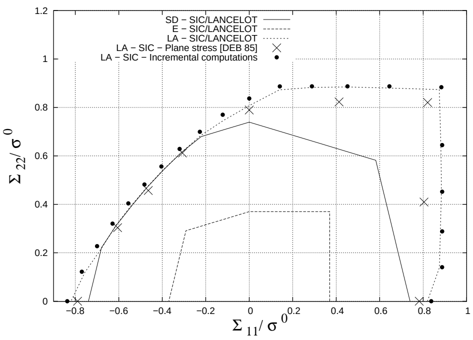

Figure 4. *Perforated sheet computed in the plane* $(\Sigma_{11}, \Sigma_{22})$ *by the static approach*

6. Concluding remarks

The proposed direct numerical method allows to obtain macroscopic admissible strength domains for elastic perfectly plastic, heterogeneous and periodic materials submitted to variable loads. The general nature of the static developments allows to study the plastic collapse due to unlimited plastic dissipation, for any 3D unit cell. The kinematic approach need however to be further investigated in order to take into account the singularity of the dissipation function at the origin.

1. This limit analysis domain has also been correlated with analytical results of the literature ([MAG 04]).

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