Stick-slip-separation waves under frictional and unilateral contact

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ABSTRACT. *The dynamical problem of a brake-like mechanical system composed of an elastic cylindrical tube in frictional contact with a rigid and rotating cylinder is considered in view of the interpretation of the phenomenon of brake squeals. The case of stick-slip-separation waves is considered here by a semi-analytical analysis of the reduced equations and by f.e.m. numerical simulations in order to complete our discussions given in [*MOI 03*] on stick-slip waves.*

RÉSUMÉ. *La réponse dynamique d'un système de cylindres coaxiaux en contact unilatéral avec frottement de Coulomb est discuté ici en vue des interprétations du phénomène de crissement des freins. Les caractéristiques des ondes adhérence-glissement-séparation sont discutées par des moyens analytiques et numériques comme prolongement de nos résultats antérieurs [*MOI 03*] concernant les ondes adhérence-glissement.*

KEYWORDS: *unilateral contact, Coulomb friction, steady sliding response, flutter instability, stick-slip-separation waves, analytical solutions, numerical simulations.*

MOTS-CLÉS : *contact unilatéral, frottement de Coulomb, glissement stationnaire, instabilité par flottement, ondes adhérence-glissement-décollement, solutions analytiques, simulations numériques.*

1. Introduction

It is well known that in the frictional contact of solids a contact point may have a slip or stick or separation regime. The study of the propagation of these zones on contact surfaces may be useful for different applications in statics and in dynamics. For example, the problem of brake noises has been intensively discussed in the literature [NAK 96, MOI 98]. The noise emittence is closely related to the friction-induced periodic vibrations. Different kinds of noises and vibrations can be identified in common brakes following their frequencies. Brake squeals result from high frequency vibration (at least 4000 Hz) and have a relatively pure spectrum composed of a few main frequencies. The source of noise is attributed to the vibration of brake components such as pad or disk and generated by friction. In this paper, an interpretation based on the fact that brake squeal is a consequence of the flutter instability of the steady sliding solution of the brake system is discussed. This instability of the steady sliding response leads to complex nonlinear dynamic responses. In particular, the possibility of dynamic bifurcation to a periodic dynamic response, in the spirit of Poincaré-Andronov-Hopf bifurcation, has been illustrated in a simple example of coaxial and rotating cylinders in frictional contact [MOI 98, MOI 03] . A simple modelling of a drum or a band brake leads to the study of the dynamical problem of a mechanical system composed of an elastic cylindrical tube in frictional contact with a rigid and rotating cylinder. This model problem has enabled us to exhibit the existence of non-trivial periodic solutions in the form of stick-slip waves propagating on the contact surface. The possibility of stick-slip-separation waves is discussed here in the same spirit in order to complete our previous results.

2. The problem of coaxial cylinders

The mechanical response in plane strain of a brake-like system composed of an elastic tube, of internal radius R and external radius R^*, in frictional contact on its inner surface with a rotating rigid cylinder of radius $R + d$ and of angular rotation Ω, is considered when the displacement is assumed to be homogeneous on the outer surface of the tube. Coulomb's law of dry friction is assumed with a constant friction coefficient f. The mismatch $d \geq 0$ is a load parameter controlling the normal contact pressures. The governing equations of the system are

$$\left\{\begin{array}{l}
\epsilon = (\nabla u)_s, \quad \text{Div}\, \sigma = \gamma \ddot{u}, \\
\sigma = \dfrac{\nu}{(1+\nu)(1-2\nu)} Tr(\epsilon)\, I + \dfrac{1}{1+\nu}\epsilon, \\
u(\xi, \theta, t) = v(\xi, \theta, t) = 0, \\
\sigma_{rr}(1, \theta, t) = -p(\theta, t), \quad \sigma_{r\theta}(1, \theta, t) = -q(\theta, t), \\
u \geq \delta, \;\; p \geq 0, \;\; p(u - \delta) = 0, \\
|q| \leq fp, \quad q(1 - \dot{v}) - fp|1 - \dot{v}| = 0,
\end{array}\right.$$

in terms of non-dimensional variables $u = \frac{\bar{u}}{R},\ \ \sigma = \frac{\bar{\sigma}}{E},\ \ r = \frac{\bar{r}}{R},\ \ \gamma = \frac{\rho R^2 \Omega^2}{E},\ \ \xi = \frac{R^*}{R},\ \ \delta = \frac{d}{R},\ \ t = \Omega \bar{t},\ \ \dot{u} = \frac{du}{dt}$. The steady sliding solution is given by $u_e = \delta \frac{1}{\xi^2 - 1}(\frac{\xi^2}{r} - r)$, $v_e = \delta f \frac{1}{\xi^2-1}(\frac{\xi^2}{r} - r)(1 + \frac{1}{\xi^2(1-2\nu)})$, $p_e = \delta \frac{1}{\xi^2-1}\frac{1}{1+\nu}(\xi^2 + \frac{1}{1-2\nu}) > 0$ and $q_e = f p_e$.

The steady sliding response is unstable. The proof of this result can be discussed under the assumption of sliding motions, in the same spirit as in the sliding of elastic layers, cf. [ADA 95] or [MAR 95]. The governing equations are nonlinear because of unilateral contact and friction conditions. Since closed-form dynamical solutions cannot be generated, two complementary approaches are followed here.

3. Semi-analytical approach

An interesting simplification to the problem has been proposed and discussed in [MOI 98, MOI 00B] when the displacement is sought in the form

$$\begin{cases} u = U(\theta,t)X(r), \quad v = V(\theta,t)X(r), \\ X(r) = \dfrac{1}{\xi^2 - 1}(\dfrac{\xi^2}{r} - r). \end{cases} \qquad [1]$$

In this approximation, the following local equations are obtained from the virtual work equation when admissible displacements are restricted to the considered expressions

$$\begin{cases} \ddot{U} - bU'' - DV' + gU = P, \\ \ddot{V} - aV'' + DU' + hV = Q, \\ P \geq 0,\ \ U - \delta \geq 0,\ \ P(U - \delta) = 0, \\ |Q| \leq fP,\ \ Q(1 - \dot{V}) - fP|1 - \dot{V}| = 0, \end{cases} \qquad [2]$$

where $'$ denotes the derivative with respect to θ and

$$\begin{cases} a = \dfrac{\tilde{a}A}{\gamma B},\ \ b = \dfrac{\tilde{b}A}{\gamma B}, \\ g = \dfrac{2\tilde{a} + 2(\xi^2 - 1)\tilde{b}}{\gamma B},\ \ h = \dfrac{2\xi^2 \tilde{b}}{\gamma B}, \\ \tilde{a} = \dfrac{1 - \nu}{(1+\nu)(1-2\nu)},\ \ \tilde{b} = \dfrac{1}{2(1+\nu)}, \\ A = -\dfrac{2\xi^2 \ln \xi}{\xi^2 - 1} + \dfrac{1 + \xi^2}{2} > 0,\ \ B = \dfrac{\xi^4 \ln \xi}{\xi^2 - 1} + \dfrac{1 - 3\xi^2}{4} > 0, \\ D = \dfrac{aC_1 - bC_2}{A},\ \ C_1 = \dfrac{2\xi^2 \ln \xi}{\xi^2 - 1} - 1 > 0, \\ C_2 = -\dfrac{2\xi^2 \ln \xi}{\xi^2 - 1} - 1 + 2\xi^2 > 0. \end{cases}$$

The coupling coefficient D between normal and tangential displacements can be positive or negative according to the values of ν and ξ. Finally, only the non-dimensional displacements on the contact surface $U(\theta, t)$ and $V(\theta, t)$ and the non-dimensional reactions $P(\theta, t)$ and $Q(\theta, t)$ remain as unknowns in the reduced equations.

The steady sliding solution is given by $U_e = \delta$, $V_e = \delta fg/h$, $P = P_e$ and $Q_e = fP_e$. The steady sliding response is unstable for the reduced system. When $f > 0$ and $D > 0$, it has been proved that a small perturbation of the steady sliding solution will lead to an exploding wave in the sense of the implied rotation, and a damping wave propagating in the opposite direction. When $f > 0$ and $D < 0$, the exploding wave propagates in the opposite sense.

It is expected that in some particular situations, there is a dynamic bifurcation of Poincaré-Andronov-Hopf's type. This means that since the steady sliding response is unstable and there is a flutter instability, the perturbed motion may eventually tend toward a periodic response. This transition has been obtained numerically in many examples, cf. [VOL 99]. A first step to clarify this idea is to search for possible periodic dynamic solutions. A periodic solution is sought in the form of a wave propagating at constant velocity:

$$U = U(\phi), \;\; V = V(\phi), \;\; \phi = \theta - ct \qquad [3]$$

where c is the non-dimensional wave velocity, U and V are periodic functions of period $T = \frac{2\pi}{k}$. The physical velocity of the wave is thus $\bar{c} = |c|R\Omega$ and the associated dynamic response is periodic of frequency $|c|k\Omega$. The propagation occurs in the sense of the rotation when $c > 0$. According to the regime of contact, a slip wave, a stick-slip wave, a slip-separation wave or a stick-slip-separation wave can be discussed. The governing equations of such a wave are:

$$\begin{cases} (c^2 - b)U'' - DV' + gU = P, \\ (c^2 - a)V'' + DU' + hV = Q, \\ P \geq 0, \;\; U \geq \delta, \;\; P(U - \delta) = 0, \\ |Q| \leq fP, \;\; Q(1 - \dot{V}) - fP|1 - \dot{V}| = 0. \end{cases} \qquad [4]$$

It has been proved that a slip wave does not exist and the existence of a solution of [4] in the form of stick-slip waves has been discussed in [MOI 98, MOI 00B]. These waves propagate in the sense of the previous exploding perturbed motions, thus opposite to the rotation of the cylinder for $D < 0$, with a frequency and a celerity independent of the rotation velocity Ω. For example, for $\sqrt{E/\rho}$ = 1000 m/s, ξ = 1.25, f = 1, R = 1 m and Ω = 100 rad/s, the celerity is 1255 m/s and the associated frequency is 10045 Hz. The amplitude of the wave is linearly proportional to the rotation Ω. It also increases with the friction coefficient f and decreases with the mismatch d. Thus, for vanishing rotations, the steady sliding solution is recovered as the limit of the dynamic response. The stick-slip solution can no longer be available if the rotation is strong enough and the possibility of stick-slip-separation waves must be considered.

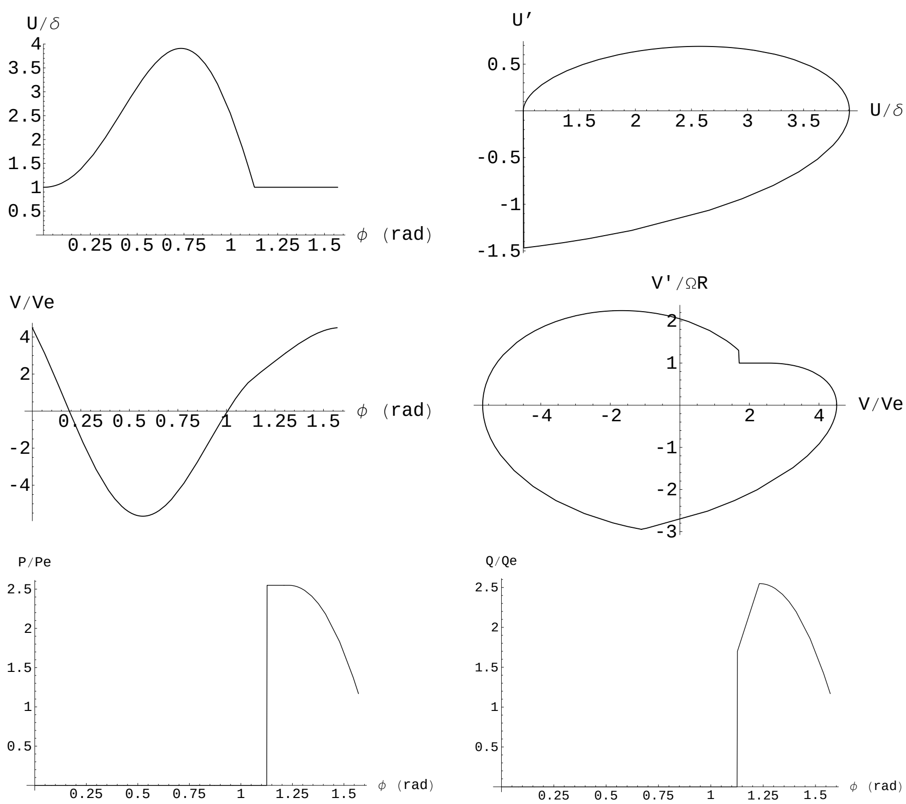

Figure 1. *Displacements U,V, phase diagrams and reactions P,Q for a stick-slip-separation wave, obtained in the semi-analytical approach for* $\nu = 0.3,\ f = 0.7, k = 4,\ \delta = 0.001,\ \Omega = 40 rad/s$

A family of stick-slip-separation waves is sought for, of period $T = 2\pi/k$, and composed of a separation zone for $\theta \in [0, \psi_1 T]$, a stick zone for $\theta \in [\psi_1 T, \psi_2 T]$ and a slip zone for $\theta \in [\psi_2 T, T]$ with $0 < \psi_1 < \psi_2 < 1$. In the slip, stick and separation zones, the governing equations are respectively

$$\begin{cases} U = \delta, \quad -DV' + g\delta = P, \\ (c^2 - a)V'' + fDV' + hV = fg\delta, \\ P \geq 0, \;\; Q = fP, \;\; 1 + cV' \geq 0. \end{cases} \qquad [5]$$

$$\begin{cases} U = \delta, \quad V' = -1/c, \\ P = g\delta + D/c, \;\; Q = hV, \\ P \geq 0, \;\; |Q| < fP. \end{cases} \qquad [6]$$

$$\begin{cases} (c^2 - b)U'' - DV' + gU = 0, \\ (c^2 - a)V'' + DU' + hV = 0, \\ U > \delta, \;\; P = Q = 0, \end{cases} \qquad [7]$$

The continuity of U, V must be ensured at the boundaries $\theta = T\psi_1, T\psi_2, T$ while U', V' must be continuous for $\theta = T\psi_1, T$ only and may be discontinuous at the transition from a separation regime to a stick regime. Finally, a nonlinear algebraic system of three unknowns c, ψ_1, ψ_2 is obtained and can be solved numerically by Mathematica for different given values of f, k, Ω and leads to a solution if all inequality conditions are satisfied. For example, figure 1 gives the displacements U, V, the phase diagrams and the contact reaction obtained when $\nu = 0.3, f = 0.7, k = 4, \delta = 0.001, \Omega = 40 rad/s$. The results are $\psi_1 = 0.716, \psi_2 = 0.784$, wave celerity $1084 m/s$.

4. Numerical approach

The explicit scheme using Lagrange multipliers, as proposed in [CAR 91] for frictional contact, is applied again as in [MOI 03]. At any time step, the velocity and acceleration vectors, $\dot{u}_m$ and $\ddot{u}_m$, are related to displacements and time-increment h following the well known β-method, This prediction step is followed by a correction step when the non-penetration condition is not satisfied. The introduced numerical damping is not a problem in the computation of the limit cycle since the energy loss of the system is compensed continuously by the rotating cylinder. However, this damping accelerates artificially the convergence rate to the limit response. It has been checked that the convergence rate is practically the same for $0.6 \leq \beta_2 \leq 0.9$ and slower for $0.5 \leq \beta_2 \leq 0.6$. On the other hand, the same limit cycles have been obtained with a smaller meshsize. Numerical simulations with various initial data have been performed in order to study the transition to a limit regime which can be a stick-slip or stick-slip-separation wave. It has been found that the limit regime may be different for two different initial conditions. The transition from a given initial state to a limit cyclic response is obtained very quickly, after only about 0.1 sec.

It is checked that a stick-slip-separation wave is effectively obtained when the mismatch is small enough or when the friction is high enough. For example, when Ω = 50 rad/s, δ = 0.0004 and f = 0.7, the limit cycle results as a stick-slip-separation wave. Figure 3 gives the numerical results on the radial displacement U for two mesh sizes using respectively 36 and 132 nodal points on the contact surface instead of a stick-slip wave if $\Omega = 1 rad/s$, cf. figure 2.

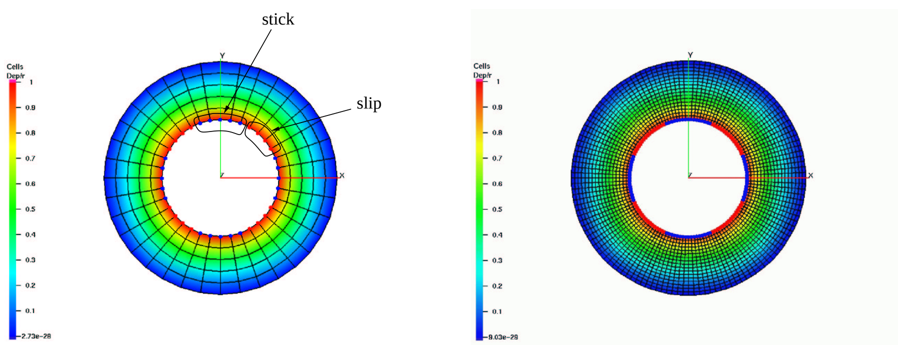

Figure 2. *A stick-slip wave in mode 4, obtained when* $\Omega = 1rad/s$, $\delta = 0.0004$ *and* $f = 0.3$. *The iso-value map of the radial displacement is shown with two mesh sizes*

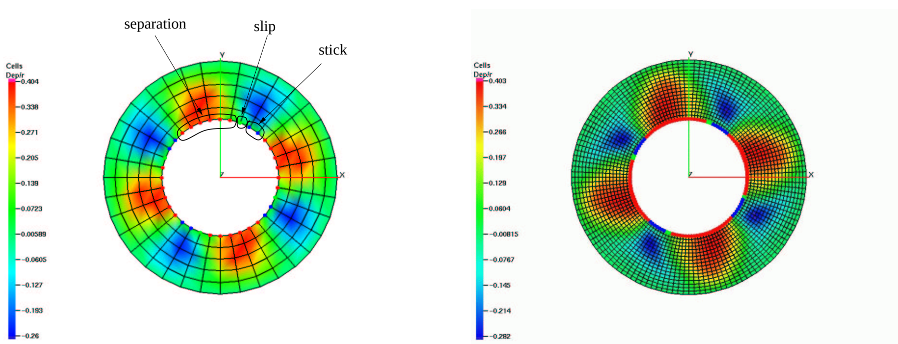

Figure 3. *A stick-slip-separation wave in mode 4, obtained when* $\Omega = 50rad/s$, $\delta = 0.0004$ *and* $f = 0.7$. *The iso-value map of the radial displacement is shown with two mesh sizes*

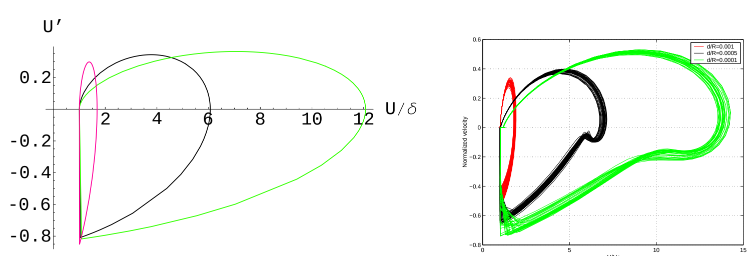

Figure 4. *Phase diagrams of the radial displacement obtained by the semi-analytical and the numerical approches, when* $\Omega = 100rad/s$ *and* $f = 0.3$ *at different load levels* $\delta = 0.0005, 0.001, 0.005$ *(green, black, red lines respectively)*

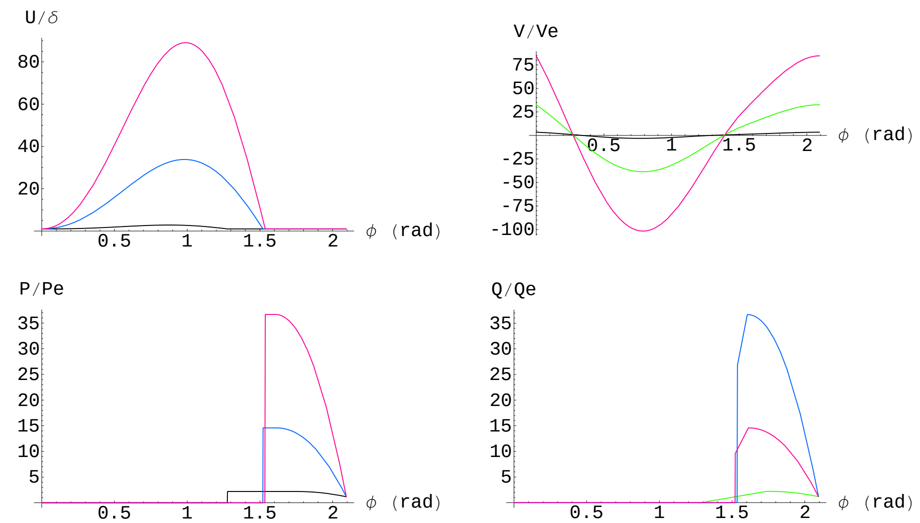

Figure 5. *Influence of the rotation* $\Omega = 32, 380, 1000 rad/s$ *(black, blue, red lines) when* $\delta = 0.001$ *and* $f = 0.9$

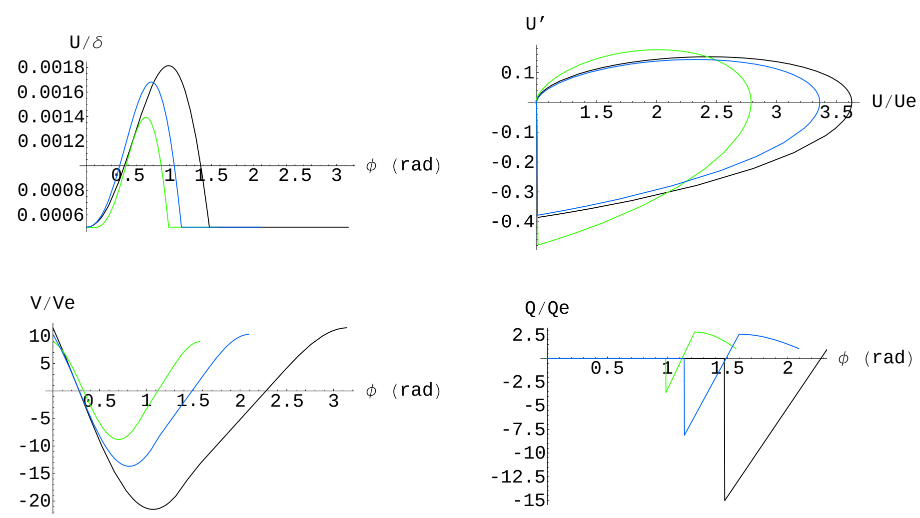

Figure 6. *Influence of the mode number k=2,3,4 (black, blue, green lines) when* $\Omega = 100 rad/s$, $\delta = 0.0004$ *and* $f = 0.7$

The numerical results can be compared to the semi-analytical solution as shown in figure 4 for the phase diagrams of the radial displacement at different values of δ. The influence of the rotation velocity and of the mode number is shown respectively in figures 5 and 6. The reader can also refer to [OUE 04] for more details on the discussions.

5. Conclusion

This discussion emphasizes the fact that the sliding of solids cannot be a smooth motion when the solids are purely elastic without any surface or volume damping mechanism such as the presence of viscosity. In statics, it is well known that the sliding of rubber leads to the formation and the propagation of Schallamach waves, cf. [SCH 71]. In dynamics of solids with frictional contact, our study explores in the same spirit the existence and propagation of stick-slip or stick-slip-separation waves.

The dynamic bifurcation from the steady sliding solution to a dynamic periodic response in the spirit of Hopf bifurcation, as illustrated in this discussion, gives an interesting direction of research for the study of noise emittence problems in various mechanical processes involving frictional contact. The brake squeal problem of car or train brakes has been interpreted in this spirit, cf. [MOI 98, MOI 00B], in particular it has been observed that the flutter modes are very closed to the measured squeal frequencies.

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