Basic ANM algorithms for path following problems

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ABSTRACT. *The Asymptotic Numerical Method (ANM) is a family of algorithms based on the computation of series to solve non-linear problems. How to use at best the informations included in the series ? How to define in an optimal way prediction-correction algorithms within ANM ? In this paper, the knowledges about these questions are reviewed. A complete bibliography is also presented, that shows a wide application field and the many variants offered by ANM algorithms.*

RÉSUMÉ. *La Méthode Asymptotique-Numérique (MAN) est une famille d'algorithmes basés sur le calcul de séries entières pour la résolution de problèmes non-linéaires. Comment utiliser au mieux les informations contenues dans une série ? Comment définir de façon optimale des algorithmes de prédiction-correction dans le cadre de la MAN ? Dans cet article, nous discutons l'état de l'art sur ces deux questions. Nous présentons aussi une étude bibliographique complète montrant un vaste champ d'application et les nombreuses possibilités offertes par la MAN.*

KEYWORDS: *Asymptotic Numerical Method, continuation, bucking, thin shells, prediction-correction.*

MOTS-CLÉS : *Méthode Asymptotique Numérique, continuation, prédiction-correction, flambage, coques minces.*

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1. Introduction

The Asymptotic Numerical Method (ANM) has proved to be an efficient method to compute solution paths of non-linear partial differential equations. It is based on the calculation of truncated series of vectors with respect to a scalar control parameter (perturbation technique). In this framework, the radius of convergence of the series and the range of validity of the obtained approximations are deduced from the series. So the step lengths are computed *a posteriori* and the procedure defines a completely adaptive computational stategy. Likely, this is the main advantage of ANM. Another advantage is the low computational cost, because the steps are generally large and only one tangent matrix triangulation is necessary to compute the whole series. Moreover, the ANM provides an explicit and accurate expression of the solution branch, contrarily to incremental-iterative approaches, such as Newton-Raphson algorithm, that lead to point by point descriptions of the branch. To our knowledge, Thomson and Walker [THO 68] were the first ones to associate perturbation technique and finite element method. Several papers were published in the seventies and the eighties [GAL 75], [KAW 76], [YOK 76], [RIK 84], the most significant contributions being due to Noor et al [NOO 81a], [NOO 81b], [NOO 85]. In the algorithms that were proposed in this period, the perturbation technique was used to build up, first a reduced basis, next an approximation of the branch that was the predictor part in a prediction-correction strategy. As analysed in [RIK 84], an efficient and general computational technique was not obtained in such a way. In this respect, the crucial points were, first the introduction of simple and efficient expansion procedures to reduce the computational cost [DAM 90],[AZR 92], [AZR 93], second some definitions of the step length from an *a posteriori* analysis of the series [COC 94b], [COC 94a], [GER 02] to get an automatic continuation algorithm, last the substitution of the truncated series by rational fractions, that are called Padé approximants [AZR 92], [COC 94c], [DEB 97], [NAJ 98], [ELH 00], [JAM 01].

Let us underline that ANM algorithms work well without any correction phase. Indeed, if the end of the step length is clearly inside the radius of convergence, the residual remains small so that the solution is very accurate along the path, even after many continuation steps. Nevertheless, it is important to foresee possible correction phases to control the quality of the solution at the step ends and to optimise the computational time. It is natural to associate a high order corrector with the high order ANM-predictor. Such high order correctors have been introduced in [DAM 99], [MAL 00], [CAD 04], via some homotopy transformations. Two classes of correction algorithms can be defined according to the matrix to be inverted : a high order Newton algorithm that requires the inversion of a new tangent matrix or algorithms using a preconditioning matrix, without any new matrix triangulation. Several predictor-corrector algorithms have been presented [LAH 02], [AGG 03], [KES 01] that are able to improve the reliability and the efficiency of ANM. In this paper, we rediscuss these ANM-strategies that can be considered as optimal in the present state of knowledge.

Many tests have established the efficiency of ANM for geometrically non-linear shell problems, especially for the statics of elastic plates and shells [LOP 98], [LOP 01], [AZR 93], [COC 94b], [COC 94a], [COC 94c], [ELH 00], including large rotations [AMM 96], [ZAH 99], [LOP 00], for non-linear vibrations [AZR 98], [AZR 99], [AZR 02], in fluid mechanics [KAW 76], [MOR 95], [TRI 96], [HAD 97], [CAD 01a], [ALL 02].

With the help of bifurcation indicators, ANM is a very efficient framework to find bifurcation points and to compute bifurcating branches [VAN 98], [BAG 03], [BOU 03] or to detect Hopf bifurcation [BEN 95], [CAD 97].

Some applications of high order iterative algorithms for linear vibrations have also been achieved to compute eigenvalue sensitivity [CHE 00] or the modal damping of viscoelastic sandwich shells [DAY 01], [DUI 03].

Theoretically, the perturbation technique can not be used for non-smooth problems, such as unilateral contact. Nevertheless, many interesting applications have been achieved, by smoothing the non-smooth problems and with the help of specific tricks to simply define the algorithms to compute the series [POT 97]. This has been applied for unilateral contact [ELH 98], [AGG 03], for plastic bodies [BRA 97], [ZAH 98] and this has permitted to consider simple benchmarks of sheet metal forming [ABI 02], sometimes with a viscoplastic constitutive law and friction [BRU 99].

Time dependent partial differential equations involve also non-smooth responses because their solutions are not analytic in time. To our knowledge, there is no very satisfactory ANM algorithm for instationary problems. Nevertheless, time-series have been applied for very small size problems [FAF 97], [BER 01] and high order iterative algorithms have permitted to reduce the number of matrix inversions in the solution of discretised partial differential equations [COC 00], [JAM 02].

Is it necessary to modify the basic ANM strategy for large scale problems ? As in other frameworks, the stakes are to limit the increasing computational time due to matrix inversion, to avoid memory overflow induced by the storage of triangulated matrices and to build up algorithms adapted to parallel computers. The subdomain concept can provide answers to these challenges[ESS 03]. Subdomain splitting can be associated with an efficient partially iterative solver, as FETI algorithm [GAL 00]. The reduced subspace technique can also be an useful way to limit the cost due to the treatment of tangent matrices [IMA 01]. Note that ANM has led to define linear solvers [CAD 01b], [ELM 02].

In this paper, the best existing ANM algorithms for path following problems are revisited. To our point of view, the optimal algorithm involves, first the computation of a rather large number of terms in a series, whose convergence is accelerated by Padé approximants, next a continuation method [ELH 00]. Moreover, to increase the reliability, this path representation has to be associated with high order iterative correctors at some step ends. The obtained predictor-corrector techniques are similar as those presented in [LAH 02]. A new corrective algorithm is also presented that works well without any new matrix inversion.

2. The Asymptotic-Numerical Method

Let us consider the following class of non-linear quadratic problems:

$$R(U,\lambda) = L(U) + Q(U,U) - \lambda F = 0 \tag{1}$$

where $L(\bullet)$ and $Q(\bullet,\bullet)$ are linear and quadratic operators, F is a given vector and $R(U,\lambda)$ is the so-called residual vector. In the previous equation, the unknowns are (U,λ). The non-linear problem [1] can represent either fluid motion equations [CAD 01a] or thin elastic shell equilibrium equations [COC 94b]. In the first case, the couple of unknowns (U,λ) includes the velocity, the pressure and the Reynolds number λ and for the second case the couple (U,λ) is the displacement, the stress tensor and the load parameter. In this paper we limit ourselves to the framework [1], but studies about more difficult problems can be found for instance in [POT 97], [ZAH 98], [ELH 98], [ABI 02].

2.1. *The perturbation method*

The principle is simply to expand the unknowns (U,λ) with respect to a path parameter $'a'$:

$$\begin{cases} U = U_0 + aU_1 + a^2U_2 + \cdots + a^nU_n \\ \\ \lambda = \lambda_0 + a\lambda_1 + a^2\lambda_2 + \cdots + a^n\lambda_n \end{cases} \tag{2}$$

where (U_0,λ_0) is a known and regular solution of equation [1] and n is the order of truncature of the series. The definition of the path parameter leads to a well-defined problem. Various choices are possible, the most convenient is the pseudo-arc length definition [NAJ 98],[COC 94b] :

$$a = \frac{1}{s^2}[< u - u_0, u_1 > + (\lambda - \lambda_0)\lambda_1] \tag{3}$$

where s is a scalar parameter, generally chosen equal to 1 (its value does not affect the asymptotic expansions [2]) and $< \bullet,\bullet >$ is a scalar product. The polynomial approximations [2] are introduced into [1] and [3]. Equating like powers of $'a'$, we obtain a set of recurrent linear equations for unknowns (U_i,λ_i):

$$\begin{cases} \text{Order } 1 & \begin{cases} L_t(U_1) = \lambda_1 F \\ < u_1, u_1 > + \lambda_1^2 = s^2 \end{cases} \\ \\ \text{Order } 2 & \begin{cases} L_t(U_2) = \lambda_2 F - Q(U_1,U_1) \\ < u_1, u_2 > + \lambda_1\lambda_2 = 0 \end{cases} \\ \cdots \\ \text{Order } n & \begin{cases} L_t(U_n) = \lambda_n F - \sum_{r=1}^{n-1} Q(U_r, U_{n-r}) \\ < u_1, u_n > + \lambda_1\lambda_p = 0 \end{cases} \end{cases} \tag{4}$$

where $L_t(\bullet)$ is the tangent operator defined as $L_t(\bullet) = L(\bullet) + Q(\bullet, U_0) + Q(U_0, \bullet)$.

2.2. *The discretization technique*

The next step is the computation of the unknowns (U_i, λ_i) with the help of a discretization technique. The finite element method is the most used technique but others discretization techniques such as the finite volume method have been used [ALL 02]. So previous linear equations are written under the following discrete form :

$$\begin{cases} \text{Order } 1 & \begin{cases} [K_t(U_0)]\{U_1\} = \lambda_1\{F\} \\ {}^t\{u_1\}\{u_1\} + \lambda_1^2 = s^2 \end{cases} \\ \text{Order } 2 & \begin{cases} [K_t(U_0)]\{U_2\} = \lambda_2\{F\} - \{FQ(1)\} \\ {}^t\{u_1\}\{u_2\} + \lambda_1\lambda_2 = 0 \end{cases} \\ \dots & \\ \text{Order } n & \begin{cases} [K_t(U_0)]\{U_n\} = \lambda_n\{F\} - \{FQ(n)\} \\ {}^t\{u_1\}\{u_n\} + \lambda_1\lambda_n = 0 \end{cases} \end{cases} \tag{5}$$

Thus equations [5] define the discretized problem at each order n. One can note that only one tangent matrix $[K_t(U_0)]$ triangulation is needed to determine the U_i's and λ_i's. In fact at each order only the right-hand sides of equations [5] needed to be evaluated. In the next section, we show that there exists an optimal order of truncature n which corresponds about to the identity between the CPU time needed to triangulate the tangent matrix and the CPU time needed to built the right-hand sides $FQ(n)$. So this optimal order is characterized by

$$\text{CPU}(K_t) \approx \text{CPU}(FQ(n)) \tag{6}$$

Once the solution (U_i, λ_i) is computed up to a chosen order of truncature n, the polynomial representation leads to a part of the solution curve. To enlarge the validity of the asymptotic expansions, one can use Padé approximants. The basic idea is to replace the polynomial approximation [2] by an equivalent rational one, in the following form:

$$\begin{cases} U_n^P(a) - U_0 = \sum_{k=1}^{n-1} \frac{P_{(n-1-k)}(a)}{Q_{(n-1)}(a)} a^k U_k \\ \lambda_n^P(a) - \lambda_0 = \sum_{k=1}^{n-1} \frac{P_{(n-1-k)}(a)}{Q_{(n-1)}(a)} a^k \lambda_k \end{cases} \tag{7}$$

where P_k and Q_k are polynoms of degree k. One can remark that all the fractions in [7] have the same denominator Q_k. This representation has been tested and evaluated in [BRA 97] and [NAJ 98] and applied in many cases. As compared with a previously proposed representation, this procedure avoids the presence of too many roots of the

denominators and this increases the reliability of the process. For additional informations and discussions, refer to [JAM 01] and to the paper of the same authors in the present volume.

2.3. *The continuation techniques*

When an approximation [2] or [7] of the solution path is known, it is rather easy to define its range of validity, without evaluating the residual along the path. In the case of series, a very simple criterion has been proposed by Cochelin [COC 94a]. The idea is to require that the last term of the series is small as compared to the first one, i.e. that their ratio is smaller than a chosen parameter δ. This leads to the following definition of the step length:

$$a_{\text{maxS}} = \left[\delta \frac{\|u_1\|}{\|u_n|} \right]^{\frac{1}{n-1}} \tag{8}$$

There are other methods to estimate the range of validity of series, that can be based on estimates of the residual [BAG 03] or on the location of the smallest pole of the analytic function [GER 02].

In reference [ELH 00], a similar idea has been proposed to define the range of validity of the rational fractions [7] as in [8]. This leads to the following definition of the maximal value of the path parameter 'a_{maxP}':

$$\delta_1 = \frac{\|U_n^p(a_{\text{maxP}}) - U_{n-1}^p(a_{\text{maxP}})\|}{\|U_n^p(a_{\text{maxP}}) - U_0\|} \tag{9}$$

where the small parameter δ_1 plays the same role as the parameter δ in expression [8]. It governs the size of the step lengths and the accuracy of the computed solution.

The relations [8] or [9] define a completely adaptive step length, because they are deduced from the computed series and consequently account for the local non-linearity of the response curve. The efficiency of such continuation algorithms has been tested in many cases ; for instance, for the procedure with Padé approximants, see [ELH 00, BRU 99, CAD 01a, ABI 02, LAH 02, AGG 03].

Because the solution path is explicitly known, a variety of *a posteriori* analyses are possible at a low computational cost. The first class of *a posteriori* analysis is based on the poles of the analytic functions [7], that are the roots of the denominator $Q_{(n-1)}(a)$. As it is known in the literature about Padé approximants, the numerators and the denominators may have very closely spaced roots, that are called "defects". A detailed procedure has been proposed to drop these defects from the final solution [ELH 00]. The smallest pole is also a measure of the radius of convergence and it can be obtained from formula [7] or from the technique presented in [GER 02]. In problems involving a bifurcation point, the presence of a pole can be considered as an indicator to localize the bifurcation point [BOU 03]. The second class of *a posteriori*

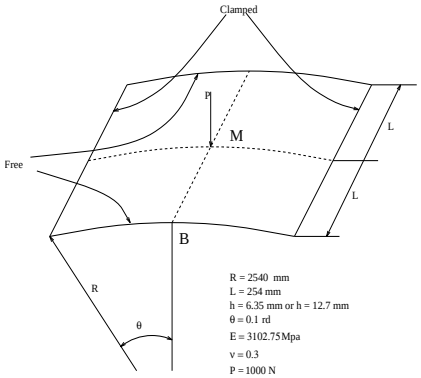
analysis is more classical and it relies on the computation of the residual. As explained in Part 5, this allows to define relevant correction procedures.

Finally, let us underline the importance of the user parameter δ_1. If it is chosen very small (10^{-7} or less), the step lengths will be short and the residual very small. If it is chosen rather large (10^{-3}, 10^{-2}), the step lengths become larger, but the quality of the solution will be generally not satisfactory at the end of the step and a correction phase will be needed.

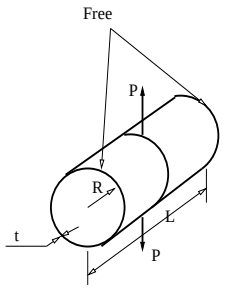
3. Numerical tests for ANM without correction

3.1. *Two benchmarks*

In this section, some practical finite element studies are performed to discuss how to manage an ANM computation. Two main parameters are offered to the user to optimize it : the order of truncature n and the parameter δ_1 that controls the size of the step lengths. Furthermore, the contribution of the Padé approximants will be presented. We consider two numerical tests. The first one is the classical cylindrical roof loaded by a single force. The second one is a cylindrical shell subjected to pinched loads and with free edges at both ends. This pinched cylinder, that involves large elastic rotations and a quasi-bifurcation, had never been studied by ANM. The geometrical and material data for these two shells are presented respectively in figures 1(a) and 1(b). Only a quarter of the cylindrical roof and an octant of the pinched cylinder are discretized due to the symmetry of geometry, loading and boundary conditions.



(a) Characteristics of the cylindrical roofs



R = 4.953 in.
L = 17.335 in.
t = 0.094 in.
E = 1.05 * 10^7 psi

(b) Characteristics of the pinched cylinder

Figure 1. *Description of the geometrically non-linear shell problems*

		Polynomial	Padé approximants
Cylindrical roof $W_{\max}$	Order 10	22	13
	Order 15	16	9
	Order 20	14	8
Pinched cylinder $\frac{P_{\max} R}{D}$	Order 10	68	27
	Order 15	52	20
	Order 20	38	20

Table 1. *Number of steps with series [2] or Padé approximants [7]. Two tests are considered. Control parameters* $\delta = \delta_1 = 10^{-5}$

For these two examples, the load-displacement curves are shown in figure 2. For the cylindrical roof the computations have been done up to a displacement at the loaded point greater than 30, whereas for the pinched cylinder a final load $\frac{PR}{D}$ greater than 120 is required. The finite element used in these two cases has been already presented in reference [ZAH 99]. It is a quadrilateral element with eight nodes for the displacement and using an additional variable to account for the strain variations throughout the thickness (EAS concept). The mesh of the cylindrical roof is composed of 9 elements, which leads to 240 dof. The computation of the pinched cylinder needs a fine mesh, so we used a mesh with 900 elements and nearly 17000 dof. In a specific case (thin roof, h=12.7 mm), the classical triangular DKT18 element has been used and the 8-nodes quadrangle in any other case.

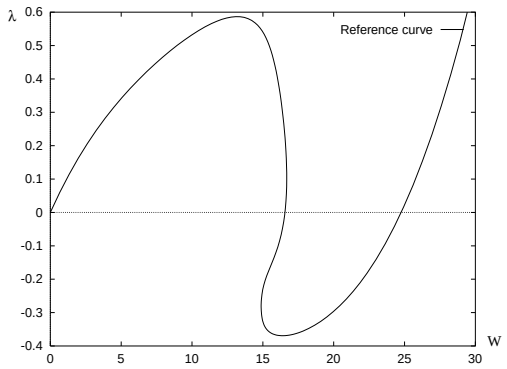
3.2. *Series or Padé approximants*

In table 1, the number of steps to get the two required response curves is presented, for various orders (10, 15, 20) and for the two representations (polynomial [2] or rational [7]). Clearly, the Padé approximants reduce strongly the number of steps, with a ratio about 50 % . A similar ratio has been got for many problems, see for instance [ELH 00], [BRU 99], [ABI 02]. Because the cost to get the fractions [7] from the series [2] is relatively small, a reduction of about 50 % in terms of CPU time has generally been obtained.

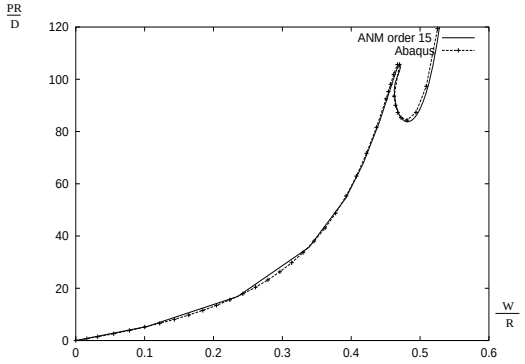
Another important feature of ANM is the decrease of the number of steps when the order of truncature increases. This implies that a relatively large truncature order is recommended, as long as the gain in terms of number of steps is not counterbalanced by an increasing cost due to new terms of the series.

3.3. *Computation time, optimal order of truncature*

The order of truncature is the first user parameter in ANM. The best choice for this order is associated with an optimisation of the computation time. Let us suppose that



(a) Thin cylindrical roof (h=6.35mm)



(b) Reference curve of the pinched cylinder

Figure 2. *Load-displacement curves at the loaded point of the geometrically non-linear shell problems*

the order n is lessened : hence, on one hand the steps become shorter and more steps are needed, but on the other hand, less terms of series are calculated and this reduces the computation time of each step. Generally this balance effect can be summarized by a curve as in figure 3.

To explain this effect, let us first analyse the computation time of one step, according to the order. This CPU time is then:

$$\text{CPU(1 step)} = \text{CPU}(K_t) + \text{CPU}(FQ(n)) \tag{10}$$

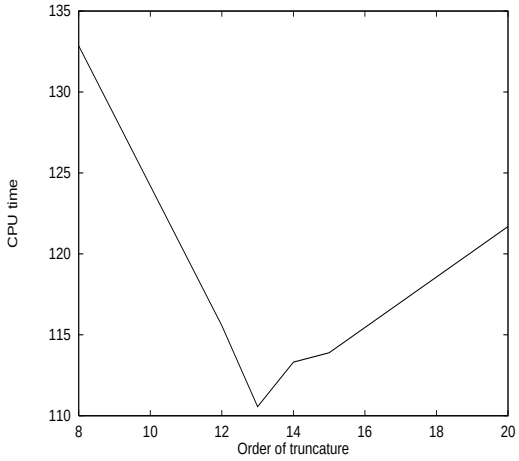


Figure 3. *A typical curve CPU time versus truncature order, from [ABI 02]. It concerns a hemispherical deep drawing process, with 5766 dof. Here the optimal order is 13*

In this respect, let us define the relative CPU time, as the ratio between the time needed to get the series (including the computation of the r.h.s. and the backward and forward substitutions) and the time of treatment of the tangent matrix (including its computation and triangulation by a direct method). So expression [10] leads to:

$$\text{CPU}^{\text{rel}}(1 \text{ step}) = 1 + \frac{\text{CPU}(FQ(n))}{\text{CPU}(K_t)} \tag{11}$$

The CPU-times $\frac{\text{CPU}(FQ(n))}{\text{CPU}(K_t)}$ are plotted in figure 4, as a function of the order. Several cases are presented, with two different physical problems and various finite element meshes,from 240 to 20 000 dof.

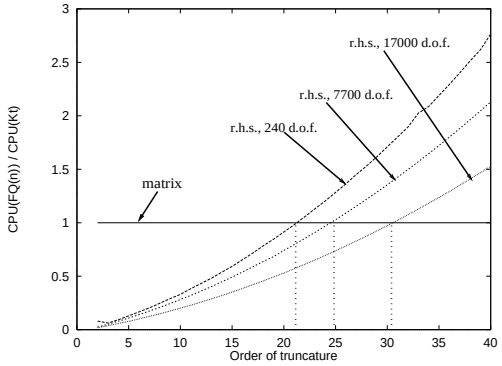
From this figure, one remarks that the cost of the series varies more or less linearly with the order (in fact it is slightly superlinear). Of course, this property depends on a proper implementation of ANM. Hence, the relative CPU cost of one step [11] can be approximated by the following formula:

$$\text{CPU}^{\text{rel}}(1 \text{ step}) = 1 + \frac{n}{\alpha} \tag{12}$$

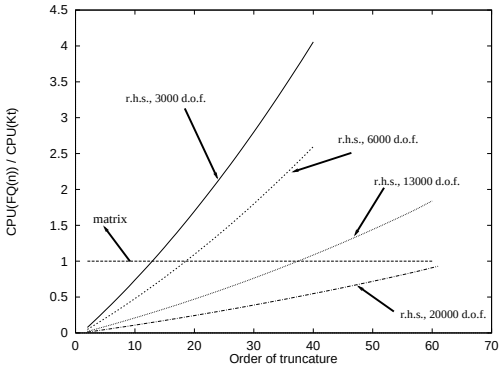
and the total cost of the whole response curve by:

$$\text{total cost} = (1 + \frac{n}{\alpha}).\text{N}_{\text{step}}(n) \tag{13}$$

where $\text{N}_{\text{step}}(n)$ is the number of steps. So a rough analysis of the computational cost can be summarized by two quantities: first the number α that corresponds to the order such that the cost of the series becomes equal to the cost of the matrix ($20 \leq \alpha \leq 30$



(a) Numerical cost of a series. Case of a symmetric tangent stiffness matrix. Shell problems of figure 2



(b) Numerical cost of a series. Case of a non-symmetric tangent matrix. Navier-Stokes equations are discretized as in [CAD 01a]

Figure 4. *Evolution of the relative CPU time to build the rhs FQ versus the order of truncature*

for the solid cases of figure 4(a), $12 \leq \alpha \leq 60$ for the fluid cases of figure 4(b)). This typical order α is generally related to the size and to the nature of the problem and of the discretization: for instance , it is large for large scale problems, and it can decrease for intricate non-linearities. The second quantity that influences the total cost is the decreasing function $\mathrm{N}_{\mathrm{step}}(n)$. The latter can only be obtained *a posteriori*. Generally it depends on the non-linearity of the response curve. Practical minimizations of the total cost have been presented in many cases in the literature, for series and Padé approximants. Generally the optimal order is not very different from the order α

	Order 10	Order 15	Order 20
α = 10	13 × 2 = 26	9 × 2.5 = 22.5	8 × 3 = 24
α = 50	13 × 1.2 = 15.6	9 × 1.3 = 11.7	8 × 1.4 = 11.2

Table 2. *Relative computational cost estimated from formula [13]. Cylindrical roof, Padé, $N_{step}(n)$ given by table 1*

appearing in [12]. For instance, let us compare the relative computational costs, table 2, that are estimated from the number of steps given in table 1 (roof case, Padé), for two values of α: α = 10,50. For the smaller value of α (10), the optimal order is 15, while it becomes 20 for the larger value of α (50), see table 2.

3.4. *Short steps or large steps ?*

The second important parameter when computing non-linear branches with ANM is the parameter δ_1 (or δ) which governs the step length but also the accuracy of the computed solutions. In figure 5 is plotted the logarithm of the residual norm versus the step number, for various values of parameter δ_1. On one hand, one can see in this figure that when the parameter δ_1 is great (for example $\delta_1 = 10^{-2}$), the number of

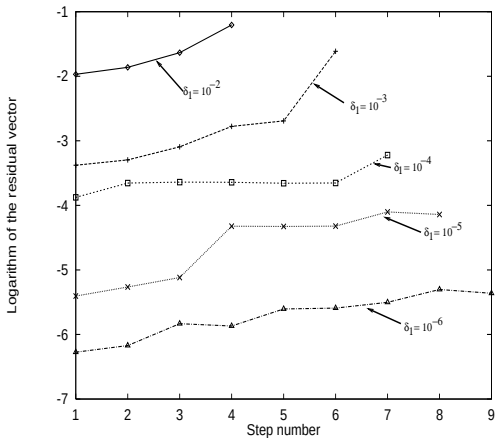


Figure 5. *Thin cylindrical roof, DKT18 element. Evolution of the logarithm of the residual vector versus the step number and for various values of parameter δ_1. Order of truncature 15*

steps to get the required displacement ($w = 30$ at the loaded point) is small (4 steps). Nevertheless the obtained solution is not very accurate. On the other hand if a small value of parameter δ_1 is chosen ($\delta_1 = 10^{-6}$), the solution is very accurate but the

number of steps is greater (9 steps). In practice, a given accuracy is required along the step or at the end of the step. For instance, with the current requirement (residual $\leq 5.10^{-3}$), the curve obtained with $\delta_1 = 10^{-2}$ is not satisfactory and corrective iterations should be scheduled at each step end. With the same requirement, only one correction is needed for $\delta_1 = 10^{-3}$ and none for δ_1 smaller than 10^{-4}. Unfortunately, this "optimal" value $\delta_1 = 10^{-4}$ cannot be known *a priori* [KES 01]. This is why it is preferable to associate a correction scheme with the continuation procedure.

4. High order correctors

If necessary, correction phases are introduced at some step ends. These corrections could be performed by traditional Newton algorithms, as done in [VAN 98, BRU 99]. Nevertheless, it is better to use high order iterative algorithms, that have been introduced recently [DAM 99, MAL 00, CAD 04].

Let (U^1, λ^1) be a given trial solution of [1]. In this point, the corresponding residual vector is greater than a chosen parameter δ_2:

$$\| R(U^1, \lambda^1) \| \geq \delta_2 \tag{14}$$

Otherwise, the correction is not done.

One can define iteratively approximed solutions (U^p, λ^p), in such a way that at the iteration $(p+1)$, the correction $(\Delta U^p = U^{p+1} - U^p, \Delta\lambda^p = \lambda^{p+1} - \lambda^p)$ should satisfy exactly the following equation:

$$R(U^{p+1}, \lambda^{p+1}) = L_t^p(\Delta U^p) + Q(\Delta U^p, \Delta U^p) - \Delta\lambda^p F + R^p = 0 \tag{15}$$

where $L_t^p(\bullet) = L(\bullet) + Q(U^p, \bullet) + Q(\bullet, U^p)$ and R^p are respectively the tangent operator and the residual vector defined at point (U^p, λ^p).

4.1. *High order iterative algorithms*

To compute the correction, one considers a homotpoty transformation, which depends on a real parameter 'ϵ' ($0 \leq \epsilon \leq 1$). So equation [15] is modified, by introducing the homotopy parameter ϵ:

$$(1-\epsilon)L^*(V^p) + \epsilon\,[L_t^p(V^p) + R^p] + Q(V^p, V^p) - \mu^p F = 0 \tag{16}$$

where the L^* is an arbitrarily chosen operator (preconditioner).

By this way, the correction (V^p, μ^p) passes continuously from 0 for $\epsilon = 0$ to the solution of [15] for $\epsilon = 1$, with $(\Delta U^p = V^p(\epsilon = 1), \Delta\lambda^p = \mu^p(\epsilon = 1))$. There is one unknown over in [15]. For instance, one can require $\mu^p = 0$ to seek a solution for the same value of the parameter, $\lambda = \lambda^1$. Within path following problems, it is better to allow a variation of parameter λ during the iteration [MAL 00]:

$$< V^p, U^{'} > + \mu^p \lambda^{'} = 0 \tag{17}$$

where $(U^{'}, \lambda^{'})$ is the slope at the end of the predictor step. Relation [17] is similar to the classical arc-length algorithms.

Next one searches the unknown (V^p, μ^p) by the perturbation method, the perturbation parameter being 'ϵ'. As compared with the prediction presented in section 2, the step end is fixed and equal to $\epsilon = 1$. If for $\epsilon = 1$, the corresponding residual of the solution is greater than a fixed parameter, for example δ_3:

$$\| R(U^{p+1} = U^p + V^p(\epsilon = 1), \lambda^{p+1} = \lambda^p + \mu^p(\epsilon = 1)) \| \geq \delta_3 \qquad (18)$$

then a new iteration is performed.

As established in [DAM 99, MAL 00], the iterative process is much more efficient if the series is replaced by Padé approximants, in the same way as for the prediction: this rational representation will be used in the exemples.

If $L^* = L_t^p$ and the expansion order is 1, the conventional Newton iterative process is recovered. Hence, this procedure with $L^* = L_t^p$ can be called high order Newton algorithm. In the same way, if the preconditioner is the tangent matrix at the first iteration ($L^* = L_t^1$), one defines a high order modified Newton algorithm. In the practice, these two procedures do not differ significantly, because the process often converges after a single iteration. Note that such iterative processes can solve any non-linear system and not only path following problems, see for instance [COC 00, JAM 02, DUI 03].

4.2. *A first application of the corrective algorithms*

Let us consider the thick cylindrical roof, i.e. the problem described in figure 1(a), but with a thickness 12.7 mm. It is discretized by DKT18 elements. The sought response curve is pictured in figure 6, as well as a prediction curve computed from $U_0 = 0$, $\lambda_0 = 0$ at order 25. A trial solution is chosen on the latter curve, that lies much beyond the validity range of the prediction. From this trial solution, one iteration has been performed with two corrective algorithms: the high order Newton algorithm ($L^* = L_t^p$) and the L^*-algorithm [16] where the preconditioner is the tangent operator at the initial point $U_0 = 0$. From figure 7, it appears that the first one converges rapidly (at order 8 or 10), while the second one does not converge after one iteration. Hence, the high order Newton algorithm is very robust and efficient, because the convergence has been obtained with a single matrix decomposition. Is it possible to obtain this convergence without any new matrix triangulation ? The reader is referred to [MAL 00], where it has been established that such iterative sequences converge at most after 2 iterations at order 15 with the L^*-algorithm.

Hence two main classes of high order iterative processes have been defined. First, the high order Newton algorithm is very efficient and very robust and it generally needs only one matrix triangulation. Next, there are many other iterative algorithms, depending on the preconditioner, that does not require any new matrix triangulation.

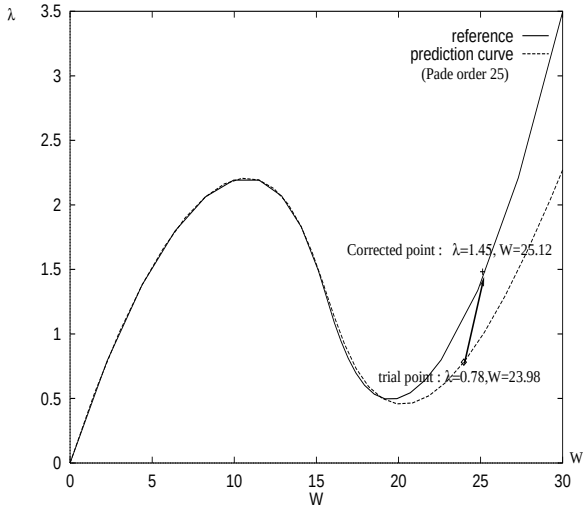


Figure 6. *Response curve of the thick cylindrical roof. The trial and corrected points. Residual vector at the trial point is :* $log_{10}(R^p) = 0.78$

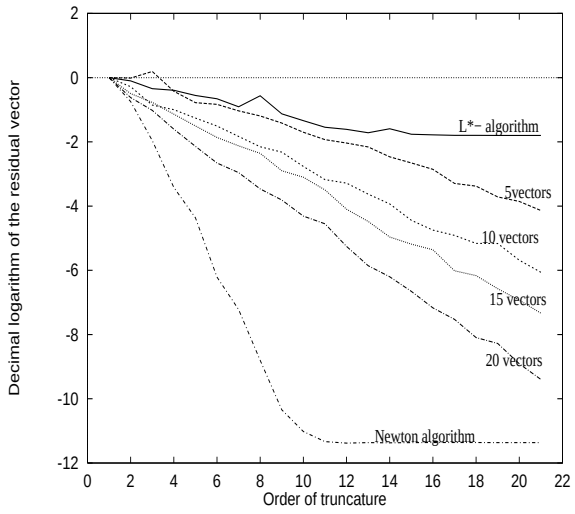


Figure 7. *Iterative process defined in figure 6. Logarithm of the residual vector versus the order of truncature for the three high order correctors*

4.3. *An improved corrector involving a reduced basis*

A new algorithm has been recently proposed [CAD 04], to combine the robustness of the high order Newton algorithm and the low cost of the L^*-algorithm. It is

defined from an arbitrary L^* operator and from a reduced basis, that can have been computed in previous steps of the calculus. As in multigrid method [ELM 02], one uses projection and prolongation operators, in order to split the equation [15]. A specific homotopy transformation is defined and this leads to couple a sort of Newton algorithm on the reduced subspace and a sort of L^*-algorithm on the whole space. The efficiency of the reduced basis is presented in figure 7, for various sizes of the basis from 5 to 20 vectors (the operator L^* being always the initial tangent operator). The considered vector basis are issued from the ANM prediction step. Clearly, the improved algorithm converges much better than the initial L^*-algorithm. The convergence can be got after one iteration, with a subspace of 20 vectors and an order n=20.

5. Some high-order predictor-corrector strategies

5.1. *Combining prediction and correction*

The high-order continuation algorithms are only concatenations of a prediction defined in section 2 and of a correction defined in section 4. Here the prediction phase is started only if the criterion [14] holds good. Four specific strategies will be considered, the three first ones being those detailed in [LAH 02]. They are sketched in table 3. The prediction curve is exactly the one presented in section 2, except in strategy 2, where the consistent tangent matrix is replaced by the one of the last correction. The maximal reliability should be obtained with the so-called "high order Newton algorithm": it combines the consistent prediction and the high order Newton correctors and coincides with the traditional Newton-Raphson algorithm for truncation orders equal to one. But it needs at least two matrix triangulations per step. The other algorithms are designed to need only one matrix triangulation per step. In the second one, this matrix is calculated at the end of the prediction and it used for the correction, if needed, and for the next prediction step. Of course, this leads to small mistakes in the prediction series. In the strategies 3 and 4, the triangulated matrix is the one of the beginning of the step. The two latter methods differ by the correction algorithm.

Five user parameters are available to manage the calculation. The truncation order of the prediction N_{pred} can be designed to minimise the computation time, as explained in section 3. Often, the order for the correction N_{corr} can be chosen lower than N_{pred}, but it must be sufficiently large to get a smaller number of correction phases (1, may be 2). The two parameters δ_2 and δ_3 characterize the required accuracy: δ_3 is level of residual to be reached after correction and δ_2 is the level of the residual to start the corrector. Last, the main parameter is δ_1, that governs the step lengths as explained in 3.4. Finally, one can summarize a step with ANM as follows:

Substep 1 : Prediction

Compute the curve $(U(a), \lambda(a))$ from an initial point (U_0, λ_0), using ANM and Padé approximants at order N_{pred}.

Substep 2: End of the prediction curve

	Prediction	Correction
Strategy 1 (high order Newton-Raphson)	Consistent	High order Newton ($L^* = L_t^p$)
Strategy 2 (high order modified Newton)	Approximated	High order modified Newton ($L^* = L_t^1$)
Strategy 3	Consistent	Matrix L^* from the beginning of the step
Strategy 4 (reduced basis)	Consistent	L^* and reduced basis from the prediction

Table 3. *Four prediction-correction strategies*

Computation of a_{maxP} by using expression [9]. The end of the prediction curve, $(U_{\text{pred}}, \lambda_{\text{pred}}) = (U(a_{\text{maxP}}), \lambda(a_{\text{maxP}}))$ depends on the control parameter δ_1.

Substep 3: Correction or not ?

Computation of the residual $R(U_{\text{pred}}, \lambda_{\text{pred}})$
- If $\|R(U_{\text{pred}}, \lambda_{\text{pred}})\| \leq \delta_2$
- Then go to substep 1 with $(U_0, \lambda_0) = (U_{\text{pred}}, \lambda_{\text{pred}})$
- Else got to substep 4

Substep 4: Correction

Compute iterative corrections $(\Delta U^p, \Delta\lambda^p)$ at order N_{corr} by Padé approximants and high order correctors defined in section 4.
- If at iteration p, $\| R(U^{p+1}, \lambda^{p+1}) \| \leq \delta_3$
- Then go to substep 1 with $(U_0, \lambda_0) = (U^{p+1}, \lambda^{p+1})$
- Else perform anoter iteration

5.2. *Evaluation of the correctors*

To evaluate the various correction strategies, let us consider again the thin cylindrical roof, discretized by eight nodes quadrilateral elements. Large values of the control parameters δ_1 are chosen ($\delta_1 = 10^{-1}$, 10^{-2}) to get a correction at most of the step ends. The other parameters of the calculus and the number of iterations are presented in table 4.

This establishes clearly the reliability of the high order Newton iterative algorithm, that converges in any case and with less matrix inversions than the traditional Newton algorithm. Especially, the strategy 2, that needs about one matrix triangulation per step is very efficient. The L^* corrector, used in strategy 3, does not converge at the second step. All these results corraborate those obtained in previous tests [LAH 02]. Nevertheless, a new efficient corrector without any new matrix triangulation is avail-

	Step 1	Step 2	Step 3	Step 4	Step 5	total
$\delta_1 = 10^{-1}$						
Strategy 1	1 (1)	1 (2)	2 (3)	1 (2)	-	5 (8)
Strategy 2	1 (1)	3 (4)	1 (1)	1 (1)	-	6 (7)
Strategy 3	11	-	-	-	-	-
Strategy 4, 15 vectors	4	4	3	7	-	18
Strategy 4, 20 vectors	3	3	4	2	-	12
Newton	2	3	4	2	-	11
$\delta_1 = 10^{-2}$						
Strategy 1	1 (1)	1 (1)	0 (0)	1 (1)	1 (1)	4 (4)
Strategy 2	1 (1)	1 (1)	1 (1)	1 (1)	1 (1)	5 (5)
Strategy 3	8	-	-	-	-	
Strategy 4, 15 vectors	2	8	0	2	4	16
Strategy 4, 20 vectors	2	3	0	2	4	11
Newton	1	2	0	2	2	7

Table 4. *Thin cylindrical roof. Number of iterations per step. Computation parameters: $\delta_2 = 10^{-2}$, $\delta_3 = 10^{-4}$, $N_{pred} = 15$, $N_{corr} = 10$ ($N_{corr} = 5$). The strategies 1 to 4 are high order algorithms defined in table 3. "Newton" refers to the traditional order 1 algorithm*

Algorithm	Pred. cost	Corr. cost	Total cost	Number of matrix
$\delta_1 = 10^{-3}$, no correction	6 × 1.5	-	9	6
$\delta_1 = 10^{-2}$, strategy 1	5 × 1.5	4 × 1.3	13.7	9
$\delta_1 = 10^{-2}$, strategy 2	5 × 1.5	4 × 0.3 +1.3	10	6
$\delta_1 = 10^{-2}$ strategy 4, 15 vectors	5 × 1.5	16 × 0.3	12.3	5
$\delta_1 = 10^{-2}$, Newton	5 × 1.5	7 × 1.0	14.5	12

Table 5. *Same data as in table 4. Comparison of various prediction-correction methods. $\alpha = 30$*

able. Indeed the reduced basis corrector, that is used in strategy 4, has converged in any case. The number of iterations is a bit large (11-18 instead of 4-6) but in cases with many dof's, a greater number of iterations is not too expensive, as compared with a matrix inversion.

5.3. *Evaluation of the predictor-corrector algorithms*

The best prediction-correction strategies are now compared, with the same test, see table 5. The total computational costs have been evaluated by formula [13], with $\alpha = 30$. Here the most rapid calculus is obtained in a case without correction ($\delta_1 =$

Step	1	2	3	4	5	Total
First iterations	2	2	2	No	2	
reduction S = 0.7	No	Yes	No		Yes	
Second iterations	-	No	-		2	10
Without reduction	2	8	0	2	4	16

Table 6. *Effect of a reduction of the step length according to the convergence of the corrector. Correction by strategy 4, 15 vectors. Reduction factor $S = 0.7$, the reduction is done when the correction does not converge after 2 iterations. Same problem, same data as in table 4, $\delta_1 = 10^{-2}$*

10^{-3}), but a similar result is obtained with strategy 2 and a correction at each step ($\delta_1 = 10^{-2}$). The high order Newton algorithm (strategy 1) is less efficient in cases where there are many iterations, but this algorithm is designed for reliability rather than for speed. Of course, corrections by traditional Newton method are possible, but this way is much slower, due to the number of matrix inversions. Last, it is possible to apply an algorithm with one matrix per step (as strategy 4), with a satisfactory computation cost.

5.4. *More adaptivity for the step length*

The simple criterion [9] defines an adaptive step length. Nevertheless, this criterion can not warrant, in any case, the required quality of solution and an easy correction if necessary. That is why, we propose further to adjust sometimes this step length, according to the convergence of the corrector. The idea is to reduce the step length with a given ratio S, by the following formula:

$$S.a_{\text{maxP}} \rightarrow a_{\text{maxP}} \tag{19}$$

in cases where the correction does not converge rapidly enough. This procedure is convenient with a low cost correction strategy, as strategy 4. This procedure has been tested for strategy 4, in the same case as in table 4. Clearly this procedure permits to limit the number of iterations (8 to 2 for the second step, see table 6) and to increase the reliability of the path following technique. The total cost is then reduced from 12.3 to 10.5 and it becomes similar as with the best algorithms (compare with table 5). Note that the same reduction procedure [19] has been applied in [AGG 03], where it is done according to the value of the residual.

5.5. *The pinched cylinder test*

Now the continuation procedures have been applied to the pinched cylinder test, with the same accuracy parameters: $\delta_2 = 10^{-2}$, $\delta_3 = 10^{-4}$ and with the orders

$N_{\text{pred}} = 20$, $N_{\text{corr}} = 15$. This computation is very difficult to manage, because of a quasi-bifurcation. For instance, when it is done with the ABAQUS-code, the standard algorithm fails and we were not able to achieve the calculus with less than 100 matrix inversions. With ANM continuation procedures, the algorithms work well if the control parameter δ_1 is not greater than $\delta_1 = 10^{-4}$. With $\delta_1 = 10^{-4}$, 14 or 15 steps are necessary and a correction is necessary only at the first step.

This correction can be achieved easily by a Newton procedure ($N_{\text{corr}} = 1$ is sufficient). On the contrary, the strategies 3 and 4 do not converge. Indeed, this first step is very large and, hence there is a great difference between the preconditioner L^* and the consistent tangent matrix. Nevertheless, it is also possible to converge with strategy 4, if one reduces the step length with a ratio S = 0.8. In this latter case, two corrections at order 15 are sufficient.

6. Conclusion and discussion

The basic features of Asymptotic-Numerical Methods to solve path following problems have been revisited. It is clearly established that the best ANM algorithms involve Padé approximants and a sufficiently large order of truncature. The calculus is mainly controlled by an user parameter δ_1. With small values of δ_1, a very high accuracy is ensured and intricate response curves can be followed with a high reliability. With larger values of δ_1, the number of steps can be reduced, but correction phases may become necessary. In any cases, such correctors have to be included in the algorithm, to improve its reliability. Various predictor-corrector techniques have been discussed. The most reliable is the so-called high order Newton-Raphson algorithm. In any case, a high order seems better than a classical first order one. Likely the most efficient is the so-called "strategy 2", that needs only one matrix triangulation per step. Other techniques, like the so-called "strategy 4" are also promising.

In this paper, the discussion has been limited to cases where a direct solver is used and to moderately large problems (less than 20.000 dof). For large scale problems, there is not yet an optimal definite strategy. Likely, domain decomposition methods [GAL 00, ESS 03] or multigrid solver [ELM 02] can be useful. Very large orders with a direct solver or small orders with an iterative one should be considered. Some informations about these points can be found in this volume.

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