Asymptotic Numerical Method for strong nonlinearities

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ABSTRACT. *Plastic constitutive laws and frictional contact conditions induce strong nonlinearities that one has to take into account in the numerical simulation of material forming processes. In this work, we present a review of the different techniques which permit the asymptotic numerical method (ANM) to be adapted to these nonlinearities. ANM needs regular relations and quadratic equations if possible. Several examples show the effectiveness of the proposed method.*

RÉSUMÉ. *Les lois de comportement plastique et les conditions de contact avec frottement représentent des fortes non-linéarités qu'il faut prendre en compte dans la simulation numérique des processus de mise en forme des matériaux. Nous présentons, dans ce travail, une revue des techniques qui permettent d'adapter la méthode asymptotique numérique (MAN) à cette situation. La MAN exige des relations régulières sous une forme quadratique de préférence. Plusieurs exemples attestent de l'efficacité de la présente méthode.*

KEYWORDS: *strong nonlinearities, asymptotic numerical method, contact, plasticity, viscoplasticity.*

MOTS-CLÉS : *fortes non-linéarités, méthode asymptotique numérique, contact, plasticité, viscoplasticité.*

1. Introduction

In the present paper, we show how the asymptotic numerical method can be applied to any kind of physical problems, especially those involving strong non linearities such as unilateral contact, friction or plasticity.

It has been shown that this method is an efficient tool to solve smooth problems such as non linear elasticity or Navier Stokes equations. The basic idea of the ANM is first to choose a convenient framework to model the mechanical problem. This requires regular functions with a moderate non linearity. A quadratic form written as follows is prefered [COC 94] :

$$R(U,\lambda) = L(U) + Q(U,U) - \lambda F = 0 \qquad [1]$$

where R is the so-called residual vector, $L(.)$ and $Q(.,.)$ are linear and quadratic operators, F is a given vector and λ is a scalar parameter. For geometrically non linear elasticity, the unknown vector U includes both the displacement field and the stress one. In this case, equation (1) involves the equilibrium condition and the constitutive relation. Next, the selected variables (U, λ) of the proposed problem are expanded into power series with respect to a scalar path parameter "a" :

$$U(a) - U_0 = \sum_{i=1}^{n} a^i U_i \qquad \lambda(a) - \lambda_0 = \sum_{i=1}^{n} a^i \lambda_i \qquad [2]$$

This transforms the starting non linear problem (1) into a recurrent sequence of linear ones admitting the same tangent operator. These linear problems can be written, for a given order "p", as follows :

$$L_t(U_p) = \lambda_p F + F_p^{nl} \qquad [3]$$

where $L_t(.) = L(.) + 2Q(U_0,.)$, is the tangent operator defined at the starting point (U_0, λ_0). These linear problems require the computation of right hand side terms $F_p^{nl} = -\sum_{r=1}^{p-1} Q(U_r, U_{p-r})$ which involve a simple sum combining the previous computed solutions. Only a simple sum is computed because the governing equations are chosen in a quadratic form. Consequently, computation time of F_p^{nl} is relatively moderate. Reference [ZAH 99] shows that time to compute about 20 terms of the series is equivalent to the one needed to evaluate and decompose the tangent stiffness matrix for a structure with 10^4 degrees of freedom. To improve the validity range of the solution, the polynomial approximation (2) is replaced by rational fractions named Padé approximants :

$$U(a) - U_0 = \sum_{i=1}^{n-1} f_i(a) a^i U_i \qquad \lambda(a) - \lambda_0 = \sum_{i=1}^{n-1} f_i(a) a^i \lambda_i \qquad [4]$$

where $f_i(a)$ are rational fractions with the same denominator [NAJ 98] [BRA 97]. This reduces significantly the number of steps to obtain the solution branch.

This technique has been applied with success to non linear shell structures [AZR 93] [ZAH 99] and Navier-Stokes equations [HAD 95] [TRI 96] [CAD 97].

This paper aims to discuss how to solve non smooth problems using ANM. An analytical solution branch of such problems is not possible if the governing equations are not expressed in an analytical form. So, in order to apply the perturbation technique, the non smooth functions are replaced by smooth ones. Furthermore, it is not straightforward to get the recurrence formulae to compute the series when the problem is not written in a simple form. Two ideas have been proposed to transform a function in a convenient form so as to apply the perturbation technique : this consists in introducing differential expressions and additional variables to set the problem into the quadratic form aforementioned. In this case the unknown U holds new variables in addition to the displacement and the stress fields. In what follows, this methodology will be applied to several non smooth problems concerning contact mechanics, friction, plasticity and viscoplasticity. Results of the proposed algorithm will be compared with that obtained with the classical iterative techniques.

2. Basic treatment of strong nonlinearities

2.1. *Main ideas*

In this section, the methodology to adapt non smooth functions to ANM is summarized. We present then the key points for contact, plasticity, viscoplasticity and friction. The methodology follows three main ideas : regularization, differential relations and additional variables.

First, let us assume that the non linear problem exhibits a function such as $y = |x|$. To apply a perturbation technique, one has to replace this function by a regular one. A possible way is to introduce the following relation : $y = (x^2 + \eta^2)^{1/2}$, where η denotes a small regularization parameter. This parameter is chosen to represent correctly the non smooth function. Figure (1) shows possible regularizations of the proposed function.

The second idea consists in introducing differential relations. Let us consider a function in the form $y = x^\alpha$ with α a non integer constant. In the ANM framework, this relation is replaced by this differential one : $xdy - \alpha ydx = 0$ which allows one to deduce simple recurrence formulae with the perturbation technique (see reference [POT 97]).

The last idea consists in introducing additional variables to set the non linear problem into a quadratic form well adapted to ANM. In this manner a function such as $F = u^3$ is replaced by :

$$\begin{cases} F = uv \\ v = u^2 \end{cases}$$

This adds a new variable v which has to be stored but allows to minimize the computation time and to make easier the computation of the series.

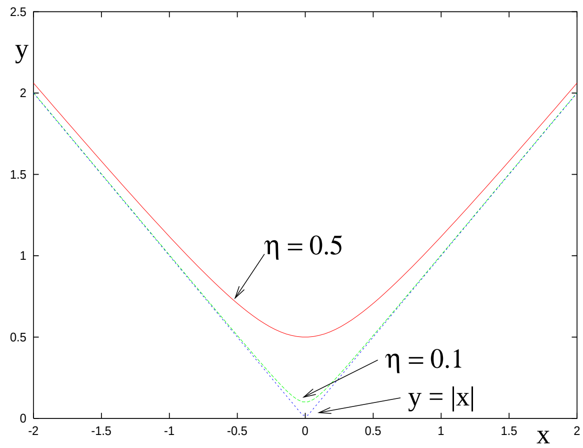

Figure 1. *Regularization of the function* $y = |x|$ *; a small value of* η *permits a good approximation of the non smooth function*

2.2. *Treatment of unilateral contact*

Contact problem is of great importance in industrial applications. This concerns metal forming, drilling, crash... Numerical studies concern specially the contact geometry, the contact laws and the algorithms able to include these requirements in the formulation [ALA 91] [SIM 92] [WRI 95].

We propose in this study to show how the contact conditions can be dealt with in the ANM framework. A simple test is presented considering the frictionless contact between a rigid straight line and a cantilever beam as shown in figure (2). As

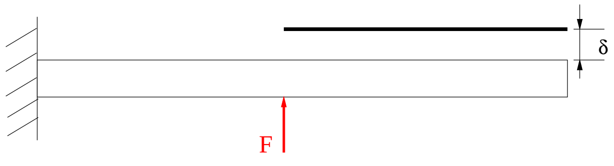

Figure 2. *Cantilever beam undergoing into contact with a rigid straight line*

the contact is considered without friction, one can write the contact conditions in the following form :

$$\begin{cases} R^c = \mathbf{R}^c.\mathbf{n} \\ R^c h = 0 \\ R^c \geq 0 \\ h \geq 0 \end{cases} \qquad [5]$$

where $\mathbf{R^c}$, h and $\mathbf{n}$ denote respectively the contact force vector, the current clearance and the normal vector to the rigid surface. These contact relations are not analytic, so they are not adapted to the ANM algorithm. For this reason, these relations are replaced by a smooth one as follows :

$$\mathbf{R}^c h = \eta_c(\delta - h)\mathbf{n} \qquad [6]$$

where δ is the initial clearance and η_c is a regularization parameter. In fact, the regularised law (6) can be seen as the result of the application of the penalty technique to (5). Figure (3) shows the influence of the regularization parameter on the contact response. Small values of η_c allow a good estimation of the contact force. But if a

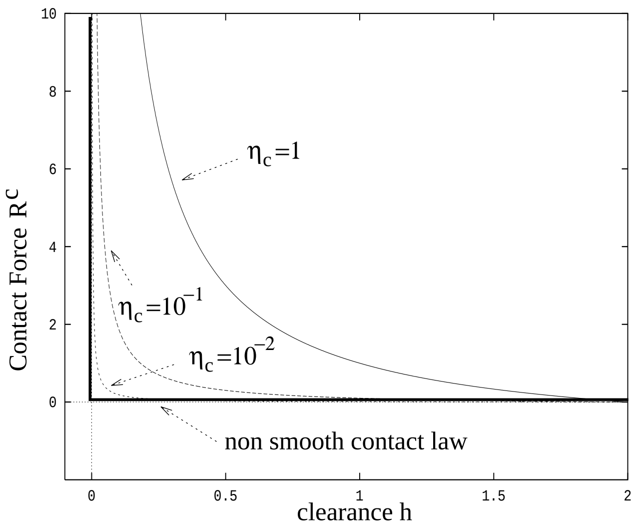

Figure 3. *Contact regularization with different values of* η_c

more general shape of the rigid surface is considered, this induces a different gap δ for each contact node. To get a uniform regularization in this case, a procedure has been proposed that defines η_c at each contact point as a function of the local initial clearance and of two given numbers : a typical contact force R^d and a typical clearance h^d [ELH 98a]. At each contact point, η_c is given by :

$$\eta_c(x) = R^d h^d/(\delta(x) - h^d) \qquad [7]$$

where x denotes the position of the considered contact point.
The contact law (6) does not allow any interpenetration. Abichou [ABI 01] has proposed another soft contact law expressed as follows :

$$\left\{ \begin{array}{l} (R^c/k + h)R^c = \eta_c(\delta - h) \\ \mathbf{R}^c = R^c\mathbf{n} \end{array} \right. \quad [8]$$

where k is a parameter defining the stiffness of the contact surface. For a large value of k, the contact law (8) tends to the harder contact relation (6). Figure (4) pictures the proposed contact relation (8) for different values of k and η_c.

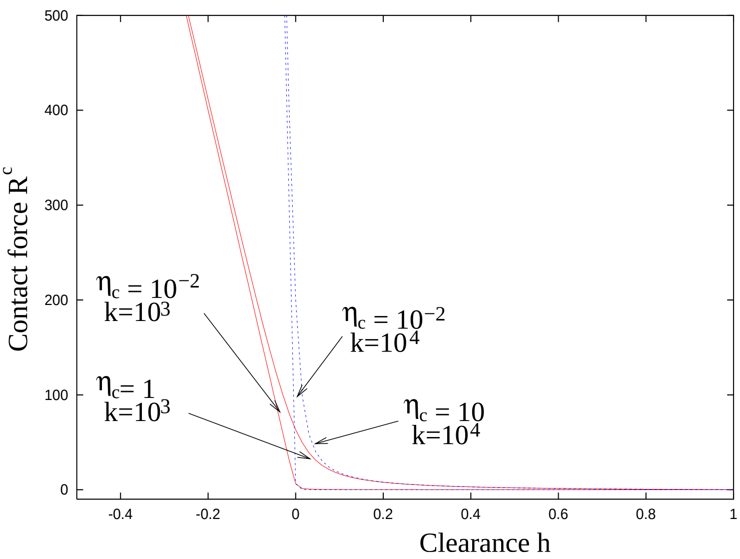

Figure 4. *Soft contact law with different values of* k *and* η_c

2.3. *Treatment of perfect deformation plasticity*

Industrial processes involve non linear material behaviour. To deal with numerical metal forming within ANM, it is necessary to adapt the constitutive laws used in such cases. In this section, a constitutive law based on an elastic-perfectly plastic behaviour is considered. The uniaxial relations can be written as follows :

$$\begin{array}{ll} \epsilon = \dfrac{\sigma}{E} & if \quad |\sigma| < \sigma_y \\ \epsilon = \dfrac{\sigma}{E} + \epsilon^p & if \quad |\sigma| = \sigma_y \end{array} \quad [9]$$

where $\epsilon, \sigma, \epsilon^p, E$ and σ_y denote respectively the strain, the stress, the plastic deformation, the Young's modulus and the yield stress. This stress-strain relation is singular at the yield limit. Then it can be replaced by an hyperbolic relation [BRA 95] :

$$E\ \epsilon = \sigma + \frac{\eta_p\ \sigma_y^2}{\sigma_y^2 - \sigma^2}\ \sigma \qquad [10]$$

Note that this hyperbolic relation involves two branches (see figure 5). The first branch corresponds to a stress which is below the yield stress σ_y and never passes over it. For small values of the regularization parameter η_p, this first branch is close to the elastic-perfectly plastic law (9) and therefore it is physically admissible. In the second case, σ is greater than σ_y which is not acceptable.

To set this law in a quadratic framework, it is sufficient to introduce two new variables : $\zeta = \dfrac{\eta_p\ \sigma_y^2}{\sigma_y^2 - \kappa}$ and $\kappa = \sigma^2$. The constitutive law is then described by the following equations :

$$\begin{cases} E\ \epsilon = \sigma + \zeta\ \sigma \\ \zeta\ (\sigma_y^2 - \kappa) = \eta_p\ \sigma_y^2 \\ \kappa = \sigma^2 \end{cases} \qquad [11]$$

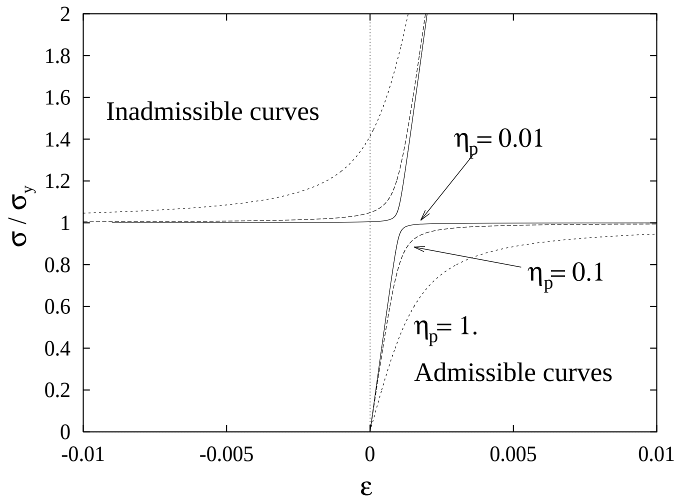

Figure 5. *Regularization of the elastic-perfectly plastic law*

2.4. *Treatment of viscoplasticity*

We present here another source of nonlinearities. It concerns the material behaviour in hot metal forming processes. A viscoplastic law based on the Norton-Hoff model is considered :

$$\sigma' = \frac{2}{3}k(\alpha + \bar{\epsilon})^n \bar{D}^{m-1}\mathbf{D}' \qquad [12]$$

The parameters k, α, n and m describe the material properties. σ' denotes the deviatoric part of the Cauchy stress, $\mathbf{D}'$ is the deviatoric strain rate tensor and $\bar{D}$ denotes the equivalent strain rate defined by the following relation :

$$\bar{D}^2 = \frac{2}{3}\mathbf{D}' : \mathbf{D}' \qquad [13]$$

$\bar{\epsilon}$ represents the cumulated plastic strain given by :

$$\frac{d\bar{\epsilon}}{dt} = \bar{D} \qquad [14]$$

Equation (12) is not adapted to a direct expansion into power series because it exhibits the same difficulties as those encountered in contact or plasticity problems. So, to adapt the viscoplastic law to the ANM framework, we apply the three procedures that have been described before. First, we introduce additional variables C, H, Q :

$$\left\{ \begin{array}{l} \sigma' = C\mathbf{D}' \\ C = \frac{2}{3}kHQ \\ H = (\alpha + \bar{\epsilon})^n \\ Q = \bar{D}^{m-1} \end{array} \right. \qquad [15]$$

The second step consists in introducing differential relations for the equations with power terms. This transforms equations (15 c) and (15 d) into :

$$\left\{ \begin{array}{l} (\alpha + \bar{\epsilon})\ dH = n\ H\ d\bar{\epsilon} \\ \bar{D}\ dQ = (m-1)\ Q\ d\bar{D} \end{array} \right. \qquad [16]$$

The last step concerns the regularization procedure. In order to avoid a singularity for a zero strain rate, the equivalent strain rate is slightly modified as follows :

$$\bar{D}^2 = \frac{2}{3}\mathbf{D}' : \mathbf{D}' + (\eta_{vp}\frac{v_c}{L_c})^2 \qquad [17]$$

where η_{vp} denotes a regularization adimensional parameter, v_c and L_c represent respectively a velocity and a length which are characteristic of the studied problem.

2.5. *Treatment of the friction law*

In metal forming processes, the contact between sheet and tool always involves friction phenomena which are not negligible in many situations. When contact occurs, a contact force appears. It can be decomposed into a normal component and a tangential one :

$$\mathbf{R}^c = \mathbf{R}_n^c + \mathbf{R}_t^c \qquad [18]$$

The normal component R_n^c is given by the following relations as it has been presented in section (2.2) :

$$\begin{cases} R_n^c = \mathbf{R}^c.\mathbf{n} \\ R_n^c = \eta_c(\delta - h)/h \end{cases} \qquad [19]$$

For the friction law, we consider the one proposed by Chenot [CHE 93] which takes into account both the contact pressure and the slip velocity. The tangential reaction is then expressed in the following form :

$$\mathbf{R}_t^c = -\mu \mid R_n^c \mid\mid v_t \mid^{q-1} \mathbf{v}_t \qquad [20]$$

where μ is the friction coefficient, $\mathbf{v}_t$ is the tangential slip velocity and q is the sensitivity coefficient to slip velocity. For $q = 0$, the Coulomb friction law is recovered.

To adapt the friction law to the ANM framework, we use the following relations :

$$\begin{cases} \mathbf{R}_t^c = -\mu \mid R_n^c \mid \mathbf{v}_q \\ v_n = (\mathbf{v} - \mathbf{v}_0).\ \mathbf{n} \\ \mathbf{v}_t = \mathbf{v} - \mathbf{v}_0 - v_n\ \mathbf{n} \\ S = \mathbf{v}_t.\mathbf{v}_t + (\omega v_c)^2 \\ P = S^{(q-1)/2} \Rightarrow SdP = \dfrac{q-1}{2} PdS \\ \mathbf{v}_q = P\mathbf{v}_t \end{cases} \qquad [21]$$

Once more, we have introduced a regularization parameter ω to avoid the singularity obtained for $\mathbf{v}_t = 0$. (v_c is a typical velocity of the problem).

2.6. *Bibliographical comments*

Over the last decade, problems involving strong non linearities have become one of the most challenging problems for the ANM. The main difficulty was to find a technique to adapt equations involving different nonlinearities and to optimize the algorithm. A first attempt to solve problems involving plasticity is due to Yokoo et al. [YOK 76]. Braikat has proposed some techniques for non linear constitutive laws [BRA 95]. Potier-Ferry et al. have introduced differential relations to set functions with power terms into more convenient forms for the perturbation technique [POT 97]. Inspired by this idea, many contributions have been then proposed : for plasticity [BRA 97] [ZAH 98] [IMA 01] [MAL 99], hyperelasticity [GAL 00], viscoplasticity [DES 97] [BRU 99], contact mechanics [ELH 98b] [AGG 03], fluid mechanics [CAD 01]. Contributions of the ANM coupling several non linearities can be found in [BRU 99] [ABI 02].

3. Computational techniques

3.1. *How to compute the series ?*

In section 2, we have shown how to adapt strongly non linear relations to the ANM framework. The transformations lead to additional variables and then to new equations. When these relations are well established, it is easy to apply the perturbation technique. Let us describe the procedure to follow for the plasticity problem discussed in section (2.3). By considering the system (11) and expanding into power series the variables $(\epsilon, \sigma, \zeta, \kappa)$ in the neighbourhood of a known solution $(\epsilon_0, \sigma_0, \zeta_0, \kappa_0)$, one obtains a sequence of linear problems.
For order 1 :

$$\begin{cases} E\ \epsilon_1 = \sigma_1 + \zeta_0\ \sigma_1 + \zeta_1\ \sigma_0 \\ \zeta_1(\sigma_y^2 - \kappa_0) - \zeta_0\ \kappa_1 = 0 \\ \kappa_1 = 2\ \sigma_0\ \sigma_1 \end{cases} \qquad [22]$$

For order n :

$$\begin{cases} E\ \epsilon_n = (1+\zeta_0)\sigma_n + \zeta_n\sigma_0 + \sum_{i=1}^{n-1} \zeta_i\sigma_{n-i} \\ \zeta_n(\sigma_y^2 - \kappa_0) - \zeta_0\kappa_n - \sum_{i=1}^{n-1} \zeta_i\kappa_{n-i} = 0 \\ \kappa_n = 2\sigma_0\sigma_n + \sum_{i=1}^{n-1} \sigma_i\sigma_{n-i} \end{cases} \qquad [23]$$

Order 1 permits to compute the tangent modulus. By substituting (22b) and (22c) into (22a), one obtains the following relation :

$$\sigma_1 = D_t\epsilon_1 \qquad [24]$$

where D_t is identified to the tangent modulus : $D_t = E/(1 + \zeta_0 + \dfrac{2\ \zeta_0\ \sigma_0}{\sigma_y^2 - \kappa_0})$.
For order $n \geq 2$, one substitutes (23b) and (23c) into (23a), to obtain the following relation :

$$E\ \epsilon_n = (1+\zeta_0+\frac{2\zeta_0\sigma_0}{\sigma_y^2 - \kappa_0})\sigma_n + \frac{\sigma_0}{\sigma_y^2 - \kappa_0}\left(\zeta_0 \sum_{i=1}^{n-1} \sigma_i\sigma_{n-i} + \sum_{i=1}^{n-1} \kappa_i\zeta_{n-i}\right) + \sum_{i=1}^{n-1} \zeta_i\sigma_{n-i} \qquad [25]$$

which can be written as follows :

$$\sigma_n = D_t\epsilon_n + \sigma_n^{nl} \qquad [26]$$

where σ_n^{nl} is a stress depending on the terms computed at the previous orders :

$$\sigma_n^{nl} = -\frac{1}{1+\zeta_0+\dfrac{2\ \zeta_0\ \sigma_0}{\sigma_y^2-\kappa_0}}[\frac{\sigma_0}{\sigma_y^2-\kappa_0}\left(\zeta_0\sum_{i=1}^{n-1}\sigma_i\ \sigma_{n-i}+\sum_{i=1}^{n-1}\kappa_i\ \zeta_{n-i}\right)+\sum_{i=1}^{n-1}\zeta_i\ \sigma_{n-i}] \qquad [27]$$

Last, ζ_n and κ_n are computed from equations (23b and 23c). The appendix details the computational procedure for a nonlinear law involving non integer power coefficient.

It is easy to implement this procedure in the case of a variational formulation of a three dimensional problem. By considering small displacements, the equilibrium equation can be written in the following form :

$$\int_\Omega {}^t\sigma : \delta\epsilon\ dv - \lambda P_e(\delta u) = 0 \qquad [28]$$

The perturbation technique leads to a set of linear problems. At order 1, the tangent problem is given by :

$$\int_\Omega {}^t\sigma_1 : \delta\epsilon\ dv = \lambda_1 P_e(\delta u) \qquad [29]$$

By considering the three dimensional version of equation (24), one obtains the following classical linear problem :

$$\int_\Omega {}^t\epsilon_1 : D_t : \delta\epsilon\ dv = \lambda_1 P_e(\delta u) \qquad [30]$$

For order n, one obtains :

$$\int_\Omega {}^t\epsilon_n : D_t : \delta\epsilon\ dv = \lambda_n P_e(\delta u) - \int_\Omega {}^t\sigma_n^{nl} : \delta\epsilon\ dv \qquad [31]$$

These linear problems are solved using the finite element method. Only the displacement field has to be discretized :

$$\{\epsilon_n\} = [B]\{q_n\}^e \qquad [32]$$

where $[B]$ is the strain-displacement matrix and $\{q_n\}^e$ is the nodal displacement vector of the element 'e'. Hence the discretized problem can be set in the following form :

$$[K_T]\{q_n\} = \lambda_n\{F\} + \bigwedge_{e=1}^{NbElt}\{F_n^{nl}\}^e \qquad [33]$$

where $\bigwedge$ is the standard assembly operator, $NbElt$ is the element number of the discretized structure, $\{F\}$ is the applied loading and $[K_T]$ is the tangent stiffness matrix :

$$[K_T] = \bigwedge_{e=1}^{NbElt}\left(\int_{\Omega_e} {}^t[B][D_t][B]\ dv\right) \qquad [34]$$

Of course, this tangent matrix is exactly the same as in a classical Newton algorithm. The only new quantity within ANM is the right hand side $\{F_n^{nl}\}^e$, that looks like a residual vector. With the present model, it is only a function of the quantity $\{\sigma_n^{nl}\}$ that is the $3D$ extension of (27) :

$$\{F_n^{nl}\}^e = -\int_{\Omega_e} {}^t[B]\{\sigma_n^{nl}\}\, dv \qquad [35]$$

Note that, for the proposed problem, the variables to be stored are the nodal displacement vector q, the stress tensor σ and the additional variables ζ and κ at each integration point : U = (q, σ, ζ, κ).

3.2. ***Computational strategies***

Since the series are computed following the procedure detailed in section (3.1), the polynomial representation is replaced by Padé approximants (4) to improve the validity range of the solution. The path following technique has been presented in [ELH 00]. The maximal value of the path parameter 'a' is defined as follows :

$$\delta_1 \simeq \frac{\|u_n(a_{max}) - u_{n-1}(a_{max})\|}{\|u_n(a_{max}) - u_0\|} \qquad [36]$$

The main parameters to compute a solution branch are the truncation order 'n' and the control parameter 'δ_1'. The optimal truncation order lies generally in the range 10-20 for problems involving a moderate number of degrees of freedom (d.o.f. $\leq 10^4$). The choice of the value of the parameter 'δ_1' is not obvious. This defines various computational strategies : one can choose a very small value of 'δ_1' to obtain the solution without corrections but this strategy may induce a large number of steps. Larger values of 'δ_1' require correction phases.

One can correct the solution using the classical Newton Raphson scheme as in [DES 97] or by performing a high order corrector algorithm recently proposed based on ANM and homotopy transformations [MAL 00] [LAH 02]. In the last case, we have implemented an algorithm with a strategy based on the matrix of correction to use the same triangulated matrix for the prediction and the correction. In this procedure referred as strategy 2 in [LAH 02], the consistent tangent matrix to compute the series (33) is replaced by the one of the previous correction phase. When the residual is greater than $\delta_2 = 10^{-2}$ one corrects the solution to obtain a weaker residual with an acceptable tolerance $\delta_3 = 10^{-4}$. Let us present a scheme of the proposed algorithm :

a : Compute the prediction curve by Padé approximants with a truncation order $n = Npred$. Except at the first step, the tangent operator used for the prediction is the one already decomposed for the correction phase of the previous step,

b : Define the end step of the prediction curve by formula (36). The step length "a_{max}" depends on the control parameter "δ_1" and the truncation order "Npred",

c : Compute the residual $\mathbf{R}(\mathbf{a_{max}})$.

If	$\|\mathbf{R}(\mathbf{a_{max}})\| \leq \delta_{\mathbf{2}} \|\lambda(\mathbf{a_{max}})\mathbf{F}\|$,
then	compute the next step (go to "**a**")
else	perform the corrections,

d : Compute the corrections with a truncation order $n = Ncorr$.
If $Ncorr = 1$, the classical modified Newton algorithm is used,
else an homotopy transformation allows us to perform the corrections with a high order and Padé approximants to obtain : $\|\mathbf{R}\| \leq \delta_{\mathbf{3}} \|\lambda\mathbf{F}\|$. In this case, generally only one iteration is needed.

Next go to "**a**".

To improve the reliability, we reduce the step length when the residual is too bad until it becomes smaller than 10^{-1}. Next a correction phase can start. For contact problems, the step length is also reduced when a penetration occurs.

4. Numerical applications

This section is devoted to numerical simulations of non linear problems involving different non linearities. Numerical results are compared to those obtained with the classical iterative algorithms. The first test concerns a simple traction of a plate to show the ability of ANM to correctly represent the non linear constitutive law based on the elastic perfectly plastic behaviour. The second test concerns the unilateral contact problem between a cantilever beam and a plane rigid surface. The last test deals with the hemispherical deep drawing coupling large displacements, viscoplasticity, contact and friction.

4.1. *Simple traction of a plate*

Let us consider a rectangular plate submitted to an uniaxial loading. The geometry is defined by a length $L = 10$, a width $l = 1$ and a thickness $h = 1$. The proposed problem involves small displacements but a non linear constitutive law. The elastic-perfectly plastic behaviour is considered with the following material data : $E = 10^5$, $\nu = 0.3$, $\sigma_y = 200$.

The stress-strain relation is regularized and then replaced by the formula (10). The regularization parameter η_p is of great importance. Figure (6) shows the influence of this parameter on the load/displacement response. To describe correctly the material behaviour, one must choose $\eta_p \leq 10^{-1}$.

The parameter δ_1 which defines the step length has to be chosen carefully. Large values of δ_1 allow one to increase the step length. But in this case a non valid solution path corresponding to a stress which passes over the yield stress σ_y can be followed. To obtain a displacement of 5% of the plate length, one needs only 9 steps using series with $n = 15$, $\delta_1 = 10^{-3}$ and $\eta_p = 10^{-2}$. The residual remains very small, less than 10^{-5}, so no correction is needed.

Note that with the classical iterative techniques, for each prediction or correction phase, the non linear constitutive equation must be solved at each integration point to compute the stress field corresponding to the displacement increment. In the ANM context, this non linear equation is strictly taken into account throughout the computation procedure. More complicated tests can be found in [ZAH 98] with various constitutive laws.

The proposed test shows the applicability of the ANM procedure to non linear constitutive laws. The unloading is not taken into account but it will be presented soon.

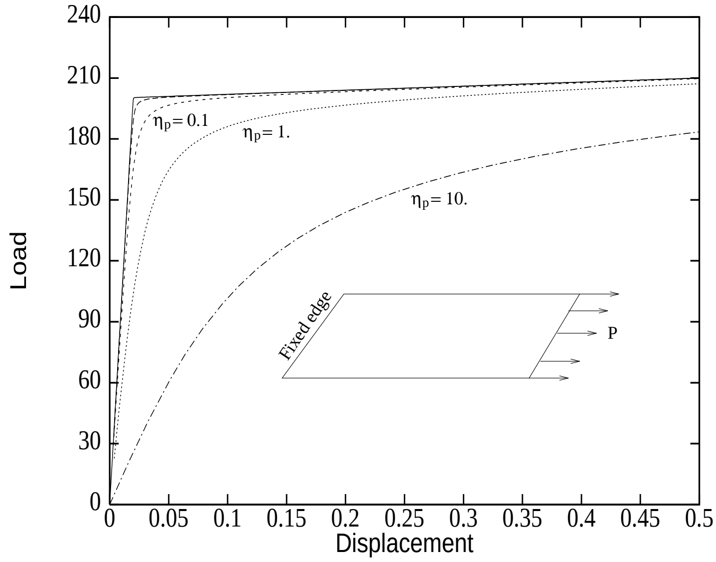

Figure 6. *Regularized elastic-perfectly plastic law : load-displacement curve of a loaded node with different values of* η_p

4.2. *Unilateral contact*

This example deals with the frictionless contact between a cantilever beam and a plane rigid surface. The structure is discretized with classical four nodes elements considering a plane stress state. Boundary conditions as well as material and geometrical data are given in figure (7). The initial clearance is the same for all the contact nodes and is equal to 2 mm. The parameter η_c introduced in section (2.2) is chosen equal to 10^{-3} allowing a consistent force distribution on the interaction line. Results of our study are compared with those of the industrial code Abaqus. This code uses the Lagrange multipliers procedure coupled with the Newton-Raphson algorithm. This example has already been discussed within ANM framework by Elhage [ELH 98b].

The proposed problem is solved using a high order prediction-correction algorithm. This associates the classical ANM with a high order corrector as proposed in [LAH 02] [AGG 03]. Two techniques of discretization have been used. In the first one, the contact force is condensed and the final linear problems (as in equation (33)) are set in terms of the nodal displacement only as in [ELH 98b]. This method is referred as "ANM + penalty" in table 1. The second technique consists in combining the perturbed lagrangian method with the ANM : the contact force is also discretized. It is referred as "ANM + perturbed lagrangian" in table 1. The latter will permit us to avoid an ill conditioning of the tangent matrix which is the typical drawback of the penalty method. Details of this calculation will be explained in a next contribution.

Figure (8) shows the vertical displacement of the node 'A' versus the loading : ANM results are in good agreement with those of Abaqus. For the prediction correction algorithm, we use a strategy based on the matrix of correction : indeed, the matrix computed for the correction phase is also used to predict the next step (strategy 2 of [LAH 02]). This allows us to perfom the computation with only one matrix decomposition per step. Because a high order is used for the correction, only one iteration is generally needed to get a small residual. The same truncation order of the series is chosen for the prediction and the correction ($n = 15$).

The computation results of the two ANM methods are reported in table (1). For different values of the parameter δ_1 which controls the size of the step length, the table gives the number of steps, the total number of the iterations and the number of matrix decompositions to obtain the whole solution path. Clearly, the robustness of the ANM is proved since the convergence is achieved whatever the considered procedure and the parameter δ_1 are. The solution branch can be obtained by decomposing only 20 tangent stiffness matrices when Abaqus needs 128 decompositions.

When the penalty method is used, no corrections are required for very small values of δ_1 (10^{-7} or 10^{-6}). In this case the classical process of the asymptotic numerical method is performed requiring at least 30 matrix decompositions. The algorithm still works for very large values of δ_1 (10^{-2} or 10^{-1}) but a correction is needed for about each step. The optimum results are obtained for $\delta_1 \geq 10^{-4}$: 20 to 22 matrix decompositions are sufficient.

By discretizing also the contact force, one obtains about the same results but without corrections for a wide range of values of δ_1 ($\delta_1 \leq 10^{-4}$). For larger values of this parameter, the algorithm needs less correction than the one based on the penalty technique.

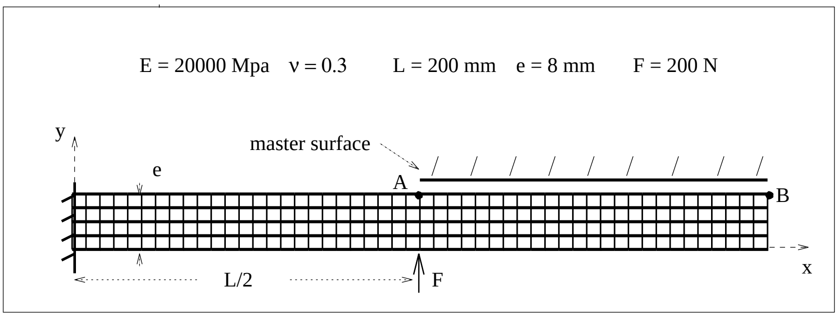

Figure 7. *Contact between an elastic beam and a rigid line*

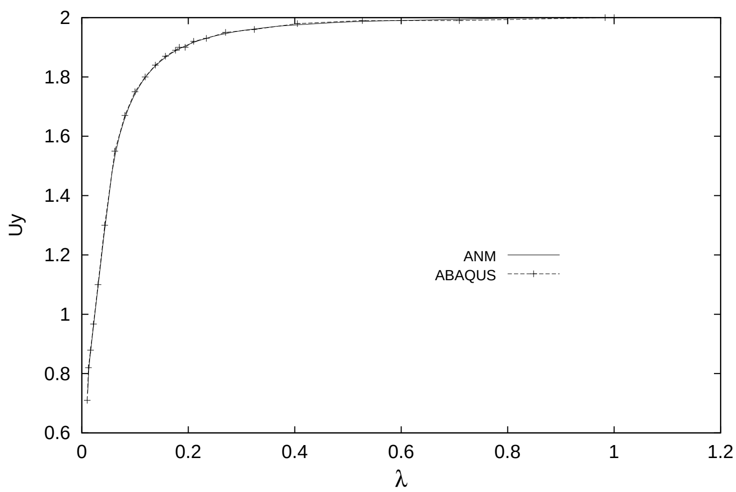

Figure 8. *Vertical displacement of the node A*

4.3. ***Hemispherical deep drawing***

The last test deals with the hemispherical deep drawing of a circular sheet (radius, ρ = 50 mm; thickness, e = 1 mm). This problem combines several non linearities concerning the geometry, the contact conditions with friction and the viscoplastic law. The material rheology is given by : $k = 180.0\ MPa\ s^m$, $\alpha = 10^{-4}$, $m = 0.8$ and $n = 0.2$. The symmetry conditions allow us to model only a section of the sheet as it is shown in figure (9). It is discretized with 50 four nodes quadrilateral elements. To take into account the material incompressibility, a mixed formulation is considered using the velocity and the pressure as two independant variables. Four Gauss integration points are used for the velocity and only one for the pressure. An updated lagrangian scheme is adopted for this problem. For the simulation, the tool which radius is 25 mm

	ANM+Penalty			ANM+perturbed lagrangian		
δ_1	$NbStep$	$NbCorr$	$NbDec$	$NbStep$	$NbCorr$	$NbDec$
10^{-7}	37	0	37	37	0	37
10^{-6}	30	0	30	30	0	30
10^{-5}	25	4	26	25	0	25
10^{-4}	20	16	21	21	0	21
10^{-3}	21	16	22	23	7	24
10^{-2}	19	19	20	19	7	20
10^{-1}	20	19	21	21	9	22

Table 1. *Contact between a cantilever beam and a rigid line, influence of the parameter δ_1. $NbStep$: number of steps, $NbCorr$: total number of corrections, $NbDec$: number of matrix decompositions*

is moved in the vertical direction until the displacement of the sheet center reaches 25 mm. The non linearities of the proposed problem involve different regularizations. In [BRU 99] the sensitivity of each regularization parameter has been studied. Note that the regularization parameter η_c of the contact conditions depends on the initial clearance δ (see section 2.2). Brunelot [BRU 99] has studied the influence of the tool velocity on the contact regularization. For two different values of this velocity $\vec{V}_0$, one can obtain very different distributions of the contact force. To avoid this drawback, the contact regularization parameter has been modified according to the tool velocity.

Let us present now a comparative study using the Coulomb friction law with $\nu = 0.3$ and $q = 0$. The tool velocity is $V_0 = 0.1\ mm\ s^{-1}$. The contact regularization is defined by (h^d, R^d) = (0.05, 20).

For the comparison, three algorithms are proposed : the first one is based on the ANM procedure with Padé approximants and a truncation order $n = 15$. For this application, the corrections have been performed at each step end with a truncation order equal to one. The second algorithm is based on the Newton Raphson iterative technique. Different step lengths $'dt'$ are used in order to improve the convergence. The last algorithm is implemented in the industrial code Abaqus, it is based on the Newton Raphson technique coupled to the Lagrange multipliers procedure to take into account the contact conditions with Coulomb friction. This latter algorithm uses an updated lagrangian formulation with adaptive step lengths.

Results of the study are reported on the table (2). Several computations have been performed with Abaqus and the best results are given on the table. Newton-Raphson algorithms require 422 or 468 decompositions of the tangent stiffness matrix to obtain the whole solution branch. The ANM algorithm is much more efficient : since it uses a high order prediction, only 24 steps are needed to obtain the same solution branch. Considering the time consuming for the right hand side F^{nl}, the computation time of one step is equivalent to the one of two matrix decompositions. For this reason, 24 steps need a computation time corresponding to about 48 matrix decompositions

[ZAH 99]. Furthermore, the ANM procedure is naturally adaptive, so the algorithm is easy to be handled by the user.

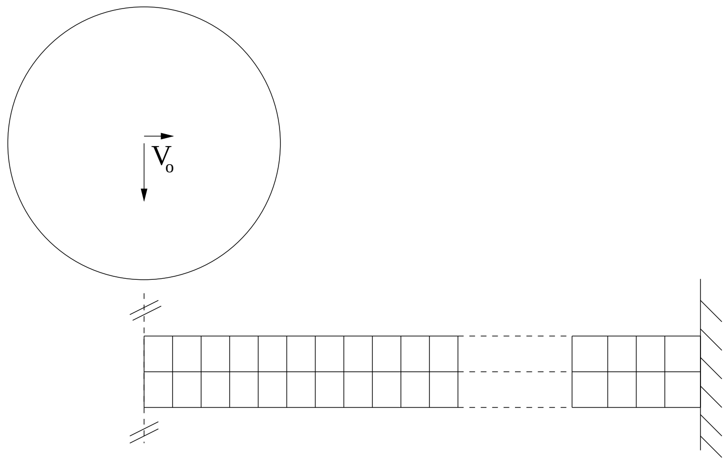

Figure 9. *Deep drawing : sketch of the problem*

		number of steps	number of decompositions
[Bru99]	dt=10.-5.-2.5	38	1559
Newton Raphson	dt=5.-2.5	53	422
	dt=2.5-1.25	104	622
Newton Raphson	(Abaqus)	102	468
A.N.M.	Padé, n=15	24	25

Table 2. *Results of the deep drawing computation*

5. Concluding remarks

Problems with strong nonlinearities have been presented in this paper in the framework of the asymptotic numerical method. The efficiency and the reliability of the ANM to solve problems involving plasticity, contact, friction and viscoplasticity are proved.

Three main ideas permits to adapt these nonlinearities to the ANM : a regularisation procedure, the introduction of differential relations and the addition of new variables. This requires more memory to store the variables but allows a significant reduction of time consuming. More robust algorithms are obtained by associating a high order corrector to the high order predictor : the number of steps is reduced significantly but one has to be careful not to follow a bad solution path because of the hyperbolic form of some regularization relations.

Contributions [ABI 02] and [BRU 99] have presented applications combining various non linearities to simulate benchmarks in metal forming processes, especially for problems with larger sizes.

Despite some recent efforts [IMA 01], the efficiency of ANM for plasticity with unloading has not been yet established. This point is crucial in view of practical applications. In our opinion, there is no fundamental difficulty to implement realistic plasticity models in the ANM framework.

6. Bibliography

[ABI 01] ABICHOU H., « Simulation de l'emboutissage à froid par une méthode asymptotique numérique », Thesis, Université de Metz, 2001.

[ABI 02] ABICHOU H., ZAHROUNI H., POTIER-FERRY M., « Asymptotic numerical method for problems coupling several nonlinearities », *Computer Methods in Applied Mechanics and Engineering*, vol. 191, 2002, p. 5795-5810.

[AGG 03] AGGOUNE W., ZAHROUNI H., POTIER-FERRY M., « High order prediction correction algorithms for unilateral contact problems », *Journal of Computational and Applied Mathematics*, , 2003, to appear.

[ALA 91] ALART P., CURNIER A., « A mixed formulation for frictional contact problems prone to Newton like solution methods », *Computer Methods in Applied Mechanics and Engineering*, vol. 92, 1991, p. 353-375.

[AZR 93] AZRAR L., COCHELIN B., DAMIL N., POTIER-FERRY M., « An asymptotic-numerical method to compute the post-buckling behaviour of elastic plates and shells », *International Journal of Numerical Methods in Engineering*, vol. 36, 1993, p. 1251-1277.

[BRA 95] BRAIKAT B., « Méthode asymptotique-numérique et fortes non-linéarités », Thesis, Université Hassan II, Casablanca, 1995.

[BRA 97] BRAIKAT B., DAMIL N., POTIER-FERRY M., « Méthodes asymptotiques numériques pour la plasticité », *Revue Européenne des Eléments Finis*, vol. 6, 1997, p. 337-357.

[BRU 99] BRUNELOT J., « Simulation de la mise en forme à chaud par une Méthode Asymptotique Numérique », Thesis, Université de Metz, 1999.

[CAD 97] CADOU J.-M., « Méthode asymptotique numérique pour le calcul des branches solutions et des instabilités dans les fluides et pour les problèmes d'interaction fluide-structure », Thesis, Université de Metz, 1997.

[CAD 01] CADOU J.-M., COCHELIN B., DAMIL N., POTIER-FERRY M., « ANM for stationary Navier-Stokes equations and with Petrov-Galerkin formulation », *International Journal of Numerical Methods in Engineering*, vol. 50, 2001, p. 825-845.

[CHE 93] CHENOT J., « La modélisation numérique des procédés de mise en forme des métaux », *La revue de métallurgie-CIT/Sciences et Génie des Matériaux*, , 1993, p. 1567-1576.

[COC 94] COCHELIN B., DAMIL N., POTIER-FERRY M., « The asymptotic-numerical method : an effi cient perturbation technique for non-linear structural mechanics », *Revue Européenne des Eléments Finis*, vol. 3, n^o 2, 1994, p. 281-297.

[DES 97] DESCAMPS J., CAO H.-L., POTIER-FERRY M., « An asymptotic numerical method to solve large strain viscoplastic problems », *Computational plasticity, Fundamentals and Applications, C.I.M.N.E.*, vol. 1, 1997, p. 393-400, D.R.J. Owen, E. Oñate and E. Hinton (Eds.), Barcelona, 1997.

[ELH 98a] ELHAGE-HUSSEIN A., « Modélisation des problèmes de contact par une méthode asymptotique numérique », Thesis, Université de Metz, 1998.

[ELH 98b] ELHAGE-HUSSEIN A., DAMIL N., POTIER-FERRY M., « An asymptotic numerical algorithm for frictionless contact problems », *Revue Européenne des Eléments Finis*, vol. 7, 1998, p. 119-130.

[ELH 00] ELHAGE-HUSSEIN A., POTIER-FERRY M., DAMIL N., « A numerical continuation method based on Padé approximants », *International Journal of Solids and Structures*, vol. 37, 2000, p. 6981-7001.

[GAL 00] GALLIET I., « Une version parallèle des méthodes asymptotiques-numériques. Application à des structures complexes à base d'élastomères », Thesis, Université Aix Marseille II, 2000.

[HAD 95] HADJI S., « Méthodes de résolution pour les fluides incompressibles », Thesis, Université de Technologie de Compiègne, 1995.

[IMA 01] IMAZATENE A., « Quelques techniques pour appliquer la MAN aux structures plastiques et aux grands systèmes », Thesis, Université de Metz, France, 2001.

[LAH 02] LAHMAM H., CADOU J.-M., ZAHROUNI H., DAMIL N., POTIER-FERRY M., « High order predictor-corrector algorithms », *International Journal of Numerical Methods in Engineering*, vol. 55, 2002, p. 685-704.

[MAL 99] MALLIL E., « Développemnt d'une méthode itérative basée sur les séries et les approximants de Padé pour le calcul non linéaire des structures », Thesis, Université Hassan II, Casablanca, 1999.

[MAL 00] MALLIL E., LAHMAM H., DAMIL N., POTIER-FERRY M., « An iterative process based on homotopy and perturbation techniques », *Computer Methods in Applied Mechanics and Engineering*, vol. 190, 2000, p. 1845-1858.

[NAJ 98] NAJAH A., COCHELIN B., DAMIL N., POTIER-FERRY M., « A critical review of asymptotic numerical methods », *Archives of Computational Methods in Engineering*, vol. 5, 1998, p. 3-22.

[POT 97] POTIER-FERRY M., DAMIL N., , BRAIKAT B., DESCAMPS J., CADOU J., CAO H., HUSSEIN A. E., « Traitement des fortes non-linéarités par la méthode asymptotique numérique », *Comptes Rendus de l'Académie des Sciences Paris, t.324, Série II b*, , 1997, p. 171-177.

[SIM 92] SIMO J., LAURSEN T., « An augmented lagrangian treatment of contact problems involving friction », *Computers and Structures*, vol. 42, n° 1, 1992, p. 97-116.

[TRI 96] TRI A., COCHELIN B., POTIER-FERRY M., « Résolution des équations de Navier-Stokes et détection des bifurcations stationnaires par une Méthode Asymptotique Numérique », *Revue Européenne des Eléments Finis*, vol. 5, n° 4, 1996, p. 415-442.

[WRI 95] WRIGGERS P., « Finite element algorithms for contact problems », *Archives of Computational Methods in Engineering*, vol. 2, n° 4, 1995, p. 1-49.

[YOK 76] YOKOO Y., NAKAMURA T., UETANI K., « The incremental perturbation method for large displacement analysis of elastic-plastic structures », *International Journal of Numerical Methods in Engineering*, vol. 10, 1976, p. 503-525.

[ZAH 98] ZAHROUNI H., DAMIL N., POTIER-FERRY M., « Asymptotic numerical method for nonlinear constitutive laws », *Revue Européenne des Eléments Finis*, vol. 7, n° 7, 1998, p. 841-869.

[ZAH 99] ZAHROUNI H., COCHELIN B., POTIER-FERRY M., « Computing finite rotations of shells by an asymptotic-numerical method », *Computer Methods in Applied Mechanics and Engineering*, vol. 175, 1999, p. 71-85.

7. Appendix

In this section we aim to explain the how differential relation discussed in section (2.1) makes easy the application of the perturbation technique in the case of nonlinear functions with power coefficients. Let us consider a non linear constitutive law inspired by the Romberg Osgood relation which can be written for a one-dimensional model as follows :

$$E\epsilon = \sigma + \alpha \left(\frac{\sigma}{\sigma_y}\right)^{\beta} \sigma \qquad [37]$$

where σ and ϵ denote the stress field and the mechanical strain. E, σ_y, α and β are material constants.
To apply the perturbation technique in a simple way, one first introduces two variables ζ and κ :

$$\zeta = \frac{\sigma}{\sigma_y} \qquad \kappa = \alpha\, \zeta^{\beta} \qquad [38]$$

Since the parameter β is not an integer, the application of the perturbation technique is not obvious. To obtain a simple form easy to implement in the ANM framework, one proposes the following differential relation :

$$\zeta\, d\kappa = \beta\, \kappa\, d\zeta \qquad [39]$$

In this manner, the starting problem (37) is now presented as follows :

$$\begin{cases} E\, \epsilon = \sigma + \kappa\, \sigma \\ \zeta = \dfrac{\sigma}{\sigma_y} \\ \zeta\, d\kappa = \beta\, \kappa\, d\zeta \end{cases} \qquad [40]$$

The variables $(\epsilon, \sigma, \zeta, \kappa)$ are expanded into power series which gives for order 1 the following system :

$$\begin{cases} E\, \epsilon_1 = \sigma_1 + \kappa_0\, \sigma_1 + \kappa_1\, \sigma_0 \\ \zeta_1 = \dfrac{\sigma_1}{\sigma_y} \\ \zeta_0\, \kappa_1 = \beta\, \kappa_0\, \zeta_1 \end{cases} \qquad [41]$$

Note that equation (41c) is deduced from order 0. By substituting (41b) and (41c) into (41a), one obtains the tangent modulus :

$$\sigma_1 = \frac{E}{1 + \kappa_0(1+\beta)} \epsilon_1 \qquad [42]$$

For order p, the linear problem deduced from (40) can be written as follows :

$$\left\{ \begin{array}{l} E\ \epsilon_p = \sigma_p + \kappa_0\ \sigma_p + \sum_{i=1}^{p-1} \kappa_i\ \sigma_{p-1} \\ \zeta_p = \dfrac{\sigma_p}{\sigma_y} \\ p\ \kappa_p\ \zeta_0 + \sum_{i=1}^{p-1} i\ \kappa_i\ \zeta_{p-1} = \beta \left(p\ \zeta_p\ \kappa_0 + \sum_{i=1}^{p-1} i\ \zeta_i\ \kappa_{p-1} \right) \end{array} \right. \qquad [43]$$

and then the constitutive relation is given in this form :

$$\sigma_p = \frac{E}{1 + \kappa_0(1+\beta)} \epsilon_p + \sigma_p^{nl} \qquad [44]$$

It is clear that the differential relation allows us to obtain a simple recurrence formula and to avoid the difficulties arising from the power coefficient β.