On two matrix-free continuation approaches for the determination of the bifurcation diagram of the von Kármán system

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ABSTRACT. *In this paper we present a combination of the asymptotic numerical continuation procedure with a preconditioned GMRES-solver as applied to the fourth-order non linear von Kármán problem. Using a mixed equal order finite element discretization, we show how our "matrix free"approach allows for an efficient determination of the non-trivial branch of the bifurcation diagram. We also show that, using a steplength estimation of Gervais and Sadiky [GER 04], one can limit himself to a third order prediction without loosing too much in the number of continuation steps.*

RÉSUMÉ. *Dans ce travail, nous proposons l'application de deux nouvelles combinaisons de l'approche asymptotique-numérique et d'un résoluteur itératif du type GMRES préconditionné, à la résolution du modèle du quatrième ordre de von Kármán. L'utilisation d'une méthode mixte de degré deux pour toutes les variables et d'un résoluteur sans matrice tangente permettent de calculer efficacement les branches non triviales du diagramme de bifurcation. En plus, nous montrons que la méthode prédicteur d'ordre trois proposée récemment par Gervais et Sadiky [GER 04] est très compétitive.*

KEYWORDS: *Mixed finite element, Newton-GMRES, ANM, Padé approximation.*

MOTS-CLÉS : *éléments finis mixtes, Newton-GMRES, MAN, approximation de Padé.*

1. Introduction

In this paper, we present and compare two continuation methods for the computation of branches of non-trivial equilibria of a von Kármán plate. The first one is based on the Asymptotic Numerical Method (ANM) while the other is a predictor-corrector continuation method based on a third order predictor. In both cases, we propose a strategy for the steplength selection and we avoid the construction of the tangent matrices through the use of an appropriately preconditioned GMRES iterative solver.

Consider a thin flat rectangular plate

$$\Omega = \{(x, y) \mid 0 \leq x \leq \ell, 0 \leq y \leq 1\},$$

subjected to a uniform compression applied in the normal direction to the vertical part of the boundary.

Figure 1. *The physical setting*

Let u denotes the vertical displacement of the median plane of the plate and

$$D_2 u = u_{x,x}. \qquad [1]$$

In view of studying large displacements, we shall assume that u is the first component of the solution of the von Kármán equations

$$\begin{cases} \Delta^2 u = [u, \phi] - \lambda D_2 u \\ \Delta^2 \phi = -[u, u] \end{cases} \qquad [2]$$

where Δ^2 is the bilaplacian and $[.,.]$ is the Poisson bracket defined by

$$[u, v] = u_{x,x} v_{y,y} + u_{y,y} v_{x,x} - 2u_{x,y} v_{x,y}. \qquad [3]$$

The unknown ϕ is the Airy stress potential and λ is a measure of the applied compression. We consider a clamped plate which means that the system [2] is supplemented with the following boundary conditions:

$$u = u_n = 0 \text{ on } \partial\Omega;$$

$$\phi = \phi_n = 0 \text{ on } \partial\Omega$$

(here the subscript n denotes the derivative in the direction normal to the boundary). For other boundary conditions we refer the reader to [HOL 84].

In the sequel, we shall propose a mixed formulation of the von Kármán equations which yields a problem of the form:

$$F(x,\lambda) := Lx + Q(x,x) + \lambda L_1 x = 0 \qquad [4]$$

$(x,\lambda) \in X \times \mathbb{R}$ where X and H are Hilbert spaces, L and $L_1 : X \to H$ are linear operators and $Q : X \times X \to H$ is a bilinear operator.

As such, the problem can be viewed as a quadratic problem which is augmented by some arclength equation and the solution $(x(s),\lambda(s))$ describes a curve in the (x,λ) space called the bifurcation diagram. To determine it, one takes advantage of the analyticity of $(x(s),\lambda(s))$ with respect to the parameter s to represent it as a power series, the coefficient of which are obtained recursively by solving linear problems in a fashion very similar to that of the classical Frobenius approach to the solution of ODE. For the method to be operational, one has to solve some intrinsically related problems concerning the radius of convergence and the relationship between the number of terms in the summation and the s-interval in which the resulting polynomial approximation gives a satisfactory result.

These questions will be addressed in the following sections which we shall now briefly sketch. In section two we provide some details on the ANM and discuss the steplength selection. In section three, we present the associated mixed formulation and its finite element discretization. Section four is devoted to the description of the solution algorithm while, in section five, we introduce a new third order predictor with which one can describe the diagram almost as efficiently as with the ANM. Finally, the paper is concluded in section five where we present and discuss in details some typical numerical results.

2. The Analytic Numerical Method and the problem of steplength selection

2.1. *A brief sketch of the ANM*

Let (x_0,λ_0) be a regular point of [4], that is to say a point where the Fréchet derivative of F, with respect to x, is invertible. In a neighbourhood of that point the solution curve is at least C^1. Given $(V,\delta) \in X \times \mathbb{R}$ a unit tangent vector to the solution path at (x_0,λ_0) chosen according to the desired orientation, we choose to parametrize the solution path with a pseudo-arclength parameter s defined by

$$s = (V, x - x_0)_X + \delta(\lambda - \lambda_0) \qquad [5]$$

Other choices of parametrization are possible and, in fact, desirable because they can lead to a larger range of validity (cf. [SAD 00] ,[GER 02]). Let us consider the augmented system

$$G(x,\lambda,s) = \left(\begin{array}{c} F(x,\lambda) \\ (V, x-x_0)_X + \delta(\lambda-\lambda_0) - s \end{array} \right) = 0. \qquad [6]$$

Since (x_0, λ_0) is a regular point of [4], the point $(x_0, \lambda_0, 0)$ is a regular point of [6] and it follows from the Implicit Function Theorem that there is an analytic function $s \mapsto (x(s), \lambda(s))$ defined in an open interval containing 0 such that $(x(0), \lambda(0)) = (x_0, \lambda_0)$ and $F(x(s), \lambda(s)) \equiv 0$. The asymptotic technique consists in expanding $(x(s), \lambda(s))$ in power series

$$(x(s), \lambda(s)) = \sum_{i=0}^{\infty} (x_i, \lambda_i) s^i \qquad [7]$$

To determine the series coefficients, we substitute it in [6]. An application of the identity principle leads to the following cascade of linear equations to be satisfied by the (x_i, λ_i):

– Order 1

$$\begin{array}{rcl} Lx_1 + \lambda_0 L_1 x_1 + Q(x_0, x_1) + Q(x_1, x_0) & = & -\lambda_1 L_1 x_0 \\ (V, x_1)_X + \delta\lambda_1 & = & 1. \end{array} \qquad [8]$$

– Order $p \geq 2$.

$$\begin{array}{l} Lx_p + \lambda_0 L_1 x_p + Q(x_0, x_p) + Q(x_p, x_0) = -\lambda_p L_1 x_0 \\ \qquad - \sum_{i=1}^{p-1} \left(\lambda_i L_1 x_{p-i} + Q(x_i, x_{p-i}) \right) \\ \qquad\qquad (V, x_p)_X + \delta\lambda_p \quad = \quad 0 \end{array} \qquad [9]$$

Computing the coefficients (x_i, λ_i), up to order N , we obtain a polynomial approximation

$$(T_N x(s), T_N \lambda(s)) = \sum_{i=0}^{N} (x_i, \lambda_i) s^i \qquad [10]$$

of the solution path through (x_0, λ_0). In the next section we deal with the task of obtaining a stepsize s^* such that this approximation is sufficiently accurate for $s \in [-s^*, s^*]$. When such a stepsize has been chosen, we use the value given by the polynomial approximation at one of the end points (selected according to the desired orientation of the path following) as a new starting point and proceed in the like manner.

2.2. *Steplength Selection*

Given a tolerance ε, the goal is to find a steplength s^* such that

$$\|((T_N x(s), T_N \lambda(s)) - (x(s), \lambda(s))\|_\infty \leq \varepsilon \text{ for } |s| < s^*. \quad [11]$$

Of course, we must have $s^* < \rho$ the radius of convergence of [7] which is typically finite.

Let us first derive an estimation of ρ. For this, let $]a, b[$ be the maximal interval of existence of the solution $(x(s), \lambda(s))$. If $-\infty < a$, when $s \to a^+$,

$$(x(s), \lambda(s), s) \to (\omega_0, \gamma_0, a)$$

a singular point of G , that is to say a point where $G(\omega_0, \gamma_0, a) = 0$ while $D_{x,\lambda} G(\omega_0, \gamma_0, a)$ is not invertible. The same result holds at b if $b < \infty$, [ALL 90]. Let $\tau_0 = a$ if $|a| = \min(|a|, b)$ and $\tau_0 = b$ otherwise. We assume that the following hypothesis is valid.

Hypothesis: On the circular boundary $C = \{z \in \mathbb{C} \mid |z| = \rho\}$ of the convergence disk, the only singularities are real.

For all the problems with a quadratic nonlinearity that we have considered, it was a valid assumption. So, assuming that it holds we have one of the following cases: a or b is the only singularity on C, a and b are the only singularities on C (in which case $|a| = b$). In any case, the radius of convergence is $\rho = |\tau_0|$. Then, under generic assumptions, it can be shown (cf. [GER 02], [SAD 00]) that $(x(s), \lambda(s))$ is not analytic at τ_0 and that, in a neighborhood of τ_0, it can be written in the following form:

$$(x(s), \lambda(s)) = \sum_{i=0}^{k-1} \alpha_i(s) \, (s - \tau_0)^{\frac{i}{k}} \quad [12]$$

where $k \geq 2$ is an integer, $\alpha_0, ..., \alpha_{k-1}$ are analytic at $s = 0$ and $s = \tau_0$ with $\alpha_0(\tau_0) = (\omega_0, \gamma_0)$ and $\alpha_1(\tau_0)$ is a non-zero tangent vector to the solution path at (ω_0, γ_0). Furthermore the tangent vector at $(x(0), \lambda(0))$ is orthogonal to $\alpha_1(\tau_0)$, so the singularity τ_0 corresponds to the first point encountered on the solution path where this orthogonal relation is satisfied [GER 02, equation (3.5)]. Let i_0 be the index of a non-zero component of $\alpha_1(\tau_0)$ and write the i_0-component of $(x(s), \lambda(s))$ as

$$P(s) = \sum_{i=0}^{k-1} \alpha_i^{i_0}(s) \, (s - \tau_0)^{i/k} \quad [13]$$

Using a classical result of Darboux, Hunter and Guerrieri ([HUN 80]) have shown that the Taylor coefficients P_j of $P(s)$ at $s = 0$ satisfy:

$$P_j\tau_0 - \frac{(j+\nu-1)}{j}P_{j-1} = O(j^{\nu-3}) \qquad [14]$$

where $\nu = -\frac{1}{k}$ is called, in complex analysis, the order of the singularity.

Neglecting the $O()$ term, two copies of this relation give:

$$\begin{array}{rcl} \frac{1}{\tau_0} & \approx & j\frac{P_j}{P_{j-1}} - (j-1)\frac{P_{j-1}}{P_{j-2}} \\ \nu & \approx & j\frac{P_j}{P_{j-1}}\tau_0 - j + 1. \end{array} \qquad [15]$$

Remark. These relations are exact for functions of the form $(s-\tau_0)^{-\nu}$ but are only asymptotic for functions such as [13]. So, it may happen that a large j is required to obtain a good estimation of the radius of convergence. In such a case, the ratio $\frac{P_{j-1}}{P_j}$ (Cauchy test) will sometimes give a good approximation of τ_0 and the second relation of [15] will give an artificial value of ν near 1.

We use [15] and two components of [7] to compute two approximations of the radius of convergence $\rho = |\tau_0|$, retaining the smallest one.

From the relations [15] we deduce [GER 02] that

$$\|(x(s),\lambda(s)) - (T_N x(s), T_N\lambda(s))\|_\infty \leq A_N(s) \qquad [16]$$

where $A_N(s) \simeq \|(x_N,\lambda_N)\|_\infty |s|^N \frac{r}{1-r}$ and $r = |\frac{s}{\tau_0}| < 1$. Thus [11] is satisfied if
$\|(x_N,\lambda_N)\|_\infty |s|^N \frac{r}{1-r} \leq \varepsilon$ for $|s| < s^*$. Some simple algebra yields

$$s^* = \overline{r}|\tau_0| \qquad [17]$$

with $\overline{r}$ the unique root in $[0,1]$ of

$$Q(r) = \|(x_N,\lambda_N)\|_\infty |\tau_0|^N r^{N+1} + \varepsilon r - \varepsilon.$$

3. A mixed formulation of the von Kármán equations

In order to apply the ANM to system [2], we must first put it in the form [4]. To this end, we shall follow the presentation given in [GER 97] and [DOS 03] which we

summarize here. For this we shall assume from now on, that the plate Ω is a convex polygon. The formulation is based on two simple principles. The first consists in a factorization of the bilaplacian through the introduction of the tensor unknowns

$$\underline{\underline{\sigma}}(u) = (\frac{\partial^2 u}{\partial x_i \partial x_j})_{i=1,2}^{j=1,2}, \quad \underline{\underline{\sigma}}(\phi) = (\frac{\partial^2 \phi}{\partial x_i \partial x_j})_{i=1,2}^{j=1,2}$$

and the second in an orthogonal decomposition of the hessians into

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^d + \text{tr}\,(\underline{\underline{\sigma}})\underline{\underline{I}},$$

where $\underline{\underline{\sigma}}^d$ is known as the deviatoric part of the tensor. To be more precise, we introduce some notations.

Instead of using tensor notation, we shall identify the deviator $\underline{\underline{\sigma}}^d$ with a vector in the following way

$$\begin{aligned}\underline{\underline{\sigma}} = \begin{pmatrix} s_{1,1} & s_{1,2} \\ s_{1,2} & s_{2,2} \end{pmatrix} &= \tfrac{1}{2}(s_{1,1}+s_{2,2}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &\quad + \begin{pmatrix} \frac{1}{2}(s_{1,1}-s_{2,2}) & s_{1,2} \\ s_{1,2} & -\frac{1}{2}(s_{1,1}-s_{2,2}) \end{pmatrix} \\ &\mapsto (\tfrac{1}{2}\,w, \underline{\sigma}^d), \quad \text{where } w = s_{1,1}+s_{2,2},\ \underline{\sigma}^d = (\tfrac{1}{2}(s_{1,1}-s_{2,2}), s_{1,2}).\end{aligned} \tag{18}$$

With this identification, we have the following identity

$$\begin{aligned}\underline{\underline{\sigma}} &\mapsto (\tfrac{1}{2} w \underline{\underline{I}}, \underline{\sigma}^d) \\ \underline{\underline{\tau}} &\mapsto (\tfrac{1}{2}\omega \underline{\underline{I}}, \underline{\tau}^d) \\ [\underline{\underline{\sigma}}, \underline{\underline{\tau}}] &= \tfrac{1}{2} w\,\omega - 2\,\underline{\sigma}^d \cdot \underline{\tau}^d\end{aligned} \tag{19}$$

Now let θ_0 be the pull-back of the boundary condition function as defined in [CIA 80] and which corresponds to the term $D_2 u$ in the system, we set

$$\Delta_0 = \Delta\theta_0, \quad S_0 = (\frac{1}{2}(\frac{\partial^2 \theta_0}{\partial x^2} - \frac{\partial^2 \theta_0}{\partial y^2}), \frac{\partial^2 \theta_0}{\partial x \partial y}).$$

If the functional spaces are

$$V = H_0^1(\Omega), \quad X = H^1(\Omega), \quad \Sigma = X^2,$$

using the $(.,.)_0$ notation for the L^2-inner product, we can define on $X \times X$, $X \times V$, and $(X \times \Sigma) \times V$, the continuous forms

$$\begin{aligned}
m(w,z) &= (w,z)_0, && w,z \in X \\
b(z,u) &= (\nabla z, \nabla u)_0, && z \in X, u \in V \\
c_1(u,s) &= \tfrac{1}{2}\left((\tfrac{\partial u}{\partial x}, \tfrac{\partial s}{\partial x})_0 - (\tfrac{\partial u}{\partial y}, \tfrac{\partial s}{\partial y})_0\right) && u \in V, s \in X \\
c_2(u,s) &= \tfrac{1}{2}\left((\tfrac{\partial u}{\partial x}, \tfrac{\partial s}{\partial y})_0 + (\tfrac{\partial u}{\partial y}, \tfrac{\partial s}{\partial x})_0\right) && u \in V, s \in X \\
t_0((w,\underline{\sigma}^d),v) &= \tfrac{1}{2}(\Delta_0\, w, v)_0 - 2(S_0 \cdot \underline{\sigma}^d, v)_0 && \underline{\sigma}^d \in \Sigma_d, v \in V \\
q((w,\underline{\sigma}^d),(\omega,\underline{\tau}^d),v) &= \tfrac{1}{2}(w\,\omega, v)_0 - 2(\underline{\sigma}^d \cdot \underline{\tau}^d, v)_0 && v \in V, \\
& && (w,\underline{\sigma}^d),(\omega,\underline{\tau}^d) \in X \times \Sigma.
\end{aligned} \quad [20]$$

The von Kármán nonlinear system can then be decomposed in the following manner: we seek a solution $(u, w, \underline{\sigma}^d, \phi, \omega, \underline{\tau}^d, \lambda)$ of the following nonlinear variational problem

$$\begin{aligned}
m(\underline{\sigma}_1^d, z) + c_1(z,u) &= 0, && \forall z \in X \\
m(\underline{\sigma}_2^d, z) + c_2(z,u) &= 0, && \forall z \in X \\
m(w,z) + b(z,u) &= 0, && \forall z \in X \\
m(\underline{\tau}_1^d, z) + c_1(z,\phi) &= 0, && \forall z \in X \\
m(\underline{\tau}_2^d, z) + c_2(z,\phi) &= 0, && \forall z \in X \\
m(\omega,z) + b(z,\phi) &= 0, && \forall z \in X \\
b(w,v) + \lambda t_0((w,\underline{\sigma}^d),v) - q((w,\underline{\sigma}^d),(\omega,\underline{\tau}^d),v) &= -(f,v)_0 && \forall v \in V \\
b(\omega,v) + q((w,\underline{\sigma}^d),(w,\underline{\sigma}^d),v) &= 0 && \forall v \in V
\end{aligned} \quad [21]$$

The first six variational equations are nothing but the expression of the relationship between the component of the auxiliary variables and the principal unknowns, the displacement u or the Airy stress ϕ as the case may be, while the last two equations are the variational form of [2]. That [21] is a correct variational representation of [2], is fully justified in [GER 97].

Now, if one uses the same P_2 conforming finite element approximation for the various spaces, approximations which will differ only by boundary conditions, one is led to a finite-dimensional non linear system of the following form:

$$
\begin{aligned}
&MS_1 + C_1U = 0\\
&MS_2 + C_2U = 0\\
&M\mathcal{T}_1 + C_1\Phi = 0\\
&M\mathcal{T}_2 + C_2\Phi = 0\\
&MW + B^tU = 0\\
&BW + \lambda\left(T_wW + T_1S_1 + T_2S_2\right) + Q_w(W,\Omega) + Q_1(S_1,\mathcal{T}_1) + Q_2(S_2,\mathcal{T}_2) = -F\\
&M\Omega + B^t\Phi = 0\\
&B\Omega - \left(Q_w(W,W) + Q_1(S_1,S_1) + Q_2(S_2,S_2)\right) = 0
\end{aligned}
\qquad [22]
$$

where $(U, W, S_1, S_2, \Phi, \Omega, \mathcal{T}_1, \mathcal{T}_2)$ is the vector of nodal-values whereas the matrices M, C_1, C_2, B, T_w, T_1, T_2 and the bilinear operators Q_w, Q_1, Q_2 are defined in the usual way in terms of the local bases of the various finite element spaces.

In view of simplifying the system, we eliminate the deviatoric variables. This leads us to the following system

$$
\begin{aligned}
MW + B^tU &= 0\\
(B + \lambda T_w)W - \lambda T_sU + Q((W,U),(\Omega,\Phi)) &= -F\\
M\Omega + B^t\Phi &= 0\\
B\Omega - Q((W,U),(W,U)) &= 0
\end{aligned}
\qquad [23]
$$

where

$$
\begin{aligned}
T_s &= T_1M^{-1}C_1 + T_2M^{-1}C_2\\
Q((W,U),(\Omega,\Phi)) &= Q_w(W,\Omega) + Q_1(M^{-1}C_1U, M^{-1}C_1\Phi)\\
&\quad + Q_2(M^{-1}C_2U, M^{-1}C_2\Phi)
\end{aligned}
\qquad [24]
$$

Finally, if we define $Z = (W, U, \Omega, \Phi)^t$ and

$$L = \begin{pmatrix} M & B^t & 0 & 0 \\ B & 0 & 0 & 0 \\ 0 & 0 & M & B^t \\ 0 & 0 & B & 0 \end{pmatrix},\ L_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ T_w & -T_s & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Q_Z(Z,Z) = \begin{pmatrix} 0 \\ Q((W,U),(\Omega,\Phi)) \\ 0 \\ -Q((W,U),(W,U)) \end{pmatrix},$$

we see that, when $f = 0$, [22] is indeed a problem of the form [4] to which one can apply the ANM according to [8] and [9], which, in this particular case reads as follows. For $p = 1, 2, \ldots, \infty$

$$\begin{aligned}
&(MW_p + B^t U_p) = 0 \\
&(B + \lambda_0 T_w)W_p - \lambda_0 T_s U_p + \\
&\quad Q((W_0,U_0),(\Omega_p,\Phi_p)) + Q((W_p,U_p),(\Omega_0,\Phi_0)) + \\
&\quad\quad \lambda_p(T_w W_0 - T_s U_0) = -\textstyle\sum_{j=1}^{p-1} \lambda_j (T_w W_{p-j} - T_s U_{p-j}) \\
&\qquad\qquad\qquad\qquad -Q((W_j,U_j),(\Omega_{p-j},\Phi_{p-j})) \\
&M\Omega_p + B^t \Phi_p = 0 \\
&B\Omega_p - 2Q((W_0,U_0),(W_p,U_p)) = \textstyle\sum_{j=1}^{p-1} Q((W_j,U_j),(W_{p-j},U_{p-j})) \\
&V \cdot Z_p + \delta\, \lambda_p = d_p
\end{aligned} \qquad [25]$$

with $d_p = 1$ for $p = 1$ and 0 otherwise.

4. Our resolution algorithm

In order to present our resolution algorithm in a simple manner, we introduce some further notations. Let us set

$$X = \begin{pmatrix} W \\ U \end{pmatrix}, Y = \begin{pmatrix} \Omega \\ \Phi \end{pmatrix}, K = \begin{pmatrix} M & B^t \\ B & 0 \end{pmatrix}, T = \begin{pmatrix} 0 & 0 \\ T_w & -T_s \end{pmatrix},$$

and denote by Q_X and Q_Y the matrices associated with the linear operators $Q((W_0,U_0),.)$ and $Q(.,(\Omega_0,\Phi_0))$ respectively. The system [25] which defines the succeeding terms of our power series can be written as

$$\begin{pmatrix} K+\lambda_0 T+Q_Y & Q_X & TX_0 \\ -2Q_X & K & 0 \\ V_X & V_Y & \delta \end{pmatrix} \begin{pmatrix} X_p \\ Y_p \\ \lambda_p \end{pmatrix} = \begin{pmatrix} G_p \\ H_p \\ d_p \end{pmatrix}. \qquad [26]$$

Eliminating the Y_p unknown, we are led to

$$\begin{pmatrix} K+\lambda_0 T+Q_Y+2Q_X K^{-1}Q_X & TX_0 \\ V_X+2V_Y K^{-1}Q_X & \delta \end{pmatrix} \begin{pmatrix} X_p \\ \lambda_p \end{pmatrix} = \begin{pmatrix} G_p-Q_X K^{-1}H_p \\ d_p-V_Y K^{-1}H_p \end{pmatrix}. \qquad [27]$$

This is the system to which we shall apply a preconditioned GMRES algorithm. According to [DOS 03], the preconditioner

$$\begin{pmatrix} K & 0 \\ 0 & 1 \end{pmatrix},$$

can be successfully used along the various branches. Typically, convergence will be very fast even with a Krylov subspace of small dimension (we generally used 15 Krylov vectors). However, it should be observed that the cost of evaluating one residual

$$\begin{pmatrix} K & 0 \\ 0 & 1 \end{pmatrix}^{-1} \left(\begin{pmatrix} K+\lambda_0 T+Q_Y+2Q_X K^{-1}Q_X & TX_0 \\ V_X+2V_Y K^{-1}Q_X & \delta \end{pmatrix} \begin{pmatrix} X_p \\ \lambda_p \end{pmatrix} - \begin{pmatrix} G_p-Q_X K^{-1}H_p \\ d_p-V_Y K^{-1}H_p \end{pmatrix} \right)$$

is not small. The computation can be summarized in the following way:

Let $X=(U,W)$ and λ be given, $TX_0, 2V_Y K^{-1}Q_X, G_p-Q_X K^{-1}H_p, d_P-V_Y K^{-1}H_p$ being computed once and for all,

1) compute $\underline{\sigma}^d = (M^{-1}C_1 U, M^{-1}C_2 U) = (S_1, S_2)$; this requires two resolutions with the matrix M;

2) compute $\lambda_0 TX = \lambda_0 (T_w W_0 + T_1 S_1 + T_2 S_2)$;

3) compute $Q_Y X$,

$$Q_Y X = Q_w(W, \Omega_0) + Q_1(S_1, \mathcal{T}_{1,0}) + Q_2(S_2, \mathcal{T}_{2,0})$$

4) compute $Y = K^{-1}Q_X X = (\Phi, \Omega)$; this requires one resolution with the K matrix;

5) compute $\underline{\tau}^d = (M^{-1}C_1\Phi, M^{-1}C_2\Phi) = (\mathcal{T}_1, \mathcal{T}_2)$; this requires two resolutions with the matrix M;

6) compute $Q_X Y$

$$Q_X Y = Q_w(W_0, \Omega) + Q_1(S_{1,0}, \mathcal{T}_1) + Q_2(S_{2,0}, \mathcal{T}_2)$$

7) compute the first component of the residual R_1 and $K^{-1}R_1$, this requires one more resolution with the K matrix.

8) compute the last component of the residual which requires one scalar product

Thus, each residual requires four linear resolutions with the matrix M and two with K. Since they are symmetric, the matrices are factorized once and for all in the form LDL^t and each resolution roughly amounts to a matrix-vector product.

5. A continuation procedure using a third order predictor

From now on, we shall stick to the notation of the last section and denote the state variable by Z instead of x.

When the nonlinearity of the problem

$$F(Z, \lambda) = 0 \qquad [28]$$

is not quadratic, the computation of high order Taylor coefficients is more complex even though it can still be performed in many situations (see [ZAH 04]). As an alternative, we present here a predictor-corrector continuation method based on the third order predictor provided by the degree 3 Taylor polynomial. This can be seen as a low order ANM, but with such a low degree expansion a correction is compulsory at each continuation step.

Let (Z_0, λ_0) be a regular point of [28]. As in section 1 we have a solution $(Z(s), \lambda(s))$ with $(Z(0), \lambda(0)) = (Z_0, \lambda_0)$. Starting at this point we want to obtain another point $(Z(s^*), \lambda(s^*))$ on the solution path with a steplength s^* as large as possible. To this end, we compute the third degree Taylor expansion at 0 of $(Z(s), \lambda(s))$:

$$(T_3 Z(s), T_3 \lambda(s)) = \sum_{i=0}^{3} s^i (Z_i, \lambda_i) \qquad [29]$$

The computation of the coefficients (Z_i, λ_i) is done as previously described and requires the resolution of three linear systems[1].

1. the superscript 0 denotes the evaluation at (Z_0, λ_0) and (V, δ) is again a unit tangent vector at (Z_0, λ_0).

5.0.0.1. Order $p = 1$

$$\begin{array}{rcl} D_Z F^0 \cdot Z_1 & = & -\lambda_1 D_\lambda F^0 \\ (V, Z_1)_V + \delta\lambda_1 & = & 1 \end{array} \qquad [30]$$

5.0.0.2. Order $p = 2, 3$:

$$\begin{array}{rcl} D_Z F^0 \cdot Z_p & = & -\lambda_p D_\lambda F^0 - F_p^s \\ (V, Z_p)_V + \delta\lambda_p & = & 0 \end{array} \qquad [31]$$

with F_p^s, $p = 2, 3$ given by

$$\begin{aligned} 2F_2^s &= D_Z^2 F^0 \cdot Z_1^2 + 2\lambda_1 D_\lambda D_Z F^0 \cdot Z_1 + \lambda_1^2 D_\lambda^2 F^0 \\ 6F_3^s &= D_Z^3 F^0 \cdot Z_1^3 + 3\lambda_1 D_\lambda D_Z^2 F^0 \cdot Z_1^2 + 6 D_Z^2 F^0 \cdot Z_1 Z_2 + 3\lambda_1^2 D_\lambda^2 D_Z F^0 \cdot Z_1 \\ &\quad + 6\lambda_1 D_\lambda D_Z F^0 \cdot Z_2 + 6\lambda_2 D_\lambda D_Z F^0 \cdot Z_1 + \lambda_1^3 D_\lambda^3 F^0 + 6\lambda_1\lambda_2 D_\lambda^2 F^0. \end{aligned}$$

These coefficients being computed, we choose as predictor the point

$$(Z_{pr}, \lambda_{pr}) = (T_3 Z(s^*), T_3 \lambda(s^*)). \qquad [32]$$

Then, we apply our GMRES-Newton type corrector to obtain the target point $(Z(s^*), \lambda(s^*))$ which is then used as a new starting point. Quite clearly, if the non-linearity is quadratic the above systems coincide with those obtained in the previous sections for the first four coefficients.

The crucial difficulty lies in the selection of a steplength s^*, as large as possible but such that [32] is close enough to the target point $(Z(s^*), \lambda(s^*))$ to have convergence of the correction method. This selection proceeds in two steps.

5.0.0.3. Step 1

We compute s_1^* by requiring that the sup-norm of the difference between the third and second orders Taylor polynomials is less or equal to the prescribed tolerance ε. Thus we get

$$s_1^* = \left(\frac{\varepsilon}{\|(Z_3, \lambda_3)\|_\infty} \right)^{1/3} \qquad [33]$$

5.0.0.4. Step 2

We seek a second steplength s_2^*, in general larger than s_1^*. Let $n + 1$ be the size of the (Z, λ) unknown. We denote by $Z_j^k, j = 0, \ldots, 3$, the k-component of the j-th coefficient and look for an $n + 1$ vector of steplengths. For any $k = 1, \ldots, n + 1$, there are two cases to be considered.

• Let's assume $Z_2^k \neq 0$ and consider the $(2,1)$-Padé approximation

$$h^k(s) = \frac{\alpha_k + \beta_k s + \gamma_k s^2}{1 + \delta_k s}$$

defined by

$$\frac{d^p h^k}{ds^p}(0) = \frac{d^p}{ds^p} Z^k(0), p = 0, 1, 2, 3.$$

These relations lead to

$$h^k(s) = Z_0^k + Z_1^k s + \frac{Z_2^k s^2}{1 + \delta_k s}$$

where $\delta_k = -\frac{Z_3^k}{Z_2^k}$. We will determine $s_{2,k}^*$ such that

$$\left| h^k(s) - \sum_{i=0}^{3} Z_i^k s^i \right| \leq \varepsilon \text{ for } 0 \leq s \leq s_{2,k}^* \qquad [34]$$

at least approximately. We see easily that

$$h^k(s) - \sum_{i=0}^{3} Z_i^k s^i = \frac{h_4^k s^4}{1 + \delta_k s} \qquad [35]$$

where h_4^k is the fourth order coefficient of the Taylor expansion of $h^k(s)$ at $s = 0$. This coefficient is given by

$$h_4^k = \frac{(Z_3^k)^2}{Z_2^k}.$$

We approximate the absolute value of the right-hand side of [35] by $|h_4^k s^4|$ (this approximation is better if $\delta_k \geq 0$). If $\delta_k \neq 0$, then with this approximation, [34] holds if

$$s_{2,k}^* = \left(\frac{\varepsilon}{|h_4^k|} \right)^{1/4}.$$

If $\delta_k = 0$, then $Z_3^k = 0$ and [34] holds for every s, in which case we put $s_{2,k}^* = \infty$.

• Let us assume $Z_2^k = 0$. If $Z_3^k \neq 0$, $h^k(s)$ does not exist and we set

$$s_{2,k}^* = \left(\frac{\varepsilon}{|Z_3^k|} \right)^{1/3}$$

analogically to the determination of s_1^*. If $Z_3^k = 0$, then we have $h^k(s) = Z_0 + Z_1 s$ and so [34] holds for every s and we set $s_{2,k}^* = \infty$.

Having $s^*_{2,k}$ for each $k \in \{1, \ldots, n+1\}$ we set

$$s^*_2 = \min\{s^*_{2,k}\,;\, k = 1, \ldots, n+1\}$$

and we take as optimal steplength

$$s^* = \max(s^*_1, s^*_2). \qquad [36]$$

This predictor-corrector method has been applied successfully in [SAD 00], [GER 04] to reputedly difficult nonlinear test problems. In the next section we present results of its application to the von Kármán problem.

6. Numerical results

For our numerical tests we consider a rectangular plate $\Omega = \{(x, y) \mid 0 \leq x \leq 2, 0 \leq y \leq 1\}$

The computational mesh is constituted of $8n^2$ triangles obtained by dividing $2n^2$ equal squares of size $h \times h$ into the four equal right triangles defined by the diagonals.

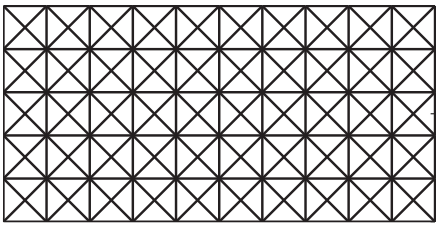


Figure 2. *The mesh*

Before presenting our results, a few technical points need to be clarified once and for all.

– As to the discretization, we have used a C^0 conforming finite element approximation of degree 2 for all the variables, on a mesh of size $h = \frac{1}{5}$. It is not our goal here to discuss that discretization and the interested reader can consult [DOS 03].

– Each resolution of a tangent system, be it for the computation of a power series coefficient or for a Newton-correction step, is performed with a preconditioned GMRES, for which we used Krylov subspaces of dimension 46 and a stopping criterion based on the reduction of the residual of a factor 10^{-8}. This very severe choice was made to maintain the influence of the solution procedure on the overall behavior of the algorithms to a minimum.

– For the computation of the approximation of the tangent vector we have used two distinct approaches. In the first case we simply used a secant approximation

$$V = \frac{u(\lambda_0 + \Delta\lambda) - u(\lambda_0)}{\Delta\lambda}, \quad \delta = 1, \quad (V, \delta) \to \frac{1}{||(V, \delta)||}(V, \delta),$$

with $\Delta = 0.1$ or 0.01[2]. In the ANM, this approximation was used only at the first stage of the continuation procedure, but, starting from the second step, the tangent vector was computed by directly differentiating the power series and by evaluating the result at the value s of the computed steplenght.

– For the sake of comparison, we repeated the computation of the first non trivial branch of the bifurcation diagram corresponding to $f = 0$ starting from the value $\lambda = 80$ and $u > 0$ at the center of the plate.

– Of course we first had to compute the solution corresponding to that state of the plate. To this end, we first performed a continuation along the perturbed diagram by putting $f = 1$ and starting with $u = 0$ and a small λ. When the value $\lambda = 80$ was reached, we set $f = 0$ and corrected the solution with our Newton-GMRES corrector thus falling back on the unperturbed diagram.

To obtain a 2D representation of the diagram, we could have chosen different characteristics of the solution. Here we have selected the maximal deflection of the plate, which on the first branch always occurs at the center. Before discussing the behaviour of our two continuation procedures, there is one more point which deserves a thorough discussion, that of the scalar product selected in the pseudo-arclength equation [5].

6.1. *Estimation of the radius of convergence and the order of the singularity*

Our estimation of the singularity τ_0 of the series [7] is based on the use of the λ-component and the i_0-component of [7] for which the index i_0 corresponds to the displacement at the central node.

In all the computations, we have used the weighted norm and scalar product defined by

$$\|(u, \phi, \lambda)\|^2 = |u|_2^2 + |\phi|_2^2 + c\lambda^2.$$

where c is a parameter which plays a major role in the computation, preventing a bad scaling of the λ-component of the tangent vectors. To illustrate this, we study its influence on the behavior of the estimations of the optimal steplength s^*, the radius of convergence ρ and the order ν of the singularity.

Our numerical results were obtained for $\lambda \approx 80.0$, using Taylor polynomials of degrees 10 and 20.

It can be seen in Table 1 that, for $c = 1$, the singularity parameter ν and the radius of convergence ρ do not seem to converge whereas the optimal step s^* is too small

2. This approach is admittedly costly since the computation of $u(\lambda_0 + \Delta\lambda)$ requires 4-5 Newton corrections. The way around this lazy method is well documented [KEL 77] and we shall count this calculation as one resolution step

	$n = 10$		$n = 20$	
	λ	u_{i_0}	λ	u_{i_0}
ρ	1.5098	9.9787	0.18804	1.4126
s^*	1.5098	9.9195	0.18804	1.4126
ν	8.6163	-9.1555	-18.005	-19.512

Table 1. *Estimations of* s^*, ρ, ν *with* $c = 1$

for the cost of the computation ($n = 20$), which results in too slow a progress of the continuation process.

	$n = 10$		$n = 20$	
	λ	u_{i_0}	λ	u_{i_0}
ρ	33.635	32.957	33.926	33.991
s^*	13.032	12.995	21.145	21.149
ν	-0.62042	-0.76463	-0.50644	-0.46234

Table 2. *Estimation of* s^*, ρ, ν *with* $c = 10$

The situation changes radically if we select $c = 10$ (Table 2). The singularity parameter ν stabilizes around $-\frac{1}{2}$, while the radius of convergence and the optimal step increase noticeably. With this value, we can proceed with the continuation without any convergence problems. However, if we further increase c to the value $c = 100$, the situation changes again.

	$n = 10$		$n = 20$	
	λ	u_{i_0}	λ	u_{i_0}
ρ	28.616	28.577	28.596	28.590
s^*	14.374	14.370	20.425	20.425
ν	-0.48793	-0.50765	-0.49560	-0.50293

Table 3. *Estimation of* s^*, ρ, ν *with* $c = 100$

Indeed, even if the singularity parameter ν is well approximated and the predicted optimal step is of the same order as for $c = 10$, the radius of convergence is smaller. To shed some light on this apparently contradictory behavior, we propose the following heuristic explanation.

Let us first recall that the radius of convergence ρ of the series is defined geometrically by the condition that, at $s = \rho$, the tangent vector to the solution curve is orthogonal to the tangent vector at $s = 0$ (see [GER 02, equation (3.5)]). Let us denote the latter by (δ, V) and the former by (μ, W). In our weighted inner product, that condition is written as

$$c\mu\delta + (V, W) = 0,$$

whatever the choice of the scalar product for the V component.

Next, let us, for the sake of explanation, assume that we are in the following simple situation: $X = I\!R$ (that is the state unknown is a simple scalar), the solution curve is as the one in figure 6, $(\lambda(0), x(0))$ is in the upper half-plane and (δ, V) is in the first quadrant. Then, the above equation can be written as

$$\frac{W}{\mu} = -\frac{c}{\frac{V}{\delta}}.$$

In our geometrically simple situation, it is clear that, a small tangent vector slope $|\frac{W}{\mu}|$ means a large ρ and conversely. Thus the smaller $c > 0$, the bigger the convergence radius. This is well illustrated in our numerical example by the results obtained for $c = 10$ and $c = 100$, but apparently contradicted by those corresponding to $c = 1$. However a closer look at the latter reveals that even for $n = 20$ the values are not converged and one should first wonder why. Well, if c is small, then $\rho >> 1$ is big which, in turn, implies that the coefficients P_j in [14] tend to zero very fast (at least like $o((\rho - \epsilon)^{-j})$ for any $\epsilon > 0$) and this appears to create numerical difficulties with the resolution of the system of asymptotic equations [15]. However, we have no clue as to the exact nature of those difficulties.

In conclusion, the choice of the weighted inner product is important in the sense that the components of the tangent vectors should be equilibrated. In this paper, the correct equilibrium was determined by trial and error and we fixed c to the value of $c = 50$.

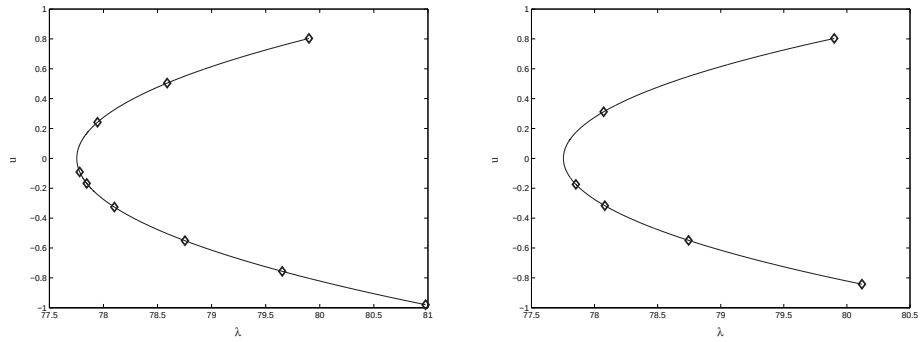
6.2. *Continuation based on the ANM*

We now evaluate the performance of the continuation procedure based on the ANM with the steplength control described in Section 2.2. We have used a tolerance $\varepsilon = 10^{-5}$ and have not corrected the resulting approximations.

The results that are presented were obtained with series truncated at orders $n = 10$ and 20. Figure 3 shows the projection in the (λ, u_{i_0})- plane of the solution path where the markers are the continuation points.

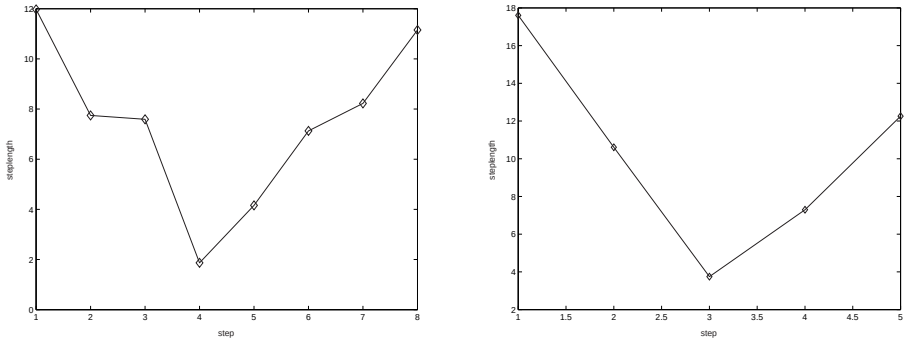
Quite clearly, the value of the selected steplength depends in a crucial way on the truncation order. To clarify this, we present a graphical comparison (see figure 4) of the steplength values obtained with $n = 10$ and $n = 20$.

In order to evaluate the precision of the prediction and to justify the absence of correction, we have computed the local error at each continuation points. For this,



Truncation order $= 10$ Truncation order $= 20$

Figure 3. *Bifurcation diagrams with different steplength selection*



Truncation order $= 10$ Truncation order $= 20$

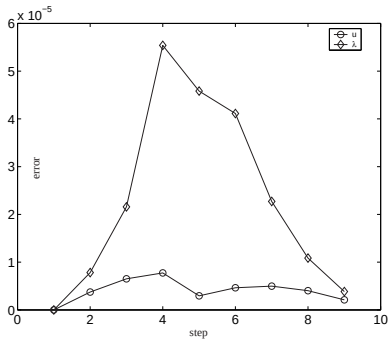
Figure 4. *Steplength evaluation at each continuation stage*

we computed the distance with respect to the $||\cdot||_\infty$ norm between the solution obtained with the ANM and the one given by the Newton method using the convergence criterion:

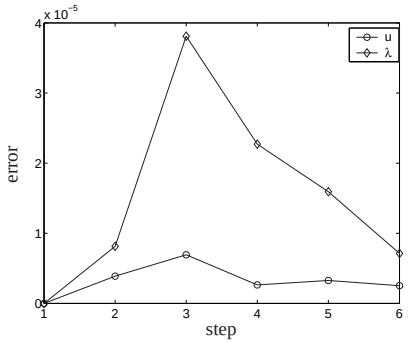
$$\|F(Z_i,\lambda_i)\|_\infty \leq 10^{-10} \text{ and } \frac{\|(Z_i,\lambda_i)-(Z_{i-1},\lambda_{i-1})\|_\infty}{\|(Z_i,\lambda_i)\|_\infty} \leq 10^{-7}. \qquad [37]$$

which is rather stringent. This led us to figure 5.

Figure 5 clearly illustrates that the approximation of the state variable Z is more accurate than that of the parameter λ, mainly around the critical point. This figure also illustrates that the local error has the same order of magnitude as the chosen tolerance.



Truncation order $= 10$



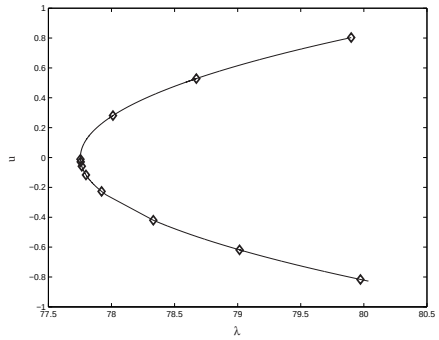
Truncation order $= 20$

Figure 5. *Prediction error on u and λ.*

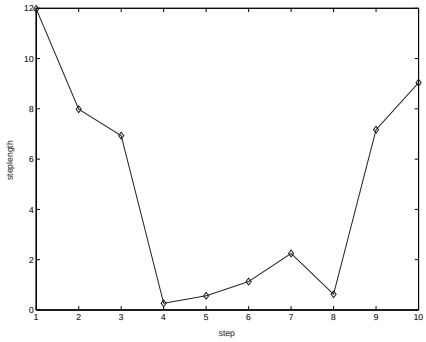
We conclude this section with a comparison between the steplenght selection approach used in this work and the one based on the formula[3]

$$s^*_{\text{Co}} = \left(\frac{\varepsilon |U_{1,h}|_2}{|U_{n,h}|_2} \right)^{\frac{1}{n-1}} \qquad [38]$$

which was proposed by Cochelin ([COC 94]). Figure 6 illustrates the bifurcation diagram and the computed steplength. In both cases it is apparent that continuation based on formula [38] requires more work in the neighborhood of the critical point.



Bifurcation diagram with $n = 10$



Steplength computation

Figure 6. *Results obtained with formula [38]*

3. $|\cdot|_2$ denotes the H^2 Sobolev semi-norm.

6.3. *Continuation with a third order predictor method*

In this last section we present a brief survey of the results obtained with the third order predictor developed in section 5.

Proceeding as for the ANM, we present in figure 7, the results obtained with two different tolerances, namely $\epsilon = 10^{-2}$ and 10^{-1}. As expected an increase of ϵ results in a bigger s^* and one may wonder how far one can go. We have made a test with $\epsilon = 1.0$ but the resulting step is too big and the method fails to converge at the second stage. The limit on epsilon is most certainly problem dependent and, in absence of further theoretical informations, it should be determined empirically.

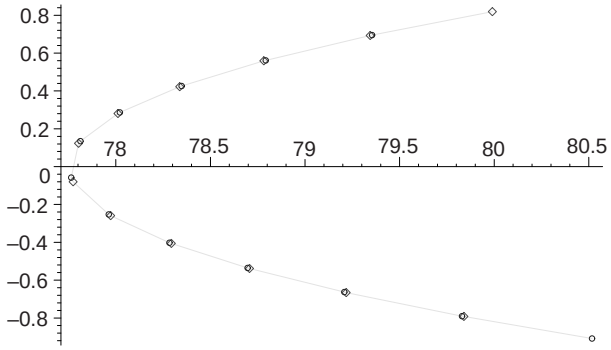


Figure 7. *Continuation with a small value of the tolerance*

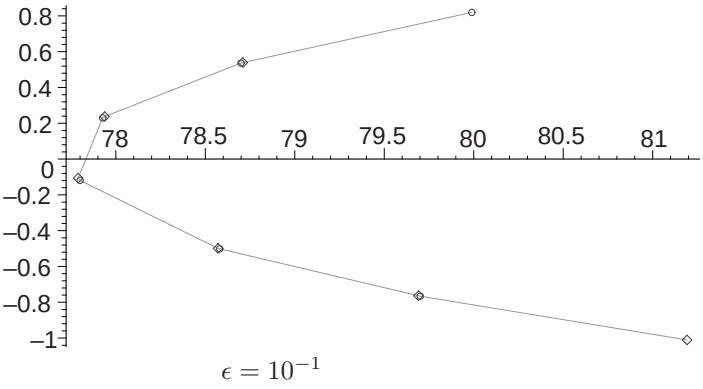


Figure 8. *Continuation with a higher value of the tolerance*

The reader may notice that, at each step, there corresponds two markers. The first of them identifies the corrected solution, while the immediate next one identifies the

	$\|\|Z_p - Z_c\|\|_\infty$	$\|\lambda_p - \lambda_c\|$	N_c
1	0.161761661905793E-01	0.389192732808397E-02	4
2	0.147410478079684E-01	0.423539887775348E-02	4
3	0.771873682796720E-02	0.896187108978097E-03	4
4	0.128629714302662E-01	0.157921069597933E-02	4
5	0.128629714302662E-01	0.157921069597933E-02	4
6	0.108588529003555E-01	0.305735175516020E-02	4
7	0.115127196732274E-01	0.299153726206214E-02	4

Table 4. *Correction size for the third order predictor*

extra solution computed for the determination of the tangent V. In order to validate the steplength computation, we have, at each step, computed the $||\cdot||_\infty$ norm of the correction in the case $\epsilon = 10^{-1}$. The results are presented in table 4 where Z_p and λ_p stands for the predicted values, Z_c and λ_c for the corrected ones while N_c denotes the number of Newton-correction steps. These results show that we are well within the prescribed tolerance and also that, contrary to the ANM, the variable λ is more easily approximated than the variable Z.

Table 4 also allows for a cost comparison with the ANM. Indeed, as noted at the beginning of this section, each resolution of a tangent system amounts to a similar application of our iterative solution procedure. Thus it seems fair to compare the two approaches on the basis of the number of such resolutions.

Let us use this basis to compare the results obtained with the ANM, with a truncation order of $N = 20$, with that of the third order predictor with $\epsilon = 10^{-2}$. According to figures 3 and 7 it takes 5 steps of the ANM and 12 steps of the third order predictor. In the first case, forgetting about the initialization, the computation of the coefficients requires 20 resolutions per step for a total of 100 resolutions. In the second case, at each step the initialization requires one resolution for the computation of the tangent vector, the computation of the coefficients takes 3 resolutions and, similarly to what happens in table 4, the final step requires 4 Newton corrections. The number of resolutions per step is thus 8, hence, for twelve steps, the total is 96 which is comparable. It should be noted that we can neglect the initialisation step in the ANM approach because we can recover the next tangent vector from the power series. If we had used the ANM with $n = 10$, the number of steps would have been 12 for a total of 120 resolutions which is even worse.

Moreover, one should note that, if the goal is to move from the state $\lambda = 80\ u > 0$ to $\lambda = 80, u < 0$ as fast as possible, then the ANM with $n = 20$ should be compared to the third-order predictor with $\epsilon = 10^{-1}$. Since in this last case, the number of steps is only 6, we see that the total count is only 48 and that the third order predictor is a clear winner. Since this predictor can also be used for non quadratic non linear problems we feel that it exhibits many qualities as a general purpose tool.

7. Conclusion

In this paper we have shown how to adapt the ANM to the resolution of the cubic von Kármán problem via an iterative solver. This solver is characterized by the fact that each iteration is rather costly, requiring the resolution of 4 linear systems involving two different matrices but also by the fact that the preconditioner can be kept fixed, hence these matrices are assembled and factorized only once. These two aspects combine to give an efficient algorithm. However, in view of the size of the problem, the control of the number of steps remains crucial. Consequently, in the first part of the paper, we focused on the question of the numerical determination of the steplength and of the radius of convergence. Comparing various approaches to steplength selection, we have shown that the one proposed in [GER 02] is more efficient, in term of the number of steps, than the one presented in [COC 94] and this, without requiring too much extra work.

Since we are working with an augmented system, where the supplementary equation is defining the arclength parameter with respect to which the series development is performed, the question of the choice of the scalar product to be used in that supplementary equation raised itself. We have shown experimentally that it is advisable to introduce a weight in the bifurcation parameter term of the inner product, in order to equilibrate the tangent vector components.

Finally we have compared the classical ANM approach to a new predictor-corrector recently developed in [GER 04]. Our simulations show that this predictor-corrector is extremely competitive for two reasons. First, it only requires the computation of the first four Taylor coefficients. However, contrary to the ANM, here a correction step is essential. Secondly, it allows for the use of a rather large steplength even if it is generally smaller than the one obtained for the ANM with a large truncation order. A cost comparison showed that if the precision requirements are stringent, the two approaches are comparable but that, if they are slightly relaxed, than the third order approach is a clear winner.

8. References

[ALL 90] ALLGOWER E. L., GEORG K., *Numerical continuation methods*, Springer-Verlag, Berlin, 1990.

[CIA 80] CIARLET P. G., RABIER P., *Les équations de von Kármán*, Springer, Berlin, 1980.

[COC 94] COCHELIN B., "A path following technique via an Asymptotic Numerical Method", *Computers and Structures*, vol. 53, num. 5, 1994, p. 1181-1192.

[DOS 03] DOSSOU K., PIERRE R., "A Newton-GMRES approach for the analysis of the post-buckling behavior of the solutions of the von Kármán equations", *SIAM Journal of scientific computing*, vol. 24, num. 6, 2003, p. 1994-2012.

[GER 97] GERVAIS J.-J., OUKIT A., PIERRE R., "Finite element analysis of the buckling and mode jumping of a rectangular plate", *Dynam. Stability Systems*, vol. 12, num. 3, 1997, p. 161-185.

[GER 02] GERVAIS J., SADIKY H., "A new steplength control for continuation with the asymptotic numerical method", *IMA J. Numer. Anal.*, vol. 22, 2002, p. 207-229.

[GER 04] GERVAIS J., SADIKY H., "A continuation method based on a high order predictor and an adaptive steplength control", *ZAMM.*, vol. 84, num. 7, 2004.

[HOL 84] HOLDER E. J., SCHAEFFER D., "Boundary conditions and mode jumping in the von Kármán equations", *SIAM J. Math. Anal.*, vol. 15, num. 3, 1984, p. 446-458.

[HUN 80] HUNTER C., GUERRIERI B., "Deducing the properties of singularities of functions from their Taylor series coefficients", *SIAM J. Appl. Math.*, vol. 39, 1980, p. 248-263.

[KEL 77] KELLER H., "Numerical Solution of Bifurcation and Nonlinear Eigenvalue Problem", RABINOWITZ P. H., Ed., *Applications of bifurcation theory*, Academic Press, New-York, 1977.

[SAD 00] SADIKY H., "Une classe de méthodes de continuation utilisant le développement de Taylor", PhD thesis, Université Laval, Québec, Canada, 2000.

[ZAH 04] ZAHROUNI H., AGGOUNE W., BRUNELOT J., POTIER-FERRY M., "Asymptotic numerical method for strong non-linearities", *Revue Européenne des Éléments Finis*, vol. 13, 2004.