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A numerical scheme to solve boundary value problems involving singular perturbation

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ABSTRACT

In this study, a numerical method is presented in order to approximately solve singularly perturbed second-order differential equations given with boundary conditions. The method uses the set of monomials whose degrees do not exceed a prescribed N as the set of base functions, resulting from the supposition that the approximate solution is a polynomial of degree N whose coefficients are to be determined. Then, following Galerkin's approach, inner product with the base functions are applied to the residual of the approximate solution polynomial. This process, with a suitable incorporation of the boundary conditions, gives rise to an algebraic linear system of size $N + 1$. The approximate polynomial solution is then obtained from the solution of this resulting system. Additionally, a technique, called residual correction, which exploits the linearity of the problem to estimate the error of any computed approximate solution is discussed briefly. The numerical scheme and the residual correction technique are illustrated with two examples.

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1. Introduction

It is quite often in science and engineering that modelling a phenomenon involves the examination of differential equations with very small (or very large) parameters. The main problem with such equations is the dramatically different behaviour exhibited by the solutions when these parameters approach zero (or infinity). This concept, known as *singular perturbation*, makes it harder to obtain accurate approximate solutions by any numerical scheme (Zhang, 2002). A historical treatment including an in-depth review of some ideas aiming to obtain accurate solutions of singularly perturbed problems can be found in (Lagerstrom & Casten, 1972). Interested reader can learn more from the fairly rich literature on the topic such as (Miller, O'Riordan, & Shishkin, 2012) and (Kevorkian & Cole, 1996).

A rudimentary insight on the concept of singular perturbation is provided by the following example from O'Malley's book (O'Malley, 1997):

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Given $\epsilon \neq 0$, the solutions of the algebraic equation $x^2 + \epsilon x - 1 = 0$, which are $x_1 = \frac{-\epsilon - \sqrt{\epsilon^2+4}}{2}$ and $x_2 = \frac{-\epsilon + \sqrt{\epsilon^2+4}}{2}$, approaches the values -1 and 1. These two values are just the solutions of the equation $x^2 - 1 = 0$, which is obtained by taking the limit of the original equation as ϵ goes to 0. This situation is termed as *regular perturbation*. On the other hand, singular perturbation can be observed when we have ϵ in the quadratic term; namely when we have the equation $\epsilon x^2 + x - 1 = 0$.

Solutions of the equation are $x_1 = \frac{-1+\sqrt{1+4\epsilon}}{2\epsilon}$ and $x_2 = \frac{-1-\sqrt{1+4\epsilon}}{2\epsilon}$. The limits of these values as $\epsilon \to 0$ are 1 and $-\infty$. However, $-\infty$ is not a solution of the limit equation, which is $x - 1 = 0$. Therefore, the equation $\epsilon x^2 + x - 1 = 0$ is an example of a singularly perturbed equation.

In this paper, we will be dealing with the following second-order differential equation considered in a real interval $[0, b]$ and given with boundary conditions:

$$\epsilon y''(x) + p(x)y'(x) + q(x)y(x) = f(x),\ y(0) = \alpha, y(b) = \beta \qquad (1)$$

Here, $p(x), q(x)$ and $f(x)$ are known real-valued functions, $\varepsilon < 1$ is a known positive perturbation parameter, while α and β are real boundary values. Problems of type (1) are encountered in numerous applications. For instance, it was used to model the motion of fluids (Prandtl, 1904) with small viscosity and human pupillary light reflex (Longtin & Milton, 1988). Solving singularly perturbed equations of this type also plays an important role in variational problems in control theory (Mohapatra & Natesan, 2011).

Although the theory of linear differential equations has been extensively developed, the case of variable coefficients does not have a general method of solution even in the second-order case. Thus, numerous attempts have been made to obtain numerical solutions of problems of type 1. To name a few of these, a uniformly convergent finite difference scheme for singular perturbation problems was presented by Kadalbajoo and Patidar (Kadalbajoo & Patidar, 2006), whereas the same goal was realised by applying a seventh-order compact difference scheme to same type of problems in (Chakravarthy, Phaneendra, & Reddy, 2007). Collocation methods were also presented for second-order singularly perturbed problems in conjunction with Bessel (Yüzbaşı, 2015) and Laguerre (Yüzbaşı, 2017) polynomials by Yüzbaşıs. In Pandit & Kumar (2014), a version of Problem (1) where a second perturbation parameter is present in the first derivative term was solved with the help of Haar wavelets. Delayed versions of Problem (1) have also been considered by various authors. For instance, a matrix method involving Chebyshev polynomials was presented in Gülsu, Öztürk, & Sezer (2011) to solve singularly perturbed delay differential equations where the delay appears in the first derivative term.

Lastly, an efficient algorithm based on finite differences was presented in Habib & El-Zahar (2007) to solve non-linear singularly perturbed differential equations of second order.

The presentation of the paper is organised as follows: The numerical scheme of the paper is explained in Section 2. Section 3 deals with a short description of a way of estimating the error of an already obtained solution, thus making it possible to improve the solution. Numerical examples are considered in Section 4, where the technique of residual correction are also demonstrated. Finally, conclusions regarding the simulation results are given in Section 5.

2. Method of solution

In this section, we will outline the procedure to be used to solve Problem (1). A similar method was previously applied to high-order Fredholm integro-differential equations in (Türkyılmazoğlu, 2014) and to same type of equations with singular kernel functions in (Yüzbaşı & Karaçayır, 2016a). In addition, a modified version of it was applied to Lotka-Volterra predator-prey system (Yüzbaşı & Karaçayır, 2017a) and Riccati equations encountered in quantum physics (Yüzbaşı & Karaçayır, 2016b).

To begin with, we assume that the exact solution $y_{\text{exact}}(x)$ of Problem (1) corresponding to a fixed value of the perturbation parameter ϵ can be written as a power series given by

$$y_{\text{exact}}(x) = \sum_{k=0}^{\infty} a_k x^k.$$

Truncating this power series after the N-th degree term and denoting the obtained polynomial of degree N by $y_N(x)$, we have

$$y_N(x) = \sum_{k=0}^{N} a_k x^k = \mathbf{X}(x)\mathbf{A}, \tag{2}$$

where

$$\mathbf{X}(x) = \begin{bmatrix} 1 & x & x^2 & \dots & x^n \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_N \end{bmatrix}^T.$$

Here, the approximate solution to be found is the polynomial $y_N(x)$ of degree N. The aim of the algorithm is to determine the unknown coefficients a_k, $k = 0, 1, \dots, N$ and the vector $\mathbf{A}$ of length $N + 1$ is just a matrix formed by these coefficients. In order to make it easier to express the derivatives of $y_N(x)$, we define the auxiliary matrix $\mathbf{B}$ to be the $(N + 1) \times$

$(N+1)$ square matrix having entries $\mathbf{B}_{i,i+1} = i$ for $i = 1, 2, \ldots, N$ and $\mathbf{B}_{i,j}$ otherwise. In a more explicit manner, we define

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & N \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

With the auxiliary matrice **B** such defined, one can easily check that the derivatives present in Problem (1) satisfy the following matrix equalities:

$$y'_N(x) = \mathbf{X}(x)\mathbf{B}\mathbf{A},\ y''_N(x) = \mathbf{X}(x)\mathbf{B}^2\mathbf{A}.$$

Next, we substitute in the differential equation of Problem (1) the matricial counterparts of the approximate solution $y_N(x)$ and its two derivatives and obtain

$$\mathbf{G}(x)\mathbf{A} = f(x), \tag{3}$$

where

$$\mathbf{G}(x) = \epsilon\mathbf{X}(x)\mathbf{B}^2 + p(x)\mathbf{X}(x)\mathbf{B} + q(x)\mathbf{X}(x).$$

Now, in order to convert the differential Equation (3) into a system of linear equations in the unknowns a_k, we take inner product of Equation (3) with the elements of the set $\mathbf{\Phi} = \{1, x, x^2, \ldots, x^N\}$. In our context, 'inner product', means the inner product in the space $L^2([0, b])$ of square integrable functions. More explicitly, if f and g are two functions from $L^2([0, b])$, then their inner product is defined by

$$\langle f, g\rangle = \int_0^b f(x)g(x)\mathrm{d}x.$$

Each inner product thus taken will give rise to a linear equation involving the unknown coefficients a_k. Since there are $N+1$ functions in the basis set $\mathbf{\Phi}$, so is the number of inner products; thus a linear system $\mathbf{WA} = \mathbf{F}$ of $N+1$ equations in $N+1$ unknowns is formed as a result. The $(N+1) \times (N+1)$ coefficient matrix **W** and the column matrix **F** of length $N+1$ are given by

$$\mathbf{W}_{i,j} = < x^{i-1}, \mathbf{G}(x)_{1,j} >,\ \mathbf{F}_{i,1} = < x^{i-1}, f(x) >,\ 0 \le i \le N,\ 0 \le j \le N.$$

More explicitly, **W** and **F** are given by

$$\mathbf{W} = \begin{bmatrix} \langle 1, \mathbf{G}(x)_{1,1} \rangle & \langle 1, \mathbf{G}(x)_{1,2} \rangle & \dots & \langle 1, \mathbf{G}(x)_{1,N+1} \rangle \\ \langle x, \mathbf{G}(x)_{1,1} \rangle & \langle x, \mathbf{G}(x)_{1,2} \rangle & \dots & \langle x, \mathbf{G}(x)_{1,N+1} \rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle x^N, \mathbf{G}(x)_{1,1} \rangle & \langle x^N, \mathbf{G}(x)_{1,2} \rangle & \dots & \langle x^N, \mathbf{G}(x)_{1,N+1} \rangle \end{bmatrix},$$

$$\mathbf{F} = \begin{bmatrix} \langle 1, f(x) \rangle & \langle x, f(x) \rangle & \langle x^2, f(x) \rangle & \dots & \langle x^N, f(x) \rangle \end{bmatrix}^T.$$

We would like the approximate solution $y_N(x)$ to satisfy the boundary conditions of Problem (1). For this reason, we now incorporate the boundary conditions $y(0) = \alpha$ and $y(b) = \beta$ into the system $\mathbf{WA} = \mathbf{F}$. The condition $y(0) = \alpha$ implies $a_0 = \alpha$, or in matricial form $\mathbf{uA} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}\mathbf{A} = \alpha$. Likewise, the condition $y(b) = \beta$ can be written as $\mathbf{vA} = \begin{bmatrix} 1 & b & b^2 & \dots & b^N \end{bmatrix}\mathbf{A} = \beta$. Now, in order to force the approximate solution $y_N(x)$ to satisfy the boundary conditions, we replace the last two rows of the system $\mathbf{WA} = \mathbf{F}$ by the matrix counterparts of these conditions and obtain a new system given by $\tilde{\mathbf{W}}\mathbf{A} = \tilde{\mathbf{F}}$. More explicitly, the last two rows of $\tilde{\mathbf{W}}$ are $\mathbf{u}$ and $\mathbf{v}$, whereas the last two entries of $\tilde{\mathbf{F}}$ are α and β. The solution of the new linear system $\tilde{\mathbf{W}}\mathbf{A} = \tilde{\mathbf{F}}$ yields the unknown coefficients a_k, hence the approximate solution

$$y_N(x) = \sum_{k=0}^{N} a_k x^k.$$

This method can be generalised to partial differential equations of any dimension in a straightforward manner. For instance, if the dimension (excluding time) of a boundary value problem given over a rectangular region is 2, the monomials in the basis set $\mathbf{\Phi}$ are replaced by $x^i y^j t^k$ where $i, j, k = 0, 1, \dots, N$ and the inner products are evaluated over a three dimensional rectangular region instead of the interval $[a, b]$. A way of incorporating the boundary conditions in vicinity of variable boundary data goes as follows: On each face of the boundary region we mark (possibly equidistant) collocation points and we impose the boundary data only at these points. This can be thought of as a mixture of the present method with collocation method utilised only on the boundary. An example of such a scheme can be found in Yüzbaşı & Karaçayır (2017b).

3. Error estimation and residual correction

In this section, we will outline a method that aims to obtain accurate solutions of any equation using an already obtained solution of it. This technique, known as *residual correction*, was first used in conjunction with

a collocation method aiming to solve ordinary differential equations given with two-point boundary conditions in Oliveira (1980). Here, we will briefly describe how it can be extended to Problem (1). Note that the process of residual correction is analogous to iterative refinement introduced by Wilkinson in Wilkinson (1963) in the case of linear algebraic equations.

Let us assume that we have obtained an approximate solution $y_N(x)$ of Problem (1). Since $y_{\text{exact}}(x)$ is the exact solution, it satisfies the differential equation and thus

$$\epsilon y_{\text{exact}}''(x) + p(x)y_{\text{exact}}'(x) + q(x)y_{\text{exact}}(x) - f(x) = 0. \tag{4}$$

We now replace $y_{\text{exact}}(x)$ by the approximate solution $y_N(x)$ and obtain

$$\epsilon y_{N''}(x) + p(x)y_{N'}(x) + q(x)y_N(x) - f(x) = R_N(x), \tag{5}$$

where $R_N(x)$ is the residual of the approximate solution $y_N(x)$. Subtracting Equation (5) from Equation (4) and rearranging yields

$$e_N''(x) + p(x)e_N'(x) + q(x)e_N(x) = -R_N(x), \tag{6}$$

where $e_N(x) = y_{\text{exact}}(x) - y_N(x)$ is the error of the approximate solution $y_N(x)$. Equation (6) is the same as the original equation with the non-homogeneous term $f(x)$ being equal to $-R_N(x)$, which we can calculate since we already know $y_N(x)$. The unknown function of this equation is the error function $e_N(x)$. Since the approximate solution satisfies the boundary conditions, we have

$$e_N(0) = y_{\text{exact}}(0) - y_N(0) = 0,\ e_N(b) = y_{\text{exact}}(b) - y_N(b) = 0. \tag{7}$$

Equation (6) given with the homogeneous boundary conditions (7) is called the 'error problem' associated with the approximate solution $y_N(x)$. Applying the method of Section 2 to this error problem for some choice of M, we can obtain an approximate solution of it, which we will denote by $e_{N,M}(x)$. This approximate solution is an *estimate* of the actual error function $e_N(x)$. Keeping in mind that $y_{\text{exact}}(x) = y_N(x) + e_N(x)$, we can use this estimate to compute a new approximate solution

$$y_{N,M}(x) = y_N(x) + e_{N,M}(x)$$

of Problem (1). The error of this new solution $y_{N,M}(x)$, called the *corrected* solution, is directly related to the accuracy of the error estimate $e_{N,M}(x)$. More explicitly, if we denote the error of $y_{N,M}(x)$ by $E_{N,M}(x)$, it is true that $E_{N,M}(x) = y\text{exact}(x) - y_{N,M}(x) = e_N(x) - e_{N,M}(x)$. Therefore, the success of residual correction directly depends on the accuracy of the error estimate $e_{N,M}(x)$. This situation will be made more clear in the example problems that will be covered in the next section.

4. Numerical examples

In this section, we will apply the numerical scheme described in Section 2 to two example problems. All the required calculations have been carried out in MATLAB. In order to assess the accuracy of our results, we will use the L_∞-norm given by

$$\| e_N(x)\|_\infty = \sup_{0\le x\le b} |e_N(x)|$$

where $e_N(x) = y_{\text{exact}}(x) - y_N(x)$ is the actual error of the approximate solution $y_N(x)$.

Example 1. Our first problem is the following ordinary differential equation with constant coefficients also examined in (Body, 2000; Yüzbaşı, 2015):

$$\epsilon y''(x) + y(x) = 0,\ y(0) = 0, y(1) = 1. \tag{8}$$

The exact solution of this problem is $y_{\text{exact}}(x) = \frac{\sin(x/\sqrt{\epsilon})}{\sin(1/\sqrt{\epsilon})}$. In order to measure the effect of both the method parameter N and the perturbation parameter ϵ on the accuracy of approximate solutions, we solved the problem using various N and ϵ values. Figure 0 compares the absolute errors of the approximate solutions obtained by various N values corresponding to $\epsilon = 2^{-4}$ and those obtained by various ϵ values corresponding to fixed value of $N = 10$. It is obvious from the two graphs that the solutions corresponding to greater N and ϵ values are more accurate as expected.

Table 1 compares the maximum absolute error values obtained by the present scheme with Bessel collocation method (Yüzbaşı, 2015) for various values of ϵ and N. The values in the table reveal that the present scheme outperforms Bessel collocation method in terms of maximum absolute error for Problem (8).

For the purpose of improving some of the already obtained approximate solutions corresponding to $\epsilon = 2^{-6}$, we applied residual correction and obtained estimates of the actual error $e_5(x)$ using $M = 6$ and $M = 7$. Similarly, we obtained estimates of the actual error $e_8(x)$ using $M = 9$ and $M = 10$. The resulting error estimates are compared with the actual

Table 1. Comparison of the present method with Bessel collocation method with respect to L_∞-norm of the error function $e_N(x)$ obtained for several values of N and ϵ in Problem (8).

	Bessel collocation method (Yüzbaşı, 2015)				Present method			
ϵ	$N=7$	$N=10$	$N=12$	$N=14$	$N=7$	$N=10$	$N=12$	$N=14$
2^{-2}	$6.141E-5$	$2.574E-8$	$1.007E-10$	$3.113E-13$	$6.895E-7$	$4.937E-11$	$4.076E-12$	$1.138E-11$
2^{-4}	$2.693E-2$	$2.832E-5$	$5.110E-7$	$6.460E-9$	$2.130E-4$	$9.168E-8$	$5.626E-10$	$3.041E-10$
2^{-6}	$5.148E-1$	$1.171E-1$	$6.665E-3$	$3.126E-4$	$3.850E-2$	$1.870E-4$	$4.735E-6$	$1.564E-6$

Table 2. Maximum absolute errors of approximate solutions of Problem (8) compared with their improvements obtained using several N and M values for $\varepsilon = 2^{-6}$.

	$N=5$	$N=5, M=6$	$N=5, M=7$	$N=8$	$N=8, M=9$	$N=8, M=10$
$\epsilon = 2^{-6}$	$4.004E-1$	$9.850E-2$	$3.850E-2$	$5.047E-3$	$1.335E-3$	$1.870E-4$

Table 3. Condition number of the coefficient matrices $\mathbf{W}$ and maximum absolute error $\| e_N\|_\infty$ corresponding to $\epsilon = 2^{-4}$ and several N values in (8).

	$N=15$	$N=20$	$N=25$	$N=30$
$\kappa(\mathbf{W})$	$2.948E17$	$1.131E18$	$4.932E18$	$4.587E18$
$\| e_N\|_\infty$	$2.819E-10$	$1.319E-9$	$1.337E-9$	$1.062E-9$

error functions in Figure 2. It is clearly seen from the graphs that the error estimates are in good agreement with the actual errors. We can improve the original approximate solutions with the help of these estimates as explained in Section 3. For instance, the corrected solution corresponding to $N = 5, M = 6$ is formed by $y_{5,6}(x) = y_5(x) + e_{5,6}(x)$. The closeness of the error estimates to the actual errors implies that the improved solutions are more accurate than the original approximate solutions, which is demonstrated in Table 2.

It is worth noting that increasing the parameter N indefinitely does not necessarily give rise to more accurate solutions. This is partly due to the coefficient matrix $\mathbf{W}$ of the linear system $\mathbf{WA} = \mathbf{F}$ in the algorithm being ill-conditioned. The condition numbers $\kappa(\mathbf{W})$ of $\mathbf{W}$ corresponding to various bigger N values are listed versus the maximum absolute errors attained corresponding to these N values in Table 3. The values in the table indicate that no improvement in accuracy is achieved when one employs the present method using N values beyond 15. Still, the high levels of accuracy attained by modest N values as shown in Figure 1 and Table 1 shows the effectivity of the present scheme.

Example 2. Secondly, let us consider the inhomogeneous equation of second order examined in (Kadalbajoo & Patidar, 2006; Yüzbaşı, 2015) given by

$$-\epsilon y''(x) + \frac{1}{x+1}y'(x) + \frac{1}{x+2}y(x) = g(x),\ y(0) = 1 + 2^{-1/\epsilon}, y(1) = 2 + e, \tag{9}$$

where $g(x) = \left(\frac{1}{x+1} + \frac{1}{x+2} - \epsilon\right)e^x + \frac{2^{-1/\epsilon}(x+1)^{1+1/\epsilon}}{x+2}$. The exact solution of this problem is known to be $y_{\text{exact}}(x) = e^x + 2^{-1/\epsilon}(x+1)^{1+1/\epsilon}$. We solved this problem for several values of the perturbation parameter ϵ and using various values for the parameter N of the numerical scheme. The absolute errors of the approximate solutions corresponding to $\epsilon = 2^{-3}$ and $\epsilon = 2^{-6}$ for various N values are depicted in Figure 2. It is understood that absolute error values

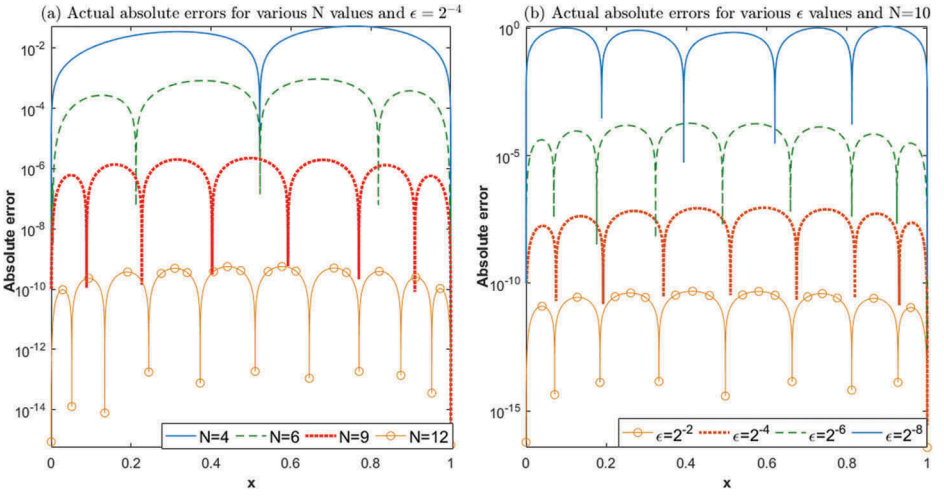

Figure 1. Graphics of the (a) actual absolute errors $|e_N(x)|$ corresponding to the approximate solutions obtained for the fixed ϵ value of 2^{-4} and (b) actual absolute errors $|e_{10}(x)|$ corresponding to the approximate solutions obtained by the fixed value of $N = 10$.

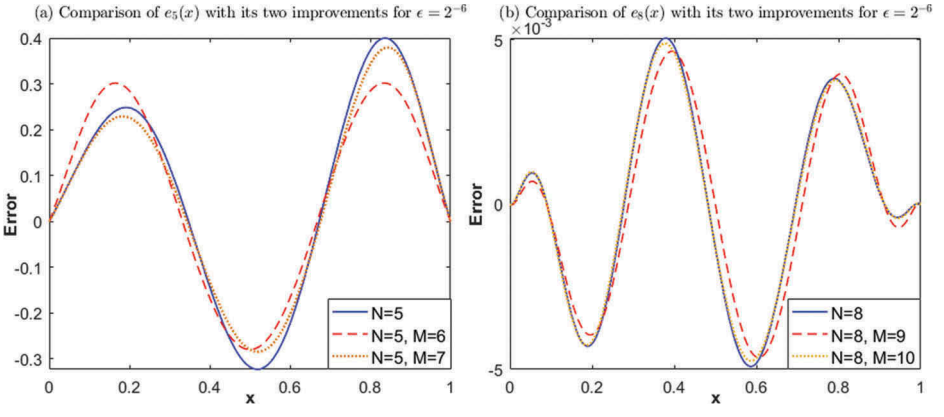

Figure 2. Graphics of (a) the actual error $e_5(x)$ compared with its two estimates (b) the actual error $e_8(x)$ compared with its two estimates obtained for $\epsilon = 2^{-6}$.

decrease substantially as we increase the parameter N, with the only exception of $N = 12$ compared to $N = 10$ in the graph on the left. Additionally, the error values corresponding to the greater perturbation parameter value of $\epsilon = 2^{-3}$ are significantly smaller than those corresponding to $\epsilon = 2^{-6}$, as expected. This situation is more apparent in Figure 3, which illustrates the absolute error values of the approximate solutions obtained for different values of the perturbation parameter ϵ corresponding to $N = 6$ and $N = 10$.

In Table 4, the maximum absolute error $\| e_N\|_\infty$ corresponding to the approximate solution $|y_N(x)|$ obtained using various values of N and ϵ are

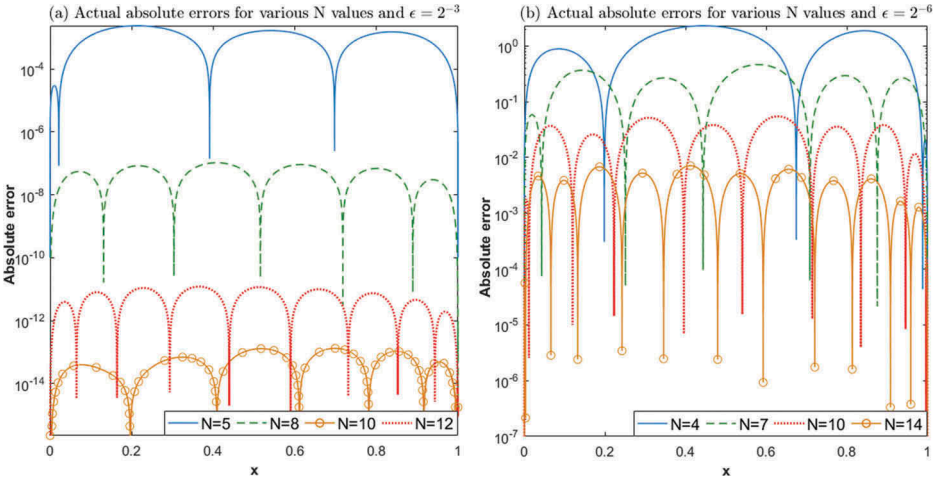

Figure 3. Graphics of the actual absolute errors obtained using various N values corresponding to (a) $\epsilon = 2^{-3}$ and (b) $\epsilon = 2^{-6}$.

Table 4. Comparison of the present method with two other methods according to the L_∞-norm of the error function $e_N(x)$ obtained using several values of N and ϵ in Problem (9).

	Bessel col. method (Yüzbaşı, 2015)			Present method			Finite dif. method (Kadalbajoo & Patidar, 2006)	
ϵ	$N = 5$	$N = 8$	$N = 10$	$N = 5$	$N = 8$	$N = 10$	$N = 64$	$N = 256$
2^{-2}	$6.272E-5$	$1.139E-8$	$1.824E-11$	$4.365E-6$	$1.237E-10$	$1.210E-13$	$7.8E-5$	$4.9E-6$
2^{-3}	$1.858E-2$	$7.610E-6$	$1.434E-11$	$2.391E-3$	$1.052E-7$	$1.323E-13$	$3.0E-4$	$1.9E-5$
2^{-4}	$1.794E-1$	$9.630E-3$	$4.495E-4$	$5.807E-2$	$3.135E-4$	$3.455E-6$	$1.1E-3$	$6.7E-5$
2^{-5}	$5.314E-1$	$1.615E-1$	$5.235E-2$	$4.029E-1$	$1.992E-2$	$1.566E-3$	$4.0E-3$	$2.5E-4$
2^{-6}	$9.319E-1$	$5.301E-1$	$3.410E-1$	$1.442E-0$	$2.281E-1$	$5.465E-2$	$1.6E-2$	$9.8E-4$
2^{-7}	$1.283E-0$	$9.404E-1$	$7.571E-1$	$3.779E-0$	$1.003E-0$	$4.390E-1$	$7.0E-2$	$3.9E-3$

compared with those of the solutions obtained by Bessel collocation method (Yüzbaşı, 2015) and mesh finite difference method(Kadalbajoo & Patidar, 2006). It is understood that results of the present scheme are almost always more accurate than those of Bessel collocation method for the same parameter values N and ϵ. On the other hand, according to the listed parameter values of N (which has different meanings in the contexts of two methods), the present scheme outperforms the mesh finite difference method for large values of ϵ whereas the situation is the opposite for small values of ϵ.

In order to have an opinion on the utility of residual correction in this problem, we attempt to improve the approximate solutions $y_4(x)$ and $y_7(x)$ for $\epsilon = 2^{-3}$ and the approximate solutions $y_5(x)$ and $y_8(x)$ for $\epsilon = 2^{-6}$. For this purpose, corresponding to the approximate solutions obtained for $\epsilon = 2^{-3}$, we estimate the error of $y_4(x)$ using $M = 5$ and $M = 6$ and the error of $y_7(x)$ using $M = 8$ and $M = 9$. As for $\epsilon = 2^{-6}$, we choose $M = 7$

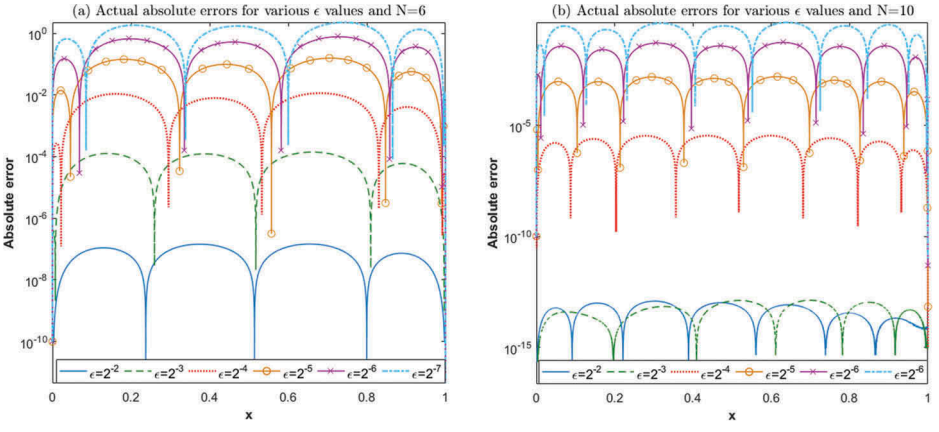

Figure 4. Graphics of the actual absolute errors obtained using various ϵ values corresponding to (a) $N = 6$ and (b) $N = 10$.

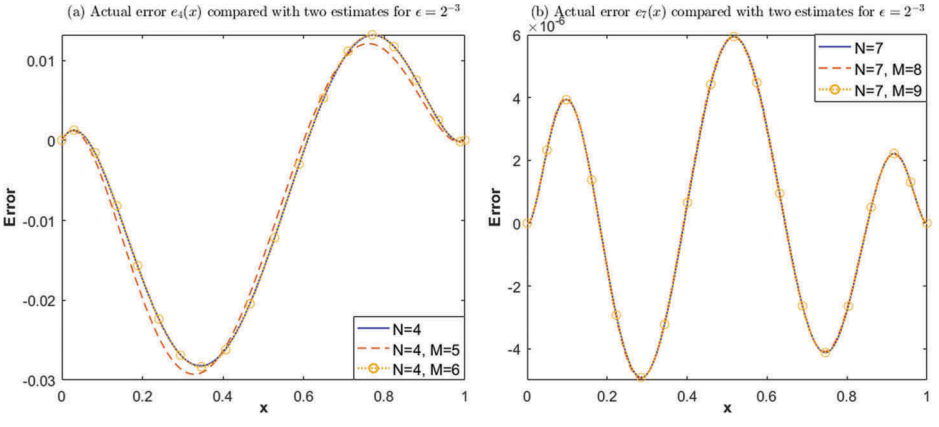

Figure 5. Graphics of (a) the actual error $e_4(x)$ compared with its two estimates (b) the actual error $e_7(x)$ compared with its two estimates obtained for $\epsilon = 2^{-3}$.

and $M = 8$ corresponding to $y_6(x)$ and $M = 10$ and $M = 12$ corresponding to $y_9(x)$. The resulting error estimates are illustrated together with the actual errors in Figures 4, 5 and 6. For each of the four (N, ϵ) pairs, the error estimates greatly agree with the actual error; furthermore, the error estimate obtained using bigger M value is even closer to the actual error.

The error estimates we have obtained can be used to compute the corrected approximate solutions as before; namely, if $e_{N,M}(x)$ is the error estimate of $y_N(x)$ obtained using M, then the corrected solution is calculated as $y_{N,M}(x) = y_N(x) + e_{N,M}(x)$. The maximum absolute errors for each of the thus computed corrected solutions corresponding to the already selected (N, ϵ) pairs are given in Table 5. Values in the table indicate a significant improvement in the accuracy of the approximate solutions as a result of residual correction.

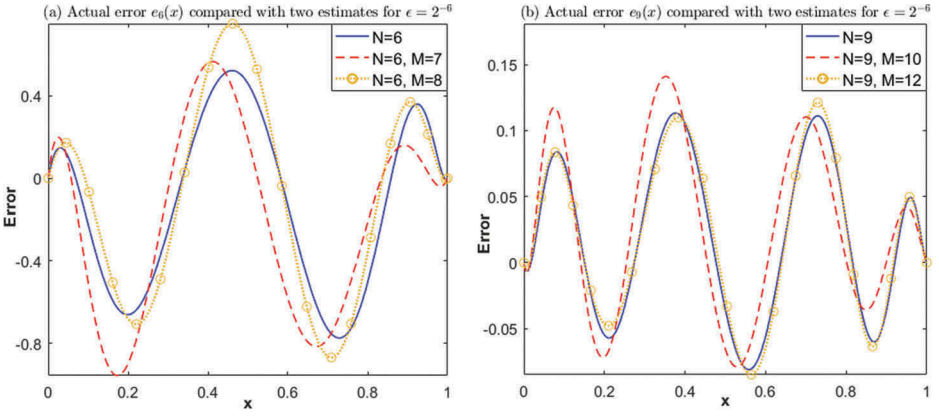

Figure 6. Graphics of (a) the actual error $e_6(x)$ compared with its two estimates (b) the actual error $e_9(x)$ compared with its two estimates obtained for $\epsilon = 2^{-6}$.

Table 5. Maximum absolute errors of approximate solutions of Problem (9) compared with their improvements obtained using several N and M values.

	$N=4$	$N=4, M=5$	$N=4, M=6$	$N=7$	$N=7, M=8$	$N=7, M=9$
$\epsilon = 2^{-3}$	$2.822E-2$	$2.391E-3$	$1.403E-4$	$5.939E-6$	$1.052E-7$	$2.895E-12$
	$N=6$	$N=6, M=7$	$N=6, M=8$	$N=7$	$N=7, M=8$	$N=7, M=9$
$\epsilon = 2^{-6}$	$7.746E-1$	$4.637E-1$	$2.281E-1$	$1.133E-1$	$5.465E-2$	$1.078E-2$

5. Conclusion

In this paper, we have presented a Galerkin-like scheme to obtain approximate polynomial solutions of the singularly perturbed second-order differential equation given with two boundary conditions. The method involves transforming the original differential equation to a linear system of algebraic equations by making proper use of Galerkin-type inner products taken with a set of base functions. We have applied the scheme to two problems. Simulation results have revealed that the accuracy of the computed approximate solutions undergoes a substantial increase as we increase the value of the parameter N of the method. In addition, although the error values naturally increase as we decrease the perturbation parameter ε, they are generally within acceptable limits even for small values of it. We have also exhibited that residual error correction can be used to estimate the error of an obtained approximation with high accuracy. Having obtained several approximate solutions, this enabled us to construct more accurate solutions compared to these original solutions, in the case of both example problems. Lastly, comparisons reveal that the present scheme outperforms two other popular methods in terms of maximum error. On the whole, simulation results make it clear that the numerical scheme presented in this paper can be relied on to give accurate approximate solutions when applied to ordinary differential equations involving singular perturbation.

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