Constitutive model of coupled damage-plasticity and its finite element implementation

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ABSTRACT. *In this work we present a general theoretical framework for developing a constitutive model capable of coupling two basic types of inelastic behaviour, plasticity and damage. We elaborate upon the main novelty with respect to the previous models of this type, which pertains to a systematic use of criteria for defining the elastic domain, both for plasticity and damage, which can be adapted to a very wide variety of engineering materials, from metals with voids on one side to concrete compaction on the other side. The numerical implementation is first presented for a simple one-dimensional case, and subsequently extended to 2D and 3D criteria which are adequate for either metals or concrete.*

RÉSUMÉ. *Dans ce travail on présente un cadre théorique général du développement des modèles de couplage de deux types de comportement anélastique, la plasticité et l'endommagement. On introduit la nouveauté principale par rapport aux modèles précédents de ce type en utilisant un critère pour définir le domaine élastique valable aussi bien pour la plasticité que pour l'endommagement, qui peut être adopté pour une grande variété des matériaux, d'une part pour les métaux poreux et, d'autre part, pour le béton en compaction. L'implantation numérique est d'abord présentée pour un cas unidimensionnel simple et ensuite généralisée pour les critères 2D et 3D, pertinents à des métaux ou des bétons.*

KEYWORDS: *coupled plasticity-damage, finite element implementation.*

MOTS-CLÉS : *plasticité-endommagement couplé, implantation éléments finis.*

1. Introduction

Two classes of constitutive models of inelastic behaviour, those of plasticity and damage, are most frequently employed for engineering materials. Plasticity model (e.g. see [HIL 50], [LUB 90] or [SIM 98] for a comprehensive account), with a clearly defined yield criterion for identifying the occurrence of plastic deformation which does not affect the elastic response which remains the same in elastic loading and unloading, is mostly applicable to simplified representation of metals and alloys. Similarly, the continuum damage model (e.g. see [LEM 88] or [KRA 96]), with its ability to account for the micro-cracking induced change of elastic response but without residual deformation upon loading, is only a simplified representation of cracking of concrete or soil-mechanics materials. It has been long recognized (e.g. see [LEM 85]) that the validity of these basic constitutive models can be significantly extended when combined into a single coupled model of damage and plasticity. The applications which can be tackled in such a manner are typically outside of the scope of each of the basic models when acting alone, ranging from the metals with voids (e.g. see [GUR 77], [NEE 84]) to compacting concrete (e.g. see [GAT 99]). Typical attempts of merging these two constitutive models have been developed along a one-track approach (e.g. see [LEM 85], [BEN 88], [SIM 87] or [JU 89], among others) where the damage model is first applied to produce the equivalent stress measure for virgin material or so called effective stress, followed by the plasticity model defining the yield criterion in terms of this effective stress. Even nowadays, the same kind of developments are being carried out to provide a reliable representation of constitutive behavior of concrete (e.g. see [MES 98], [JOH 99] or [MAH 00], among others), each proposing a rather complex computational procedure to obtain stresses and tangent moduli.

In this work we depart from these previous developments to propose a new kind of coupled damage-plasticity model which has the following desirable features:
(i) both plasticity and damage model are built around the corresponding criteria indicating the presence of inelastic deformation. For the former this amounts to constructing the classical yield criterion only in terms of total rather than the effective stress, whereas for the latter this implies using a somewhat less standard continuum damage formulation with a clearly defined damage criterion.
(ii) the numerical computation on both plasticity and damage part of the model can be carried out in parallel (even making use of the presently available parallel computer architecture) and their final results can then be merged through a local iterative procedure which imposes the uniqueness of stresses.

The outline of the paper is as follows. In the next section we briefly present the main idea of coupling the plasticity and damage in a single constitutive model, starting with a simple one-dimensional setting and extending it to a more general case. Numerical implementation is discussed in Section 3 and several numerical examples are presented in Section 4. In Section 5 we state some closing remarks.

2. Theoretical formulation

For clarity, we first start with a simple one-dimensional setting allowing for only non-trivial component of the stress tensor and the strain tensor. It is mostly a matter of tensor calculus to develop the corresponding 2D or 3D version of any equation governing the model. However, the particular forms of the yield or damage criterion provide a very significant wealth of possible applications in higher dimensional case, making the model useful for both metals and soil-mechanics materials.

2.1. *Basic hypothesis*

The one-dimensional constitutive model of coupling the damage and plasticity can be built on 3 basic hypotheses: additive decomposition of the total strain field, the strain energy and finally the plasticity and damage criteria. The first of them implies that the total deformation can be additively decomposed into elastic part ϵ^e, plastic part ϵ^p and damage part ϵ^d, leading to

$$\epsilon = \epsilon^e + \epsilon^p + \epsilon^d. \qquad [1]$$

We note in passing that the damage models rarely make use of the notion of the damage strain, with few exceptions (e.g. see [CAR 97] or [YAZ 90]) dealing only with rate form of the equation (1) .

The second ingredient of the model governing elastic response is specified in terms of strain energy. Assuming the simplest quadratic form in terms of the corresponding state variables we can write the strain energy as the sum of elastic, damage and plastic parts:

$$\Psi(\epsilon, \epsilon^d, D, \xi^d, \epsilon^p, \xi^p) = \Psi^e(\epsilon^e) + \Psi^d(\epsilon^d, D) + \Xi^p(\xi^p) + \Xi^d(\xi^d), \qquad [2]$$

where,

$$\Psi^e(\epsilon^e) \quad = \quad \frac{1}{2}\epsilon^e C^e \epsilon^e \qquad [3]$$

In equations (2) and (3) above, the elastic modulus is denoted with C^e, ϵ^p and D are internal variables of plastic strain and damage compliance, ξ^p and ξ^d are strain like internal variables describing the hardening phenomenon for plasticity and damage, respectively, and $\Xi^p(\xi^p)$ and $\Xi^d(\xi^d)$ are the corresponding hardening functions. Instead of working with the damage strain energy $\Psi^d(\cdot)$, it is convenient to appeal to the Legendre transformation (e.g. see [STR 86]) to define complementary energy, which is postulated as a quadratic form in stress with

$$
\begin{aligned}
\chi^d(\sigma, D) &:= \sigma\epsilon^d - \Psi^d(\epsilon^d, D) \\
&= \frac{1}{2}\sigma D\sigma \ ,
\end{aligned}
\qquad [4]
$$

The final group of basic model ingredients is provided to specify the elastic domain, where no change of internal variables takes place, along with the yield criteria Φ^p and damage criteria Φ^d with

$$
\begin{aligned}
\Phi^p(\sigma, q^p) &= |\sigma| - (\sigma_y^p - q^p) \leq 0 \\
\Phi^d(\sigma, q^d) &= |\sigma| - (\sigma_f^d - q^d) \leq 0 \ ,
\end{aligned}
\qquad [5]
$$

where q^p and q^d are stress like variables describing the hardening phenomena, σ_y is the yield limit and σ_f is the fracture limit.

We next show that all the remaining ingredients can be derived from these three basic ones simply by the standard thermodynamics consideration and the principles of maximum plastic and maximum damage dissipation. To start, one can provide the local form of the second principle of thermodynamics (e.g. see [LUB 90], [LEM 88]), stating that the total inelastic dissipation is always non-negative. Subsequently, by making use of the results in (2) to (4) we can further write

$$
\begin{aligned}
0 \leq \dot{\mathcal{D}} &= \sigma\dot{\epsilon} - \dot{\Psi} \\
&= (\sigma - \frac{\partial \Psi^e}{\partial \epsilon^e})\dot{\epsilon^e} + \overbrace{\sigma\dot{\epsilon^p} - \frac{\partial \Xi^p}{\partial \xi^p}\dot{\xi^p}}^{\dot{\mathcal{D}}^p} + \dot{\sigma}(\frac{\partial \chi^d}{\partial \sigma} - \epsilon^d) \\
&\quad + \underbrace{\frac{\partial \chi^d}{\partial D}\dot{D} - \frac{\partial \Xi^d}{\partial \xi^d}\dot{\xi^d}}_{\dot{\mathcal{D}}^d},
\end{aligned}
\qquad [6]
$$

where $\mathcal{D}^p$ and $\mathcal{D}^d$ denote, respectively, plastic and damage dissipation.

In an elastic process with no change of plastic variables, with $\dot{\epsilon}^p = 0$ and $\dot{\xi^p} = 0$, and no change of damage variables, with $\dot{D} = 0$ and $\dot{\xi^d} = 0$, which further implies that no dissipation takes place, with $\dot{\mathcal{D}}^p = 0$, $\dot{\mathcal{D}}^d = 0$ and $\dot{\mathcal{D}} = 0$, it follows from (6) above that the stress can be obtained from the elastic strain energy:

$$
\sigma := \frac{\partial \Psi^e}{\partial \epsilon^e} = C^e\epsilon^e \qquad [7]
$$

and that the damage strain is defined through stress and the current value of he damage compliance:

$$\epsilon^d := \frac{\partial \chi^d}{\partial \sigma} = D\sigma. \qquad [8]$$

Assuming that the stress constitutive equation in (7) and damage strain definition in (8) remain the same in an inelastic process we can conclude from dissipation inequality in (6) that

$$0 < \dot{\mathcal{D}} = \underbrace{\sigma\dot{\epsilon^p} + q^p\dot{\xi^p}}_{\dot{\mathcal{D}}^p} + \underbrace{\frac{1}{2}\sigma\dot{D}\sigma + q^d\dot{\xi^d}}_{\dot{\mathcal{D}}^d}, \qquad [9]$$

where we introduced stress-like variables q^p and q^d which are as thermodynamically conjugate to hardening variables ξ^p and ξ^d and which can be computed from

$$\begin{aligned} q^p &= -\frac{\partial \Xi^p}{\partial \xi^p} \\ q^d &= -\frac{\partial \Xi^d}{\partial \xi^d}. \end{aligned} \qquad [10]$$

2.2. *Plasticity model*

In an inelastic process where the plastic module is activated we can appeal to the principle of maximum plastic dissipation (e.g. see [HIL 50] or [LUB 90]) to conclude that among all plastically admissible fields of stress and stress-like hardening variable we can choose those which render the maximum of the plastic dissipation. This can be formally presented as a constrained minimization problem which can also be recast in a non-constrained form by making use of the Lagrange multiplier procedure (e.g. see [STR 86])

$$\left.\begin{aligned} \sigma = &\quad ? \\ q^p = &\quad ? \end{aligned}\right\} \Longrightarrow \quad min_{\sigma,q^p:\Phi^p(\sigma,q^p)=0}(-\dot{\mathcal{D}}^p(\sigma,q^p))$$

$$\Updownarrow$$

$$min_{\forall\sigma,q^p}(L^p(\sigma,q^p,\dot{\gamma}^p), \qquad [11]$$

where $\dot{\gamma}^p$ is the plastic multiplier and

$$L^p = -\dot{\mathcal{D}}^p(\sigma,q^p) + \dot{\gamma}^p\Phi^p(\sigma,q^p). \qquad [12]$$

The Kuhn-Tucker optimality conditions (e.g. see [STR 86]) for this problem can be readily obtained as:

$$
\begin{aligned}
0 = \frac{\partial L^p(\sigma, q^p, \dot{\gamma}^p)}{\partial \sigma} &= -\dot{\epsilon}^p + \dot{\gamma}^p \frac{\partial \Phi^p}{\partial \sigma} \Longrightarrow \dot{\epsilon}^p = \dot{\gamma}^p \frac{\partial \Phi^p}{\partial \sigma} \\
0 = \frac{\partial L^p(\sigma, q^p, \dot{\gamma}^p)}{\partial q^p} &= -\dot{\xi}^p + \dot{\gamma}^p \frac{\partial \Phi^p}{\partial q^p} \Longrightarrow \dot{\xi}^p = \dot{\gamma}^p \frac{\partial \Phi^p}{\partial q^p},
\end{aligned} \quad [13]
$$

which can be interpreted as the evolution equations of the internal variables of the plastic model with the Lagrange multiplier $\dot{\gamma}^p$ as the plastic multiplier. If we assume that the latter takes a zero value in an elastic process, the last Kuhn-Tucker equation associated with the Lagrangian in (11) leads to the standard form of the loading /unloading conditions which can be written as

$$
\dot{\gamma}^p \geq 0 \; ; \; \Phi^p \leq 0 \; ; \; \dot{\gamma}^p \Phi^p = 0. \quad [14]
$$

The positive value of the plastic multiplier can be obtained from the plastic consistency condition, imposing for a state of plastic loading the plastic admissibility on the subsequent state, resulting with

$$
0 = \dot{\Phi}^p = \frac{\partial \Phi^p}{\partial \sigma} \dot{\sigma} + \frac{\partial \Phi^p}{\partial q^p} \dot{q}^p. \quad [15]
$$

From (10) and (13) we get:

$$
\begin{aligned}
\dot{q}^p &= \frac{\partial q^p}{\partial \xi^p} \dot{\xi}^p \\
&= -\dot{\gamma}^p \frac{\partial^2 \Xi^p}{\partial {\xi^p}^2} \frac{\partial \Phi^p}{\partial q^p},
\end{aligned} \quad [16]
$$

and from (7) and (13):

$$
\begin{aligned}
\dot{\sigma} &= C^e (\dot{\epsilon} - \dot{\epsilon}^d - \dot{\epsilon}^p) \\
&= C^e (\dot{\epsilon} - \dot{\epsilon}^d) - \dot{\gamma}^p C^e \frac{\partial \Phi^p}{\partial \sigma}.
\end{aligned} \quad [17]
$$

Combining the equations (15), (16) and (17) we obtain:

$$
\dot{\gamma}^p = \frac{\frac{\partial \Phi^p}{\partial \sigma} C^e (\dot{\epsilon} - \dot{\epsilon}^d)}{\frac{\partial \Phi^p}{\partial \sigma} C^e \frac{\partial \Phi^p}{\partial \sigma} + \frac{\partial \Phi^p}{\partial q^p} \frac{\partial^2 \Xi^p}{\partial {\xi^p}^2} \frac{\partial \Phi^p}{\partial q^p}} \quad [18]
$$

Finally, using (17) and (18) we determine the elasto-plastic consistent tangent modulus (with damage kept fixed) that

$$\dot{\sigma} = [C^e - \frac{C^e \frac{\partial \Phi^p}{\partial \sigma} \frac{\partial \Phi^p}{\partial \sigma} C^e}{\frac{\partial \Phi^p}{\partial \sigma} C^e \frac{\partial \Phi^p}{\partial \sigma} + \frac{\partial \Phi^p}{\partial q^p} \frac{\partial^2 \Xi^p}{\partial \xi^{p2}} \frac{\partial \Phi^p}{\partial q^p}}](\dot{\epsilon} - \dot{\epsilon}^d). \qquad [19]$$

2.3. *Damage model*

The damage model can also be cast in an equivalent format as the one given for the plasticity, although that is not the usual manner of presenting it (e.g. see [LEM 88] or [KRA 96]). Namely, in the case where the damage model is activated while the plasticity remains inactive, we can appeal to the principle of maximum damage dissipation to select among all admissible value of stress and hardening damage variables those which maximize the damage dissipation. The latter can be formulated as an constrained minimization problem and further transformed into an unconstrained one by using the Lagrange multiplier method

$$\left.\begin{array}{rl} \sigma = & ? \\ q^d = & ? \end{array}\right\} \Longrightarrow \quad min_{\sigma, q^d : \Phi^d(\sigma, q^d) = 0}(-\dot{\mathcal{D}}^d(\sigma, q^d))$$
$$\Updownarrow$$
$$min_{\forall \sigma, q^d}(L^d(\sigma, q^d), \dot{\gamma}^d), \qquad [20]$$

where $\dot{\gamma}^d$ is the Lagrange multiplier for damage and

$$L^d = -\dot{\mathcal{D}}^d(\sigma, q^d) + \dot{\gamma}^d \Phi^d(\sigma, q^d). \qquad [21]$$

The Kuhn-Tucker optimality conditions corresponding to the damage Lagrangian can readily be obtained as

$$\begin{array}{rcl} 0 = \dfrac{\partial L^d(\sigma, q^d, \dot{\gamma}^d)}{\partial \sigma} & = & -\dot{D}\sigma + \dot{\gamma}^d \dfrac{\partial \Phi^d}{\partial \sigma} \Longrightarrow \dot{D} = \dot{\gamma}^p \dfrac{1}{\sigma} \dfrac{\partial \Phi^d}{\partial \sigma} \\ 0 = \dfrac{\partial L^d(\sigma, q^d, \dot{\gamma}^d)}{\partial q^d} & = & -\dot{\xi}^d + \dot{\gamma}^d \dfrac{\partial \Phi^d}{\partial q^d} \Longrightarrow \dot{\xi}^d = \dot{\gamma}^d \dfrac{\partial \Phi^d}{\partial q^d} \end{array} \qquad [22]$$

which specify the evolution equations for the internal variables of the damage model. To complete this description of the Kuhn-Tucker optimality condition we also provide the loading/unloading conditions with

$$\dot{\gamma}^d \geq 0 \; ; \; \Phi^d \leq 0 \; ; \; \dot{\gamma}^d \Phi^d = 0, \qquad [23]$$

where the zero value of the damage multiplier is introduced for an elastic process, where no change of damage internal variables would take place. The positive value of the damage multiplier, which occurs for the case of damage loading can be computed from the damage consistency condition, by imposing the admissibility on the subsequent state to get

$$0 = \dot{\Phi}^d = \frac{\partial \Phi^d}{\partial \sigma}\dot{\sigma} + \frac{\partial \Phi^d}{\partial q^d}\dot{q}^d. \quad [24]$$

From (10) and (22) we have

$$\begin{aligned} \dot{q}^d &= \frac{\partial q^d}{\partial \xi^d}\dot{\xi}^d \\ &= -\dot{\gamma}^d \frac{\partial^2 \Xi^d}{\partial {\xi^d}^2}\frac{\partial \Phi^d}{\partial q^d}, \end{aligned} \quad [25]$$

and from (8) and (22) we get

$$\begin{aligned} \dot{\sigma} &= (\dot{D^{-1}\epsilon^d}) \\ &= D^{-1}\dot{\epsilon}^d - D^{-1}\dot{D}\overbrace{D^{-1}\epsilon^d}^{\sigma} \\ &= D^{-1}\dot{\epsilon}^d - \dot{\gamma}^d D^{-1}\frac{\partial \Phi^d}{\partial \sigma}, \end{aligned} \quad [26]$$

where we used the relation $(\dot{D^{-1}}) = -D^{-1}\dot{D}D^{-1}$. Combining equations (24),(25) and (26) we obtain:

$$\dot{\gamma}^d = \frac{\frac{\partial \Phi^d}{\partial \sigma}D^{-1}\dot{\epsilon}^d}{\frac{\partial \Phi^d}{\partial \sigma}D^{-1}\frac{\partial \Phi^d}{\partial \sigma} + \frac{\partial \Phi^d}{\partial q^d}\frac{\partial^2 \Xi^d}{\partial {\xi^d}^2}\frac{\partial \Phi^d}{\partial q^d}}. \quad [27]$$

Hence using (26) and (27) we get the damage consistent tangent modulus as

$$\dot{\sigma} = [D^{-1} - \frac{D^{-1}\frac{\partial \Phi^d}{\partial \sigma}\frac{\partial \Phi^d}{\partial \sigma}D^{-1}}{\frac{\partial \Phi^d}{\partial \sigma}D^{-1}\frac{\partial \Phi^d}{\partial \sigma} + \frac{\partial \Phi^d}{\partial q^d}\frac{\partial^2 \Xi^d}{\partial {\xi^d}^2}\frac{\partial \Phi^d}{\partial q^d}}]\dot{\epsilon}^d. \quad [28]$$

2.4. *Elasto-plastic-damage coupling*

Finally, for the case where both plasticity and damage models are active one can apply simultaneously the principle of maximum plastic and maximum damage dissipation to recover both sets of results as presented in (7), (13) and (15) for plasticity and (8), (22) and (24) for damage. This would imply in particular that we can write two different forms of the stress rate equation, one for plasticity and another for damage, according to

$$\dot{\sigma} = C^{ep}(\dot{\epsilon} - \dot{\epsilon}^d), \qquad [29]$$

where

$$C^{ep} = \begin{cases} C^e & ; \dot{\gamma}^p = 0 \\ C^e - \frac{C^e \frac{\partial \Phi^p}{\partial \sigma} \frac{\partial \Phi^p}{\partial \sigma} C^e}{\frac{\partial \Phi^p}{\partial \sigma} C^e \frac{\partial \Phi^p}{\partial \sigma} + \frac{\partial \Phi^p}{\partial q^p} \frac{\partial^2 \Xi^p}{\partial \xi^{p2}} \frac{\partial \Phi^p}{\partial q^p}} & ; \dot{\gamma}^p > 0 \end{cases} \qquad [30]$$

and

$$\dot{\sigma} = C^{ed} \dot{\epsilon}^d, \qquad [31]$$

where

$$C^{ed} = \begin{cases} D^{-1} & ; \dot{\gamma}^d = 0 \\ D^{-1} - \frac{D^{-1} \frac{\partial \Phi^d}{\partial \sigma} \frac{\partial \Phi^d}{\partial \sigma} D^{-1}}{\frac{\partial \Phi^d}{\partial \sigma} D^{-1} \frac{\partial \Phi^d}{\partial \sigma} + \frac{\partial \Phi^d}{\partial q^d} \frac{\partial^2 \Xi^d}{\partial \xi^{d2}} \frac{\partial \Phi^d}{\partial q^d}} & ; \dot{\gamma}^d > 0 \end{cases} \qquad [32]$$

Therefore, from equality of stress rates in (29) and (31) we can obtain the damage strain rate according to

$$C^{ep}(\dot{\epsilon} - \dot{\epsilon}^d) = C^{ed} \dot{\epsilon}^d \Longrightarrow \dot{\epsilon}^d = [C^{ep} + C^{ed}]^{-1} C^{ep} \dot{\epsilon}. \qquad [33]$$

Replacing the last result in (31) we can rewrite the stress rate equation as

$$\dot{\sigma} = C^{epd} \dot{\epsilon}, \qquad [34]$$

where C^{epd} is the elasto-plastic-damage consistent tangent modulus is given as

$$C^{epd} = C^{ed}[C^{ep} + C^{ed}]^{-1} C^{ep}. \qquad [35]$$

3. Numerical Implementation

In this section we address the numerical implementation issues of the proposed damage-plasticity model, within the framework of one-step, implicit backward Euler

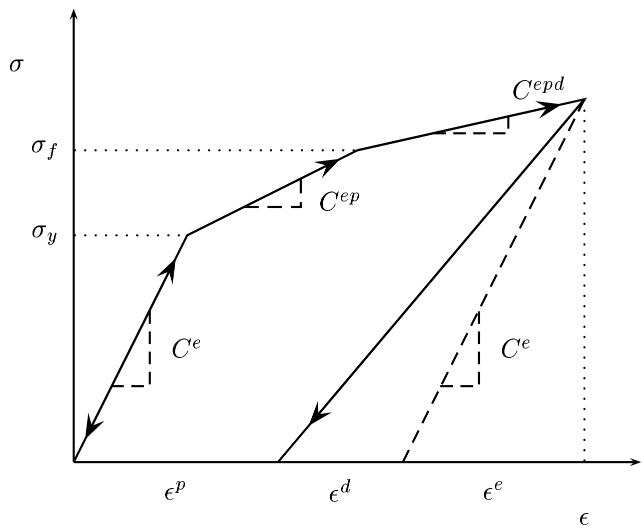

Figure 1. *The $\sigma - \epsilon$ diagram for the one-dimensional elasto-plastic-damage constitutive model. We first reach the yield surface and then the fracture threshold surface. By unloading we can observe both phenomena, the appearance of plastic strains and decreasing of elastic properties. We also give the graphical representation of the strain decomposition,* $\epsilon = \epsilon^e + \epsilon^p + \epsilon^d$

time-integration scheme. The central problem of computational plasticity can thus be presented as follows:

Central problem of computational damage-plasticity

Given: $\epsilon_n = \epsilon(\cdot, t_n)$, $\epsilon_n^p = \epsilon_n^p(\cdot, t_n)$, $\epsilon_n^d = \epsilon_n^d(\cdot, t_n)$, $\xi_n^p = \xi_n^p(\cdot, t_n)$, $D_n = D(\cdot, t_n)$, $\xi_n^d = \xi^d(\cdot, t_n)$, $\Delta t = t_{n+1} - t_n$

Find: ϵ_{n+1}, ϵ_{n+1}^p, ϵ_{n+1}^d, ξ_{n+1}^p, D_{n+1}, ξ_{n+1}^d

Such that:

$$A_{el=1}^{n_{el}}[\mathbf{f}_{n+1}^{el,int} - \mathbf{f}_{n+1}^{el,ext}] = \mathbf{0},$$

$$\mathbf{f}_{n+1}^{el,int} = \int_{l_e} \mathbf{B}^{el} \sigma(\epsilon_{n+1}, \epsilon_{n+1}^p, \xi_{n+1}^p, D_{n+1}, \xi_{n+1}^d, \epsilon_{n+1}^d) dx,$$

$$\mathbf{f}_{n+1}^{el,ext} = \int_{l_e} \mathbf{N}^{el} b dx + [\mathbf{N}^{el} \bar{t}]_{\partial l_e},$$
$$\dot{\gamma}_{n+1}^{p} \geq 0,\ \Phi_{n+1}^{p}(\sigma_{n+1}, q_{n+1}^{p}) \leq 0,\ \dot{\gamma}_{n+1}^{p} \Phi_{n+1}^{p} = 0$$
$$\dot{\gamma}_{n+1}^{d} \geq 0,\ \Phi_{n+1}^{d}(\sigma_{n+1}, q_{n+1}^{d}) \leq 0,\ \dot{\gamma}_{n+1}^{d} \Phi_{n+1}^{d} = 0. \quad [36]$$

In (36) above the first equation expresses equilibrium of the complete system, whereas the other two are the equations imposing the plastic and damage admissibility of the computed stress state at time t_{n+1}. The symbol $A_{el=1}^{n_{el}}$ denotes the standard finite element assembly procedure summing up all element contributions (e.g. see [BAT 90]), $\mathbf{f}^{el,int}$ and $\mathbf{f}^{el,ext}$ represent the internal and external forces of the element $'el'$, $\mathbf{N}^{el}$ is element displacement variation and $\mathbf{B}^{el}$ the corresponding strain variation. The essential role of the finite element method and the commonly used numerical integration scheme is to drastically simplify the central problem reducing it to corresponding numerical integration points which are chosen according to the given finite element mesh.

Furthermore, in accordance to the usual operator split procedure (e.g. see [SIM 98]), we can separate the solution of the equilibrium equations from the one carried out to find the admissible stress state. The first group of equations, namely the equilibrium equations are solved iteratively in order to provide the current guess of the total strain field. If the Newton method is employed, one would need the tangent stiffness matrix, or more precisely the elasto-plastic-damage tangent modulus (e.g. see [BEN 88]). However, this is not essential and many other iterative schemes can be used for such a purpose. For any global solution procedure we are given the best iterative value for total strain field at each integration point, $\epsilon_{n+1}^{(i)}$, where the superscript $'(i)'$ denotes the iteration counter. The central problem thus reduces to computing the corresponding values of the internal variables which will provide an admissible stress field. This computation is accomplished by a local iterative scheme, with iteration counter denoted by $'(k)'$, where two ingredients of the model exploited independently produce the current values of plastic and damage strains, $\epsilon_{n+1}^{p(k)}$ and $\epsilon_{n+1}^{d(k)}$, and the corresponding stress field according to the following expression:

$$\sigma_{n+1} = C^{e}(\epsilon_{n+1}^{(i)} - \epsilon_{n+1}^{p(k)} - \epsilon_{n+1}^{d(k)}). \quad [37]$$

As already indicated, the computation of this kind is carried out independently on plasticity and damage part of the model, producing the stress values, σ_{n+1}^{p} and σ_{n+1}^{d}, respectively. At the final stage, we then impose that these two stresses coincide, which provides the converged values of the plastic and damage deformation.

3.1. *Plasticity computation*

The computation for the plasticity model is carried out by the procedure described in this section. By keeping the damage variable fixed and, in particular, the current value of the damage strain, $\epsilon_{n+1}^{d(k)}$, frozen, we first assume that the step remains elastic (with respect to the potential change of plasticity model internal variables), which allows us to compute the elastic trial state:

$$\sigma_{n+1}^{p,trial} = C^e(\epsilon_{n+1}^{(i)} - \epsilon_n^p - \epsilon_{n+1}^{d(k)}) \quad [38]$$

$$q_{n+1}^{p,trial} = q(\xi_n^p). \quad [39]$$

If the trial elastic step is indeed plastically admissible in the sense that

$$\Phi_{n+1}^{p,trial} = \Phi^p(\sigma_{n+1}^{p,trial}, q_{n+1}^{p,trial}) \leq 0,$$

we conclude that the total step is the exact solution with $\dot{\gamma}_{n+1}^p = 0$. In the opposite, we need to correct this result for the stress by finding the positive value of the plastic multiplier which will re-establish the plastic admissibility. To that end we first integrate by backward Euler scheme the evolution equation in (13) to obtain the correction of internal variables:

$$\begin{aligned} \epsilon_{n+1}^p &= \epsilon_n^p + \gamma_{n+1}^p \frac{\partial \Phi_{n+1}^p}{\partial \sigma_{n+1}^p} \\ \xi_{n+1}^p &= \xi_n^p + \gamma_{n+1}^p \frac{\partial \Phi_{n+1}^p}{\partial q_{n+1}^p} \\ q_{n+1}^p &= q^p(\xi_{n+1}^p), \end{aligned} \quad [40]$$

where

$$\gamma_{n+1}^p = \dot{\gamma}_{n+1}^p \Delta t \quad [41]$$

and the corresponding value of the stress,

$$\begin{aligned} \sigma_{n+1}^p &= C^e(\epsilon_{n+1}^{(i)} - \epsilon_{n+1}^p - \epsilon_{n+1}^{d(k)}) \\ &= C^e(\epsilon_{n+1}^{(i)} - \epsilon_n^p - \epsilon_{n+1}^{d(k)}) - C^e(\epsilon_{n+1}^p - \epsilon_n^p) \\ &= \sigma_{n+1}^{p,trial} - C^e \gamma_{n+1}^p \frac{\partial \Phi_{n+1}^p}{\partial \sigma_{n+1}}, \end{aligned} \quad [42]$$

which should be tested for the plastic admissibility,

$$\Phi_{n+1}^p = \Phi^p(\sigma_{n+1}^p, q_{n+1}^p) = 0 \ . \quad [43]$$

Using the yield surface function defined in (5) and the equations (40) and (42), (43) can be rewritten as

$$\Phi^p_{n+1} = |\sigma^{p,trial}_{n+1}| - C^e \gamma^p_{n+1} - (\sigma^p_y - q^p(\xi^p_{n+1})) = 0 \; . \qquad [44]$$

For a convex hardening function (e.g. saturation hardening) we can always solve the last equation using the Newton procedure to get converged value γ^p_{n+1} from which we can calculate

$$\begin{aligned} \epsilon^p_{n+1} &= \epsilon^p_n + \gamma^p_{n+1} sign(\sigma^{p,trial}_{n+1}) \\ \xi^p_{n+1} &= \xi^p_n + \gamma^p_{n+1} \\ \sigma^p_{n+1} &= \sigma^{p,trial}_{n+1} - C^e \gamma^p_{n+1} sign(\sigma^{p,trial}_{n+1}) \; . \end{aligned} \qquad [45]$$

3.2. *Damage computation*

We then turn towards the damage part of the model to compute the potential evolution of the damage internal variables. We start again this computation by assuming the elastic trial step and keeping the damage variables the same as in the previous step, which leads to

$$\sigma^{d,trial}_{n+1} = D^{-1}_n \epsilon^{d(k)}_{n+1} \qquad [46]$$

$$q^{d,trial}_{n+1} = q(\xi^d_n). \qquad [47]$$

We recall that the damage strain value at t_{n+1}, ϵ^d_{n+1}, is determined by the iteration process, where for either plastic and damage calculation at $k-th$ iteration we keep the damage strain value frozen at $\epsilon^{d(k)}_{n+1}$.

If the damage admissibility of the trial step is confirmed with

$$\Phi^{d,trial}_{n+1} = \Phi^d(\sigma^{d,trial}_{n+1}, q^{d,trial}_{n+1}) \leq 0 \; ,$$

the trial step is indeed the exact solution with $\gamma^d_{n+1} = 0$. In the opposite, one needs to compute the corresponding positive value of the damage multiplier that will re-establish the damage admissibility. More precisely, integrating by backward Euler scheme the damage variables evolution equations in (20) we can write

$$\begin{aligned} \sigma^d_{n+1} &= D^{-1}_{n+1} \epsilon^{d(k)}_{n+1} \\ &= \sigma^{d,trial}_{n+1} - D^{-1}_n \gamma^d_{n+1} \frac{\partial \Phi^d_{n+1}}{\partial \sigma^d_{n+1}} \\ D_{n+1} &= D_n + \gamma^d_{n+1} \frac{1}{\sigma^d_{n+1}} \frac{\partial \Phi^d_{n+1}}{\partial \sigma^d_{n+1}} \end{aligned}$$

$$\begin{aligned} \xi^d_{n+1} &= \xi^d_n + \gamma^d_{n+1} \frac{\partial \Phi^d_{n+1}}{\partial q^d_{n+1}} \\ q^d_{n+1} &= q^d(\xi^d_{n+1}). \end{aligned} \quad [48]$$

By using the damage function form introduced in (5) and the updates in (48) above, the damage admissibility condition can be written as

$$\Phi^d_{n+1} = |\sigma^{d,trial}_{n+1}| - D_n^{-1} \gamma^d_{n+1} - (\sigma^d_f - q^d(\xi^d_{n+1})) = 0. \quad [49]$$

Solving this non-linear equation by the Newton method gives us the value of the damage multiplier γ^d_{n+1} and hence the final values of stresses and the damage internal variables:

$$\begin{aligned} \sigma^d_{n+1} &= \sigma^{d,trial}_{n+1} - D_n^{-1} \gamma^d_{n+1} sign(\sigma^{d,trial}_{n+1}) \\ D_{n+1} &= D_n + \gamma^d_{n+1} \frac{1}{|\sigma^d_{n+1}|} \\ \xi^d_{n+1} &= \xi^d_n + \gamma^d_{n+1} \\ q^d_{n+1} &= q^d(\xi^d_{n+1}). \end{aligned} \quad [50]$$

In equation (46) the trial stress is calculated assuming that the initial value of the damage variable is non-zero, $D_0 > 0$, to avoid a potential problem of dividing by zero. By this assumption we have to change the elastic property values in the program to $\tilde{C}^e$, so that the real elastic properties of the model do not change:

$$C^e = (D_0 + \tilde{C}^{e\ -1})^{-1}. \quad [51]$$

However, by taking the value of D non-zero, we slightly change the damage hardening variable evolution since $D(\xi^d = 0) > 0$. Thus, D_0 becomes another material parameter for the hardening evolution. In our calculations we took $D_0 = C^{e-1}$, which means that within the present model calculations the elastic properties have to be $\tilde{C}^e = 2C^e$.

3.3. *Coupling*

We note that the computation on the plastic model described in the previous section is carried out in a completely independent manner from the computation on the damage part of the model summarized in (46) to (50); In particular these two computational procedures can be advanced in parallel with no exchange of results, dedicating to each its particular processor on a parallel computer, to carry out the corresponding part of the work.

The final results produced by the two parts of the model should finally be compared against one another and any discrepancy, or residual, should be eliminated with

$$r(\epsilon_{n+1}^{d(k)}) := \sigma_{n+1}^{p}(\epsilon_{n+1}^{d(k)}) - \sigma_{n+1}^{d}(\epsilon_{n+1}^{d(k)}) = 0. \quad [52]$$

The last equation is then solved by an iterative procedure to which we supply the results available from the two parts to perform the next iterative step with

$$0 = r_{n+1}^{(k)} - C_{n+1}^{ep}\Delta\epsilon_{n+1}^{d(k)} - C_{n+1}^{ed}\Delta\epsilon_{n+1}^{d(k)} \quad [53]$$

leading to

$$\begin{aligned} \Delta\epsilon_{n+1}^{d(k)} &= \frac{r_{n+1}^{(k)}}{C_{n+1}^{ep} + C_{n+1}^{ed}} \\ \epsilon_{n+1}^{d(k+1)} &= \epsilon_{n+1}^{d(k)} + \Delta\epsilon_{n+1}^{d(k)}. \end{aligned} \quad [54]$$

The computation then continues with this improved value of the damage strain until the convergence in (52) is finally achieved. The number of iterations depends significantly on the choice of the starting value of the damage strain, $\epsilon_{n+1}^{d\ (k=0)}$. As the first guess we should not take the last converged value, ϵ_n^d, since the damage strain evolves also in an elastic process ($\epsilon^d = D\sigma \neq const.$ even if $D = const$). So, we first assume that process remains elastic, which means that the damage variable, D, does not change and the trial stress equals:

$$\sigma_{n+1}^{trial} = \sigma_n + (D_n + C^{e\ -1})^{-1}(\epsilon_{n+1} - \epsilon_n). \quad [55]$$

By definition the corresponding damage strain is equal to:

$$\epsilon_{n+1}^{d\ (k=0)} = D_n \sigma_{n+1}^{trial}. \quad [56]$$

The value above is then taken as the starting value of the damage strain iteration process to solve the non-linear case (equations (52) to (54)).

In (54) above we need the corresponding values of the elasto-plastic as well as the elasto-damage moduli, which can be easily computed from (45) for plasticity and from (50) for damage to obtain:

$$\frac{\partial \sigma_{n+1}^{p}}{\partial(\epsilon_{n+1} - \epsilon_{n+1}^{d})} =: \quad C_{n+1}^{ep} = \begin{cases} C^e & ; \gamma_{n+1}^{p} = 0 \\ \frac{C^e (\frac{\partial^2 \Xi^p}{\partial \xi^{p2}})_{n+1}}{C^e + (\frac{\partial^2 \Xi^p}{\partial \xi^{p2}})_{n+1}} & ; \gamma_{n+1}^{p} \geq 0 \end{cases}$$

$$\frac{\partial \sigma^d_{n+1}}{\partial \epsilon^d_{n+1}} \quad =: \quad C^{ed}_{n+1} = \begin{cases} D_n^{-1} & ; \gamma^d_{n+1} = 0 \\ \frac{D_n^{-1}(\frac{\partial^2 \Xi^d}{\partial \xi^{d2}})_{n+1}}{D_n^{-1} + (\frac{\partial^2 \Xi^d}{\partial \xi^{d2}})_{n+1}} & ; \gamma^d_{n+1} \geq 0 \end{cases} . \qquad [57]$$

The given values of tangent moduli for both parts of the model are also of direct use for constructing the tangent modulus of the coupled model, according to

$$C^{ep}_{n+1}(d\epsilon_{n+1} - d\epsilon^d_{n+1}) = C^{ed}_{n+1} d\epsilon^d_{n+1} \qquad [58]$$

$$\begin{aligned} \Longrightarrow d\epsilon^d_{n+1} &= \frac{C^{ep}_{n+1}}{C^{ep}_{n+1} + C^{ed}_{n+1}} d\epsilon_{n+1} \\ \Longrightarrow d\sigma_{n+1} &= \underbrace{\frac{C^{ep}_{n+1} C^{ed}_{n+1}}{C^{ep}_{n+1} + C^{ed}_{n+1}}}_{C^{epd}} d\epsilon_{n+1} . \end{aligned} \qquad [59]$$

The tangent elasto-plastic-damage modulus given in (59) is used to compute the corresponding element contribution to the global stiffness matrix for the Newton iterative scheme applied in solving global equations with

$$\begin{aligned} A^{n_{el}}_{el=1} [\mathbf{K}^{el,(i)}_{n+1} (\mathbf{d}^{el,(i+1)}_{n+1} - \mathbf{d}^{el,(i)}_{n+1}) = \mathbf{f}^{el,ext}_{n+1} - \mathbf{f}^{el,int}_{n+1}] \\ \mathbf{K}^{el,(i)}_{n+1} = \int_{l_e} \mathbf{B}^{el} C^{epd}_{n+1} \mathbf{B}^{el,T} dx. \end{aligned} \qquad [60]$$

The computational procedure presented in the foregoing can be generalized to 2D or 3D problems with no modification other than dealing with the second order tensors used for stress and strain, and the fourth order tensor for the tangent modulus. The outline of such a computational procedure is given in Appendix.

4. Numerical simulations

In this section we present the results of a couple of numerical simulations. In order to illustrate the versatility of the model to represent large spectrum of different materials, one of the example draws from porous metal failure and another from compacting concrete .

4.1. *Criteria for porous metals in tension*

The porous metal coupled model was built along the lines of the pioneering work of Gurson ([GUR 77]), however with important difference regarding the present model, which has the ability to describe the closing of pores at unloading. Postulating that it is only spherical part of stress which determines the porosity, the damage criterion is given as

$$\Phi^d(\boldsymbol{\sigma}, q^d) = \langle tr(\boldsymbol{\sigma}) \rangle - (\boldsymbol{\sigma}_f^d - q^d), \qquad [61]$$

where $tr(\boldsymbol{\sigma})$ denotes the trace of the tensor $\boldsymbol{\sigma}$ and $< \cdot >$ the Macauley brackets:

$$< x >= \begin{cases} x & ; x \geq 0 \\ 0 & ; x < 0 \end{cases} . \qquad [62]$$

Here we neglect the possibility that the material can be damaged in compression. To model the plasticity of metal matrix, we used the von Mises criterion:

$$\Phi^p(\boldsymbol{\sigma}, q^p) = \sqrt{dev(\boldsymbol{\sigma}) : dev(\boldsymbol{\sigma})} - (\sigma_y^p - q^p), \qquad [63]$$

where $dev(\boldsymbol{\sigma})$ denotes the deviatoric part of the tensor $\boldsymbol{\sigma}$, $dev(\boldsymbol{\sigma}) \equiv \boldsymbol{\sigma} - \frac{1}{3}tr(\boldsymbol{\sigma})$.

From the choice of the criteria it follows that the evolution of damage variables depends only upon the spherical part of the stress tensor and the evolution of plastic variables upon its deviatoric part. Hence, the two nonlinear phenomena appear uncoupled in strain space. This is the direct consequence of the initial physical presumption that the opening of the micro-cracks is due to positive spherical part of the stress and sliding of crystal planes due to the stress deviator. The former corresponding to damage and the latter to plasticity.

Finally, we use an exponential hardening law for either phenomenon, plasticity and damage,

$$\begin{aligned} q^p(\xi^p) &= (\sigma_y^p - \sigma_\infty^p)(1 - e^{-b^p \xi^p}) \\ q^d(\xi^d) &= (\sigma_f^d - \sigma_\infty^d)(1 - e^{-b^d \xi^d}), \end{aligned} \qquad [64]$$

where σ_∞^p and σ_∞^d are saturation values of stress, whereas b^p and b^d are the material parameters governing the rate of saturation.

4.1.1. *Results*

The model is illustrated on an example of a rectangular plate with a circular hole in the middle, submitted to a simple tension test. By exploiting symmetry conditions, only one quarter of the model is used in the analysis (see Figure 2).

The material properties taken in the calculation were the following; (i) for elasticity: Young's modulus, $E = 240GPa$ and the shear modulus, $\mu = 92GPa$; (ii) for plasticity: yield stress, $\sigma_y = 170MPa$, hardening limit stress, $\sigma_\infty^p = 210MPa$

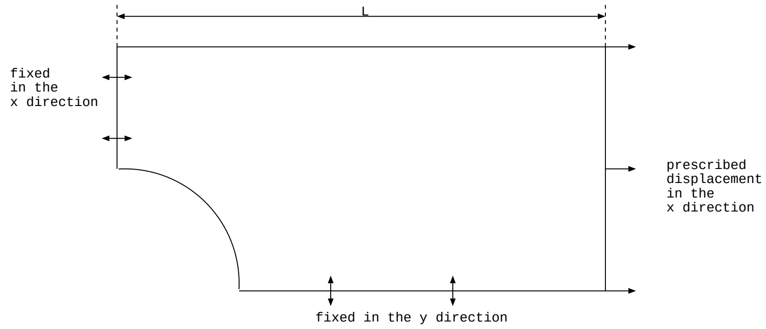

Figure 2. *One quarter of the specimen, used in calculation and prescribed boundary conditions*

and saturation parameter, $b^p = 50$; (iii) for damage: fracture stress, $\sigma_f = 170MPa$, hardening limit stress, $\sigma^d_\infty = 210MPa$ and saturation parameter, $b^d = 50$.

In Figures 3 and 4 we show how the spreading of plastified and damaged regions will change with the other phenomenon being activated. Different stages of activation of either plasticity or damage models are illustrated by contours of hardening variables ξ^p and ξ^d, respectively.

We observe the complete disappearance of shear band (Figure 3), a typical response of metals or alloys with von Mises criterion, when damage is also taken into account. Besides, we notice that in the case where both phenomena are activated either region is reduced to a smaller volume, but the differences between the maximum and minimum value of ξ^p and ξ^d is larger. With other words, the phenomena are, when activated simultaneously, more localized.

4.1.2. *Convergence characteristics*

Since the plastic and damage model state variables computations are carried out independently, the convergence rate of each one is not affected by another. In particular, the implementation of the von Mises model is completely the same as for the plasticity model and thus keeps its robustness in the coupled model as well. Moreover, since the damage model is formulated in a completely analogous way to the plasticity, that is using the fracture criterion and the principle of the maximum dissipation, we obtain comparable computational efficiency and similar robustness. In both cases, for saturation hardening models defined in (64), we need $2-8$ iterations to converge with the Newton procedure applied to each model.

The iterative procedure for the coupled plasticity-damage model, assuring the final uniqueness of stresses, requires in general only a few iterations to converge. The convergence criteria at k-th iteration can be written as:

a)

b)

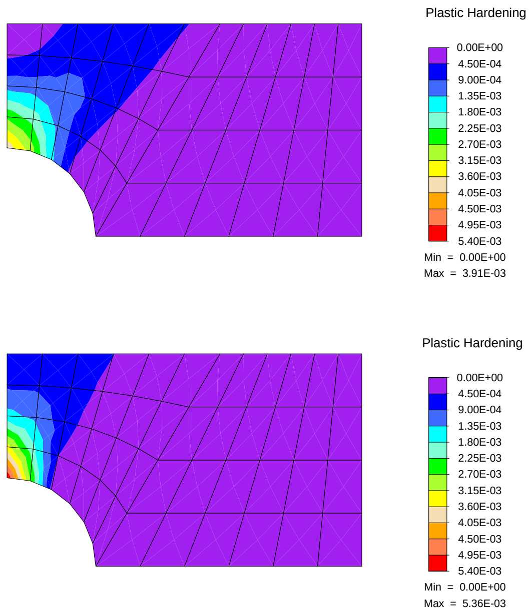

Figure 3. *Contours of the plastic hardening variable ξ^p, when a) only plasticity is activated and b) both plasticity and damage are activated*

$$\sqrt{\sum_{i,j=1}^{3} (\sigma^p(\epsilon^{d(k)}) - \sigma^d(\epsilon^{d(k)}))^2} \leq \eta \, min(\sigma_y, \sigma_f), \quad [65]$$

where η is a chosen tolerance. A typical sequence of the intermediate results obtained in the iterative process is presented in Table 1.

iter.	$\|\boldsymbol{\sigma}^p - \boldsymbol{\sigma}^d\|^2$	$\boldsymbol{\epsilon}_{11}^d$	$\boldsymbol{\epsilon}_{22}^d$	$\boldsymbol{\epsilon}_{12}^d$
1	$2.400521\ 10^{17}$	0.01681690	0.006922074	−0.002569930
2	$7.003845\ 10^{12}$	0.01614148	0.007330000	−0.002210263
3	$3.758989\ 10^{9}$	0.01615158	0.007325925	−0.002293375
4	$1.504535\ 10^{2}$	0.01615126	0.007326051	−0.002292093
5	$9.815405\ 10^{-5}$	0.01615126	0.007326049	−0.002292127

Table 1. *Iterative values for stress difference norm and damage strain computed by plasticity-damage model.*

a)

b)

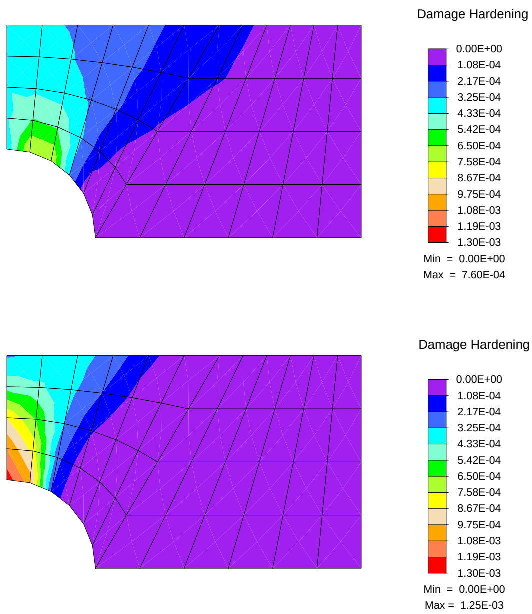

Figure 4. *The value of the damage hardening variable* ξ^d*, when a) only damage is activated and b) both, plasticity and damage are activated*

The solution uniqueness for the model of this kind is guaranteed if the plasticity and damage criteria do not concern the same inelastic mechanisms. Nevertheless, uniqueness of the solution does not ensure the easy convergence. We have remarked that for the plastic and damage criteria too similar, we need smaller time steps to converge, while the convergence is remaining quadratic.

4.2. *Criteria for compacting concrete*

It is quite interesting that the same kind of model can be used to represent compacting concrete behavior, although the latter can be seen as placed at the opposite end of the spectrum with respect to porous metal behavior described by Gurson's criterion. Namely, while the previous study deals with the increase of metal porosity and softening response under tension loading, in this example we model the decrease of concrete porosity under hydrostatic pressure loading and increasingly hardening response of compacting concrete [GAT 99].

Since we do not necessarily seek to use the most elaborate model for concrete capable of providing a very good agreement between the numerical simulations and experimental results, we use rather a simple model for plasticity component in terms of the Drucker-Prager criterion with no hardening:

$$\Phi^p(\boldsymbol{\sigma}) = \sqrt{dev(\boldsymbol{\sigma}) : dev(\boldsymbol{\sigma})} - tan(\alpha)\,\frac{1}{3}tr(\boldsymbol{\sigma}) - \sqrt{2/3}\,\sigma_f^p, \qquad [66]$$

where $tan(\alpha)$ is a (positive) material constant, roughly representing internal friction.

The damage model component is chosen to describe an essential mechanism of compacting concrete "damage" leading to an increase in resistance. Therefore, the damage criterion for concrete compaction concerns only the spherical part of the stress:

$$\Phi^d(\boldsymbol{\sigma}, q^d) = tr(\boldsymbol{\sigma}) - (\sigma_f^d - q^d), \qquad [67]$$

where σ_f^d is the damage elastic limit and q^d is the damage hardening variable. By introducing a linear hardening law for the damage:

$$q^d(\xi^d) \quad = \quad -K^d \xi^d \qquad [68]$$

with fairly large value of hardening modulus K^d we hope to provide a reliable representation of concrete behavior in compaction.

4.2.1. *Results*

The computations are performed for the same model of a perforated plate (see Figure 2), but this time submitted to compression loading. The numerical values for the material parameters taken in the calculation are the following: (i) elasticity: Young's modulus $E = 240GPa$ and the shear modulus $\mu = 92GPa$; (ii) plasticity: yield stress, $\sigma_y = 170MPa$, material parameter, $tan\alpha = 0.6$; (iii) damage: fracture stress, $\sigma_f = 210MPa$, $K^d = 200$.

In Figure 5 we show how the damaged regions change with or without the plasticity component activated, by tracing the contours of the damage hardening variable ξ^d. We observe that the damage hardening is much less present when both model components are activated.

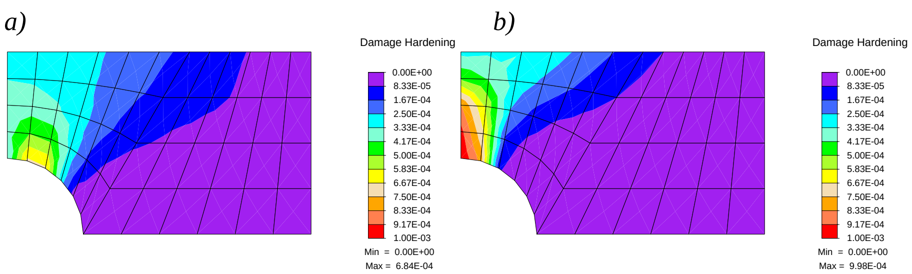

Figure 5. *The value of the damage hardening variable ξ^d, when a) only damage is activated and b) both, plasticity and damage are activated*

In Figure 6 we show contours of the largest stress component in terms of its absolute value, σ_{11}, for three different cases where only the plasticity, only the damage or both model components are activated. It is for the latter case when the stress response appears least localized.

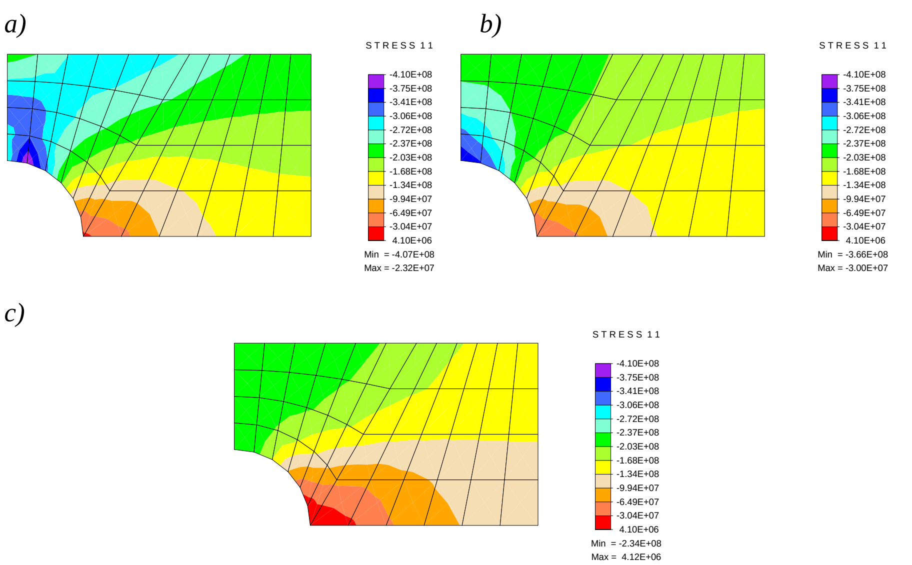

Figure 6. *The value of the stress σ_{11}, when a) only plasticity is activated, b) only damage is activated and c) both plasticity and damage are activated*

5. Conclusions

The general theoretical framework for developing a coupled plasticity-damage constitutive model presented herein allows to accommodate a wide variety of materials, as illustrated on porous metal and compacting concrete. The main novelty of the model with respect to the standard coupled models of this kind pertains to treating each model component, plasticity and damage, in an independent manner attributing to each its own yield or damage criteria. This kind of feature may especially be advantageous when the behavior of one or both components of the coupled model is well under control. In the opposite case, in order to simulate a real material behaviour, we need to identify the material parameters, either in the phenomenological sense by comparing it with experimental results or by using a microscale-based representation to provide the material parameters for a particular component.

Another important novelty introduced herein concerns the numerical implementation, where the plastic and damage component state variables computations are carried out independently from each other. The latter allows that the classical return mapping algorithm providing the quadratic convergence rates is directly used for plasticity component computations. Moreover, since the damage model is formulated in a completely analogous way to the plasticity, that is using the damage criterion and the principle of the maximum dissipation, the same type of return mapping algorithm can be used to obtain computational efficiency and robustness comparable to those of plasticity component.

The chosen strategy of separating the computations for two components to reduce the model complexity is in general taxed mildly, since the additional computational cost concerns typically only a few iterations to obtain the correct value of damage strain and to converge the stress provided by two components to the same value.

Acknowledgements

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Appendix : Numerical algorithm of the elasto-plastic-damage model

given

$\boldsymbol{\epsilon}_n^{(i)}, \boldsymbol{\epsilon}_n^p, \mathbf{D}_n, \xi_n^p, \xi_n^d$

find

$\boldsymbol{\sigma}_{n+1}^{(i)}, \boldsymbol{\epsilon}_{n+1}^{p(i)}, \mathbf{D}_{n+1}^{(i)}, \xi_{n+1}^{p(i)}$ and $\xi_{n+1}^{d(i)}$

elastic computation

$$\boldsymbol{\sigma}_{n+1}^{trial} = \mathbf{C}^e(\boldsymbol{\epsilon}_{n+1}^{(i)} - \boldsymbol{\epsilon}_n^p - \boldsymbol{\epsilon}_{n+1}^{d(k=0)}) = (\mathbf{C}^{e\ -1} + D_n)^{-1}(\boldsymbol{\epsilon}_{n+1}^{(i)} - \boldsymbol{\epsilon}_n^p)$$

$$\boldsymbol{\epsilon}_{n+1}^{d(k=0)} = \mathbf{D}_n \boldsymbol{\sigma}_{n+1}^{trial}$$

$\boldsymbol{\epsilon}_{n+1}^{d(k)}$ $\Downarrow$ **plastic computation**	$\boldsymbol{\epsilon}_{n+1}^{d(k)}$ $\Downarrow$ **damage computation**
$\Phi_{n+1}^{p,trial} = \Phi^p(\boldsymbol{\sigma}_{n+1}^{trial}, q_n^p)$	$\Phi_{n+1}^{d,trial} = \Phi^d(\boldsymbol{\sigma}_{n+1}^{d}, q_n^d)$
if $\Phi_{n+1}^{p,trial} \leq 0$:	if $\Phi_{n+1}^{d,trial} \leq 0$:
$\boldsymbol{\sigma}_{n+1}^p = \boldsymbol{\sigma}_{n+1}^{trial}$	$\boldsymbol{\sigma}_{n+1}^d = \boldsymbol{\sigma}_{n+1}^{trial}$
$\boldsymbol{\epsilon}_{n+1}^p = \boldsymbol{\epsilon}_n^p$	$\mathbf{D}_{n+1} = \mathbf{D}_n$
$\xi_{n+1}^p = \xi_n^p$	$\xi_{n+1}^d = \xi_n^d$
$q_{n+1}^p = q_n^p$	$q_{n+1}^d = q_n^d$
if $\Phi_{n+1}^{p,trial} > 0$:	if $\Phi_{n+1}^{d,trial} > 0$:
solve	solve
$\Phi_{n+1}^p(\boldsymbol{\sigma}_{n+1}^p(\gamma_{n+1}^p), q_{n+1}^p(\gamma_{n+1}^p)) = 0$	$\Phi_{n+1}^d(\boldsymbol{\sigma}_{n+1}^d(\gamma_{n+1}^d), q_{n+1}^d(\gamma_{n+1}^d)) = 0$
where	where
$\boldsymbol{\sigma}_{n+1}^p = \boldsymbol{\sigma}_{n+1}^{trial} - \gamma_{n+1}^p \mathbf{C}^e \frac{\partial \Phi_{n+1}^p}{\partial \boldsymbol{\sigma}}$	$\boldsymbol{\sigma}_{n+1}^d = \boldsymbol{\sigma}_{n+1}^{trial} - \gamma_{n+1}^d \mathbf{D}_n^{-1} \frac{\partial \Phi_{n+1}^d}{\partial \boldsymbol{\sigma}}$
$\xi_{n+1}^p = \xi_n^p + \gamma_{n+1}^p$	$\xi_{n+1}^d = \xi_n^d + \gamma_{n+1}^d$
$q_{n+1}^p = q^p(\xi_{n+1}^p)$	$q_{n+1}^d = q^d(\xi_{n+1}^d)$
$\boldsymbol{\epsilon}_{n+1}^p = \boldsymbol{\epsilon}_n^p + \gamma_{n+1}^p \frac{\partial \Phi_{n+1}^p}{\partial \boldsymbol{\sigma}}$	$\mathbf{D}_{n+1} = \mathbf{D}_n + \frac{\gamma_{n+1}^d}{\Phi_{n+1}^d + \sigma_f - q_{n+1}^d} \frac{\partial \Phi_{n+1}^d}{\partial \boldsymbol{\sigma}} \otimes \frac{\partial \Phi_{n+1}^d}{\partial \boldsymbol{\sigma}}$
$\Downarrow$ $\boldsymbol{\sigma}_{\mathbf{n+1}}^{\mathbf{p}}$	$\Downarrow$ $\boldsymbol{\sigma}_{\mathbf{n+1}}^{\mathbf{d}}$

If $\boldsymbol{\sigma}_{n+1}^p \neq \boldsymbol{\sigma}_{n+1}^d$:

correction of $\boldsymbol{\epsilon}_{n+1}^{d(k)}$

$$\Delta\boldsymbol{\epsilon}_{n+1}^{d(k)} = (\mathbf{C}^{ep} + \mathbf{C}^{ed})^{-1}(\boldsymbol{\sigma}_{n+1}^p - \boldsymbol{\sigma}_{n+1}^d)$$

$$\boldsymbol{\epsilon}_{\mathbf{n+1}}^{\mathbf{d(k+1)}} = \boldsymbol{\epsilon}_{\mathbf{n+1}}^{\mathbf{d(k)}} + \boldsymbol{\Delta\epsilon}_{\mathbf{n+1}}^{\mathbf{d(k)}}$$