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Magnetohydrodynamic pipe flow in annular-like domains

Canan Bozkaya and Münevver Tezer-Sezgin

Department of Mathematics, Middle East Technical University, Ankara, Turkey

ABSTRACT

The magnetohydrodynamic (MHD) pipe flow in annular-like domains with electrically conducting walls is investigated using both the extended-domain-eigenfunction method (EDEM) and the boundary element method (BEM). EDEM aims to reformulate the original problem on an extended symmetric domain obtained by transforming the inner boundary to a smaller circle towards the centre of the pipe, so that an eigenfunction solution can be obtained theoretically. By collocating only the inner circular boundary, the solution is transformed back to the original inner wall, which can be regarded as a semi-theoretical solution. On the other hand, BEM is a boundary only nature technique which transforms the differential equation into a boundary integral equation using the fundamental solution of the differential equation. Calculations are carried out for increasing values of Hartmann number (M) in annular-like domains with several shapes of inner wall at various wall conductivities. It is observed that although the results obtained by EDEM and BEM are very compatible for small M, EDEM is computationally less expensive and faster in convergence compared to BEM. However, BEM gives more accurate results than EDEM for large M due to the accumulation of numerical errors close to inner boundary in EDEM.

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MHD pipe flow; EDEM; BEM; annular-like domain

1. Introduction

The analytical solutions to elliptic boundary value problems (BVPs) on domains with complex geometries are very rarely obtainable. Thus, these type of problems are preferably solved by some traditional numerical techniques such as the finite difference method (FDM), the finite element method (FEM) and the boundary element method. However, researchers recently have considered some alternate semi-analytic approaches to solve elliptic BVPs. One of these numerical implementations is the Trefftz method (Herrera, 2000) which is a boundary approximation method and utilises eigenfunctions of the differential operator to construct a finite sum approximation to the elliptic BVP. Cheung, Jin, and Zienkiewicz (1991) and Li (2008) proposed Trefftz method for the solution of Helmholtz equation and for the solution of Helmholtz equation with degeneracy, respectively. The method of embedding is suggested for the solution of a linear

CONTACT Canan Bozkaya bcanan@metu.edu.tr

partial differential equation in terms of eigenfunction expansions in an arbitrary domain by Shankar (2005) and in unbounded and multiply connected domains by Shankar (2006) to overcome the limitations on the geometry of the domain. Another semi-analytic method, which is called extended-domain-eigenfunction method, has been introduced by Aarão, Bradshaw-Hajek, Miklavcic, and Ward (2010) for solving elliptic BVPs on annular-like domains. In this technique, the original domain of the problem is embedded into an extended or larger one with simple boundaries where an eigenfunction solution like in the Trefftz method can be generated using standard techniques such as separation of variables. When the solution in the larger domain is restricted to the original domain, one can obtain the solution of the original problem. In the work Aarão et al. (2010), the Laplace's equation in an annular-like domain is solved by EDEM. Later, the EDEM is also employed to solve the modified Helmholtz BVPs in annular-like domains by Aarão, Bradshaw-Hajek, Miklavcic, and Ward (2011).

In this paper, we aim to extend the implementation of the extended-domain-eigenfunction method to solve the magnetohydrodynamic flow in annular-like domains, for the first time to the best of author's knowledge. The MHD is the discipline combining the classical fluid mechanics and electrodynamics. The MHD effects are widely exploited in technical devices (e.g. in pumps, flow meters, generators) and industrial processes in metallurgy, material processing, chemical industry, industrial power engineering and nuclear engineering. The flow of an incompressible, viscous, electrically conducting fluid in pipes gives rise to coupled convection–diffusion type equations in velocity and induced magnetic field. Due to this coupling, the analytical solutions for the MHD flow equations are available only for some simple geometries subject to simple boundary conditions. Therefore, some numerical techniques have been used for the solution of MHD flow problems in ducts with no holes under various wall conductivities, namely: FDM (Seungsoo & Dulikravich, 1991; Sheu & Lin, 2004), FEM (Barrett, 2001; Gardner & Gardner, 1995), BEM (Bozkaya & Tezer-Sezgin, 2007; Carabineanu, Dinu, & Oprea, 1995; Liu & Zhu, 2002; Tezer-Sezgin & Bozkaya, 2008), spectral method (Carabineanu & Lungu, 2006) and meshless methods (Bourantas, Skouras, & Loukopoulos, 2009; Loukopoulos, Bourantas, Skouras, & Nikiforidis, 2011). On the other hand, the MHD problem inside a circular pipe when both the pipe wall and the surrounding medium are electrically conducting, and have small magnetisations compared to the fluid inside the pipe has been simulated using BEM in Tezer-Sezgin and Han-Aydın (2013). The same problem when the thickness of the pipe wall is taken into account, has been also solved using the dual reciprocity BEM in the work of Han-Aydın and Tezer-Sezgin (2014). In the present study, we also employ the BEM for the solution of MHD flow in annular-like domains not only to compare but also to validate the results obtained by the semi-theoretical EDEM, since there are no analytical solutions for MHD flow in annular-like domains.

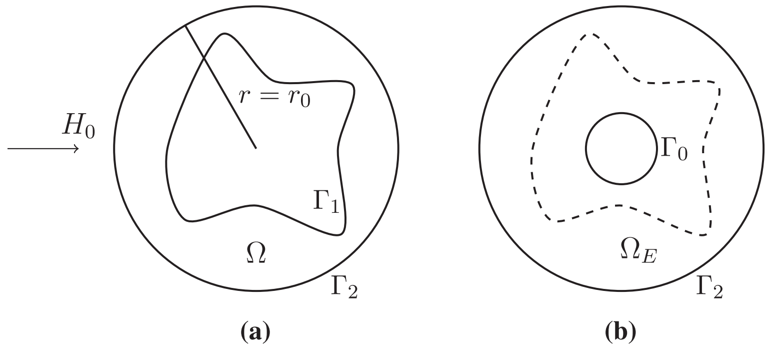

Figure 1. Cross-section of the annular-like pipe.
Note: (a) original domain Ω between Γ_1 and Γ_2, (b) extended domain Ω_E between Γ_0 and Γ_2.

2. Problem definition and governing equations

The equations governing the steady, laminar, fully developed flow of an incompressible, viscous, electrically conducting fluid in an annular-like pipe subject to a constant and uniform horizontally applied magnetic field of intensity H_0, are the same as those MHD duct flow equations given in Dragoş (1975), and can be expressed in non-dimensional form as

$$\left.\begin{aligned} \nabla^2 V + M\frac{\partial B}{\partial x} &= -1 \\ \nabla^2 B + M\frac{\partial V}{\partial x} &= 0 \end{aligned}\right\} \quad \text{in } \Omega\,, \tag{1}$$

where Ω is the annular-like domain (the cross-section of the pipe) between the inner boundary Γ_1 and the outer boundary Γ_2 as shown in Figure 1(a). The unknowns $V(x, y)$ and $B(x, y)$ are respectively the dimensionless velocity and induced magnetic field in the z-direction which is the axis of the annular pipe. Hartmann number M is defined by $M = H_0 L_0 \sqrt{\sigma}/\sqrt{\mu}$, where L_0 is the characteristic length, σ and μ are the electrical conductivity and the coefficient of viscosity of the fluid, respectively.

The corresponding boundary conditions are given as

$$V = 0 \text{ on } \Gamma = \Gamma_1 \cup \Gamma_2,\ B = k \text{ on } \Gamma_1, \text{ and } B = -\frac{x}{M} \text{ on } \Gamma_2\,, \tag{2}$$

where k is a constant which indicates the conductivity of the inner pipe wall. The Equation (1) can be transformed into two modified Helmholtz equations through appropriate transformations. First, the governing Equation (1) are decoupled into two homogeneous equations with the change of variables $w_1 = V + B + x/M$, $w_2 = V - B - x/M$. Thus, the resulting equations and the corresponding boundary conditions become as follows:

$$\left.\begin{array}{l}\nabla^2 w_1 + M\dfrac{\partial w_1}{\partial x} = 0 \\ \nabla^2 w_2 - M\dfrac{\partial w_2}{\partial x} = 0\end{array}\right. \text{ in } \Omega, \qquad \begin{array}{ll} w_1 = k + \frac{x}{M}, \quad w_2 = -k - \frac{x}{M} & \text{on } \Gamma_1 \\ w_1 = w_2 = 0 & \text{on } \Gamma_2 . \end{array} \tag{3}$$

Then, by the transformations $u_1 = w_1 e^{\nu x}$ and $u_2 = w_2 e^{-\nu x}$ with $\nu = M/2$, one can obtain the following modified homogeneous Helmholtz equations with the corresponding boundary conditions

$$\begin{array}{l}\nabla^2 u_1 - \nu^2 u_1 = 0 \\ \nabla^2 u_2 - \nu^2 u_2 = 0\end{array} \text{ in } \Omega, \quad \begin{array}{ll} u_1 = \left(k + \dfrac{x}{M}\right) e^{\nu x}, \quad u_2 = -\left(k + \dfrac{x}{M}\right) e^{-\nu x} & \text{on } \Gamma_1 \\ u_1 = u_2 = 0, & \text{on } \Gamma_2. \end{array} \tag{4}$$

Once the problem (4) is solved for u_1 and u_2, the solution in terms of the original variables velocity V and induced magnetic field B can be obtained backward by the formulas:

$$V = \frac{1}{2}(e^{-\nu x} u_1 + e^{\nu x} u_2), \qquad B = \frac{1}{2}\left(e^{-\nu x} u_1 + e^{\nu x} u_2 - 2\frac{x}{M}\right) . \tag{5}$$

3. Numerical methods

3.1. Application of EDEM

In this section, the extended-domain-eigenfunction method is briefly presented for the solution of MHD flow in annular-like pipe. The EDEM based on the methodology introduced in Aarão et al. (2010), (2011), is employed for the elliptic BVP involving the modified Helmholtz equation

$$\begin{array}{c}\text{Original} \\ \text{Problem}\end{array} \quad A : \begin{cases} \nabla^2 u - \nu^2 u = 0 \text{ in } \Omega \\ u|_{\Gamma_1} = f_1 \\ u|_{\Gamma_2} = 0 , \end{cases} \tag{6}$$

where $f_1 \in L^2(\Gamma_1)$ is a given function and Ω is the annular-like domain which is enclosed by the curves $\Gamma_1 : r = t_1(\theta)$ and $\Gamma_2 : r = t_2(\theta) = r_0$ (see Figure 1(a)) in polar coordinates. Both Γ_1 and Γ_2 are centered at the origin. Here, u can be considered as either u_1 with $f_1 = (k+x/M)e^{\nu x}$ or u_2 with $f_1 = -(k+x/M)e^{-\nu x}$ as given in Equation (4). The basic idea of EDEM is to extend the original domain Ω to a larger annular domain, denoted by Ω_E, such that the new extended domain contains greater symmetry. This extension is done by choosing two concentric circles Γ_2 and $\Gamma_0 : r = t_0(\theta) = r_1 \leq \min(t_1(\theta))$ (that is, the curve Γ_0 is enclosed by Γ_1) (see Figure 1(b)). Then, a related BVP is formulated as

$$\begin{array}{c}\text{Extended} \\ \text{Problem}\end{array} \quad A_E : \begin{cases} \nabla^2 w - \nu^2 w = 0 \text{ in } \Omega_E \\ w|_{\Gamma_0} = f_0 \\ w|_{\Gamma_2} = 0 \end{cases} \tag{7}$$

in which f_0 is initially unknown.

Due to the symmetry of the extended domain, an eigenfunction solution to problem A_E can be obtained by using separation of variables technique. The most general solution to the modified Helmholtz equation satisfying homogeneous Dirichlet condition on the outer boundary Γ_2 is (Polyanin, 2002)

$$w(r,\theta) = \sum_{m=0}^{\infty} \left(K_m(\nu r) - \frac{K_m(\nu b) I_m(\nu r)}{I_m(\nu b)} \right) (A_m \cos(m\theta) + B_m \sin(m\theta)), \quad (8)$$

where I_m and K_m are mth order modified Bessel functions of the first and the second kind, respectively. The unknown coefficients $\{A_m, B_m\}_{m=0}^{\infty}$ are determined through the application of the boundary condition at Γ_0 as in the Fourier series representations. However, since the boundary condition f_0 is not yet specified, as an alternative the boundary condition on the original inner boundary Γ_1 is used to determine the coefficients $\{A_m, B_m\}$. This is accomplished by defining an invertible mapping F of f_1 on Γ_1 to f_0 on Γ_0, such that the solution w of problem A_E, when restricted to Ω, is the solution of problem A, (Aarão et al., 2010). Thus, an f_0 is required such that $w|_\Omega = u$ and $w|_{\Gamma_1} = f_1$.

Now, to determine the coefficients $\{A_m, B_m\}$, first the expansion (8) is truncated to a finite sum

$$w(r,\theta) = \sum_{m=0}^{L} \left(K_m(\nu r) - \frac{K_m(\nu b) I_m(\nu r)}{I_m(\nu b)} \right) (A_m \cos(m\theta) + B_m \sin(m\theta)) \quad (9)$$

then, we consider a finite number $(2L+1)$ of points on the inner boundary Γ_0. This identifies $2L+1$ unique points, $\{x_j = (\theta_j, t_1(\theta_j))\}_{j=1}^{2L+1}$, on the original inner boundary Γ_1 corresponding to those points chosen on Γ_0. Imposing the boundary conditions $w|_{\Gamma_1} = f_1$ in the system (9) for those $2L+1$ points results in $2L+1$ equations in terms of the unknown coefficients $\{A_m, B_m\}_{m=0}^{L}$ (where $B_0 \equiv 0$), which can be written as the matrix equation $Az = C$, (Aarão et al., 2011), where

$$A = \begin{bmatrix} \alpha_{01} & \alpha_{11} & \cdots & \alpha_{M1} & \beta_{11} & \cdots & \beta_{L1} \\ \alpha_{02} & \alpha_{12} & \cdots & \alpha_{L2} & \beta_{12} & \cdots & \beta_{L2} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \alpha_{0,2L+1} & \alpha_{1,2L+1} & \cdots & \alpha_{L,2L+1} & \beta_{1,2L+1} & \cdots & \beta_{L,2L+1} \end{bmatrix}, \quad z = \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_L \\ B_1 \\ \vdots \\ B_L \end{bmatrix}, \quad (10)$$

$$C = \begin{bmatrix} f_1(r(\theta_1), \theta_1) \\ f_1(r(\theta_2), \theta_2) \\ \vdots \\ f_1(r(\theta_{2L+1}), \theta_{2L+1}) \end{bmatrix}$$

with $\alpha_{ij} = g_i(r(\theta_j))\cos(i\theta_j)$, $\beta_{ij} = g_i(r(\theta_j))\sin(i\theta_j)$, and for non-zero ν,

$$g_i(r(\theta_j)) = K_i(\nu r(\theta_j)) - K_i(\nu b)I_i(\nu r(\theta_j))/I_i(\nu b), \;\; i = 1, \ldots, L\,.$$

The unknown coefficients $\{A_m, B_m\}_{m=0}^{L}$ are found by solving the system (10) for z, see (Aarão et al., 2011). Then, the solution w within the original domain Ω, i.e. $u = w|_\Omega$, is obtained by direct application of Equation (9). Thus, the solutions u_1 and u_2 given in problem (4) are obtained by taking $f_1 = (k + x/M)e^{\nu x}$ and $f_1 = -(k + x/M)e^{-\nu x}$, respectively, in the problem A given in (6). Once u_1 and u_2 are obtained, the original unknowns, velocity V and the induced magnetic field B, can be determined back through relationships in (5).

3.2. Application of BEM

A direct BEM with constant elements is also applied to the MHD flow problem in annular-like pipes for comparison with EDEM. BEM transforms the differential equation defined in a domain into an equivalent boundary integral equation using the fundamental solution of the governing equation and discretizes only the boundary of the problem under consideration. The application of BEM to problem A in (6) by using the fundamental solution $u^* = \frac{K_0(\nu r)}{2\pi}$ of the modified Helmholtz equation, r being the magnitude of the distance vector between the source and field points, results in (Tezer-Sezgin & Dost, 1994).

$$-c_S u(S) + Hu + Gq = 0\,, \tag{11}$$

where H and G are the BEM matrices with entries

$$H_{ij} = \frac{\nu}{2\pi}\int_{\Gamma_j} K_1(\nu r)\frac{\partial r}{\partial n}\mathrm{d}\Gamma_j \text{ and } G_{ij} = \frac{1}{2\pi}\int_{\Gamma_j} K_0(\nu r)\mathrm{d}\Gamma_j$$

and $q = \dfrac{\partial u}{\partial n}$. Here, the constant $c_S = \theta_S/2\pi$ where θ_S is the internal angle at the source point S. After the imposition of the corresponding boundary conditions given in Equation (4) for the unknowns u_1 and u_2 in (11), one can obtain solutions u_1, u_2 and their normal derivatives on the boundaries Γ_1 and Γ_2. The interior values for u_1 and u_2 can be obtained by taking $c_S = 1$ in Equation (11). Then, the velocity V and the induced magnetic field B are computed through Equation (5).

4. Results and discussions

The two-dimensional MHD flow subject to an external horizontally applied magnetic field is considered in annular-like domains of which inner wall has several geometry (e.g. an ellipse, a square and a sinusoidal loop). In all cases, the outer pipe wall is determined by a circle $\Gamma_2 : r = r_0$ and in calculations r_0 is taken as $r_0 = 3$. The numerical simulations are carried out for various values of Hartmann number (($0.2 \leq M \leq 20$ in EDEM and BEM) and ($M \leq 300$ in BEM)), and the inner boundary is taken as either insulated ($k = 0$) or with conductivity $k = 1$. The results are presented in terms of equi-velocity and current lines including a comparison of the EDEM and BEM methods. The EDEM solutions are obtained using maximum $2L+1 = 79$ collocation points for the discretization of the inner boundary Γ_1, while in BEM maximum $N = 250$ constant boundary elements are used for the discretization of the boundary $\Gamma = \Gamma_1 \cup \Gamma_2$.

4.1. Problem 1: annular-like domain with elliptic inner pipe wall

In the first problem, we consider the MHD pipe flow in an annular-like domain with elliptic inner pipe wall $\Gamma_1 : x^2/a^2 + y^2/b^2 = 1$ (a and b are the semi-axes of the ellipse) and circular outer wall $\Gamma_2 : r = r_0$. The elliptical inner boundary and the circular outer boundary can be written in polar coordinates (r, θ), respectively, as $\Gamma_1 : \quad r = t_1(\theta) = ab/\sqrt{a^2 \sin^2(\theta) + b^2 \cos^2(\theta)}$, and $\Gamma_2 : \quad r = t_2(\theta) = r_0$, for $\theta \in [0, 2\pi)$.

The comparison of the EDEM and BEM solutions under the effect of increasing Hartmann number ($M = 0.2$, 8, 16, 20) is displayed by taking $a = 1$, $b = 1/2$ in Figure 2 when the inner wall is insulated ($k = 0$) and in Figure 3 when the inner wall is with conductivity $k = 1$. In the case of insulated inner wall, flow is symmetrically divided into four loops when the magnetic field effect is negligibly small ($M = 0.2$). This symmetry is preserved for higher values of $M = 8$, 16, 20 but the loops along the vertical centreline vanish. On the other hand, the symmetry about the vertical centreline $x = 0$ in both the velocity and the induced magnetic field is destroyed for each value of M when k is increased from 0 to 1 (see Figures 2 and 3). Hartmann layers in V on the portions of inner and outer walls which are perpendicular to the applied magnetic field are well observed for both $k = 0$ and $k = 1$. The fluid concentrates around the inner pipe in the direction of applied magnetic field as M increases and the fluid becomes stagnant in the rest of the region for both $k = 0$ and $k = 1$. However, the induced magnetic field profile at $k = 1$ alters significantly compared to the case of $k = 0$ for increasing M. That is, the current lines circulate and form a thick boundary layer around the inner wall in the direction of applied magnetic field as M increases. These figures show that the solutions obtained by EDEM and BEM are in very good agreement. It is also observed that for low values of M the EDEM with coarse discretization produces as accurate results as does the BEM. On the other hand, the EDEM solution suffers from large numerical errors particularly close to Γ_1 when M is increasing (see Figure 2), and the rate of the formation of boundary layer is slower than the rate in BEM. Increasing the number of points in EDEM does not improve this rate significantly. However, BEM is able to give quite accurate results by using an adequate number of boundary elements for higher values of Hartmann number as shown in Figure 4. Hartmann layers are more pronounced and the current lines are distributed symmetrically about the

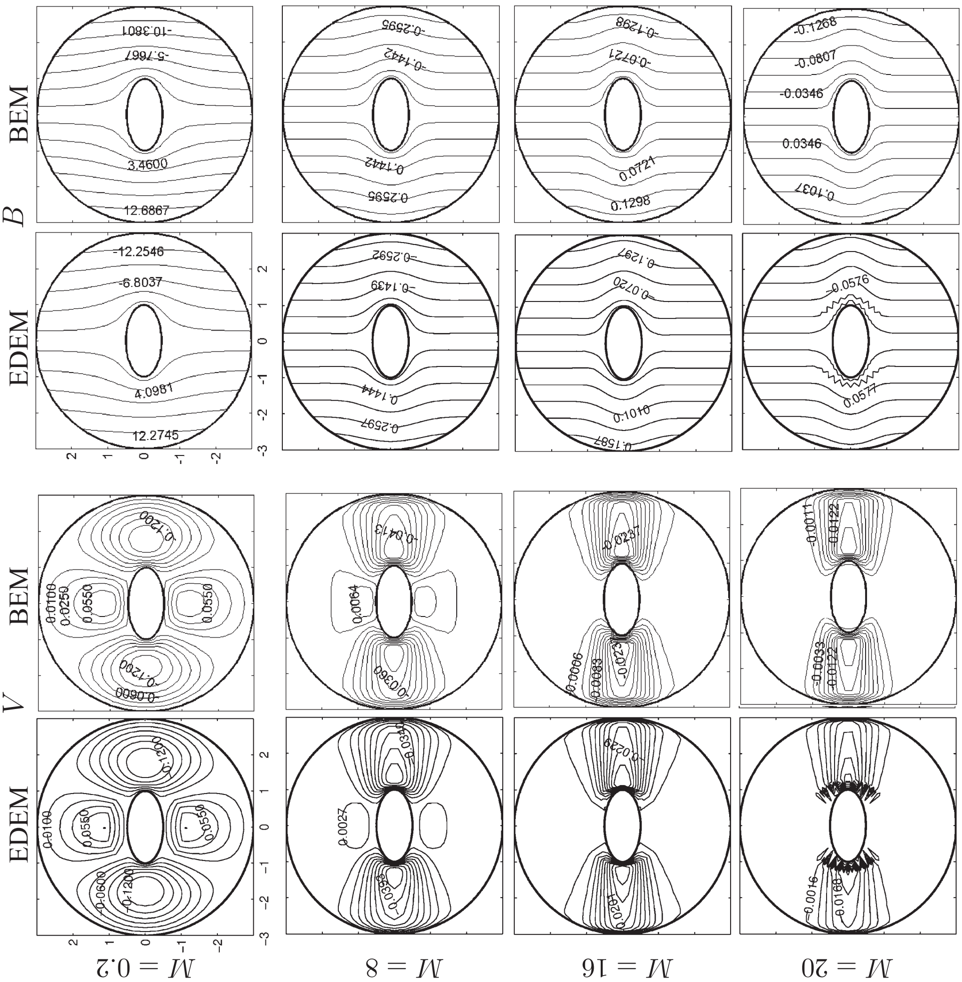

Figure 2. Effect of $M(=0.2, 8, 16, 20)$ on V and B when $k=0$ by EDEM and BEM.

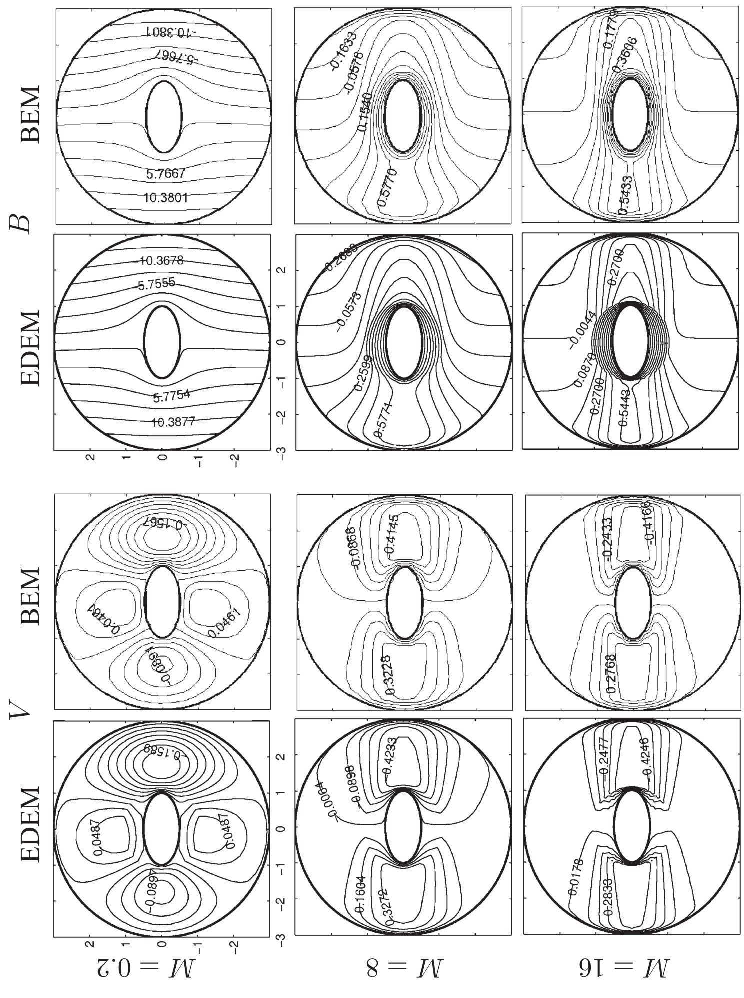

Figure 3. Effect of $M(=0.2, 8, 16)$ on V and B when $k = 1$ by EDEM and BEM.

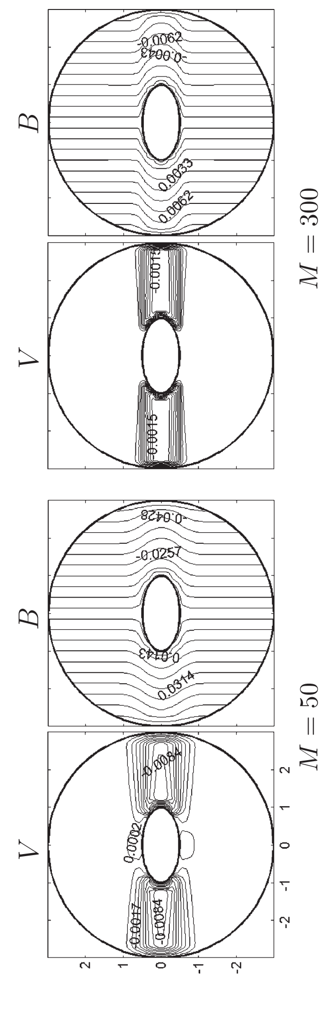

Figure 4. V and B when $M(=50, 300)$ and $k=0$ by BEM.

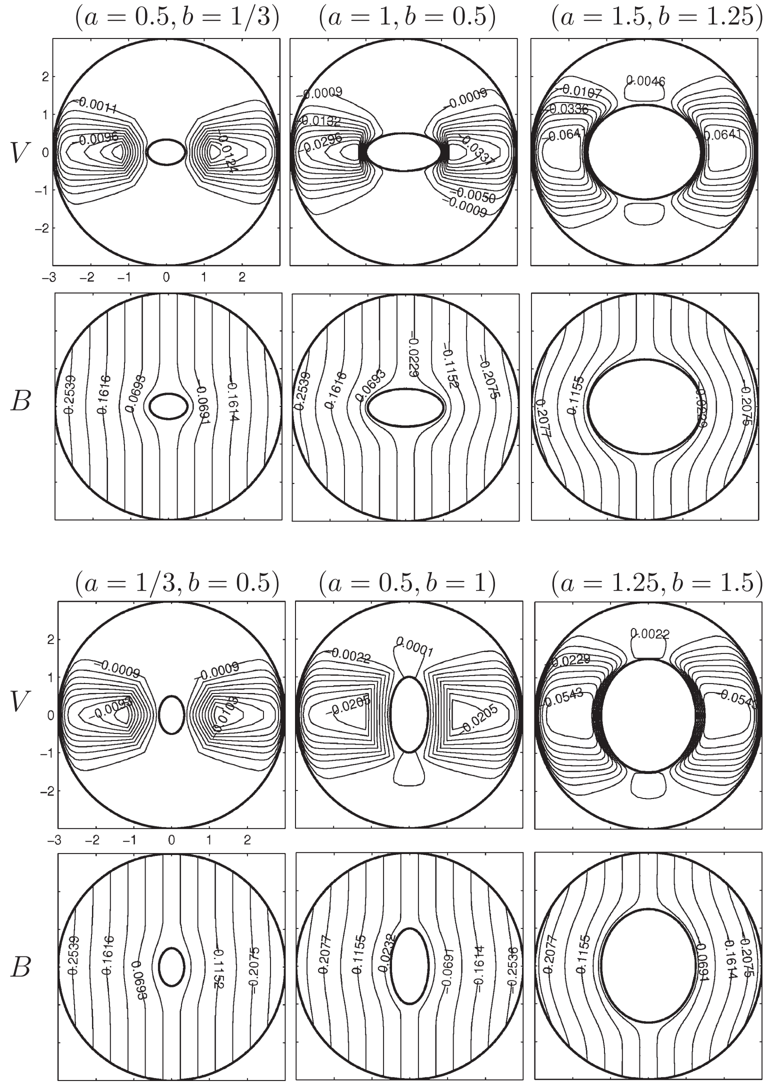

Figure 5. Effect of the size of the ellipse on V and B when $k = 0$, $M = 10$ by EDEM.

vertical centreline $x = 0$. Moreover, both the velocity and induced magnetic field values decrease with an increase in M, indicating the retarding effect of the external magnetic field. As M increases, the velocity becomes uniform with stagnant flow except in the region of Hartmann layers where the fluid action takes place.

The effect of the size of the elliptic inner wall on the velocity and the induced magnetic field is also investigated by keeping $M = 10$ and $k = 0$. Figure 5 displays the EDEM solution in terms of equi-velocity and current lines at various semi-axes values (a, b) of inner ellipse. It is observed that as the size of the ellipse increases there is no significant change in the profiles and the magnitudes of the velocity and induced magnetic field. However, when the area of the ellipse gets

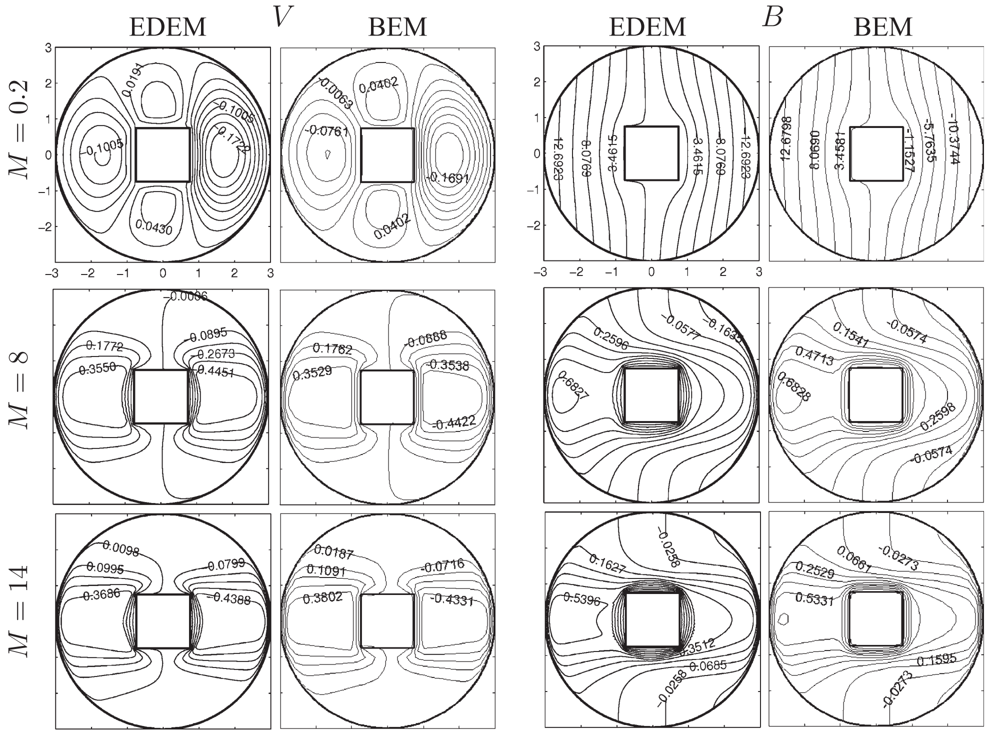

Figure 6. Effect of $M(=0.2, 8, 14)$ on V and B when $k = 1$ by EDEM and BEM.

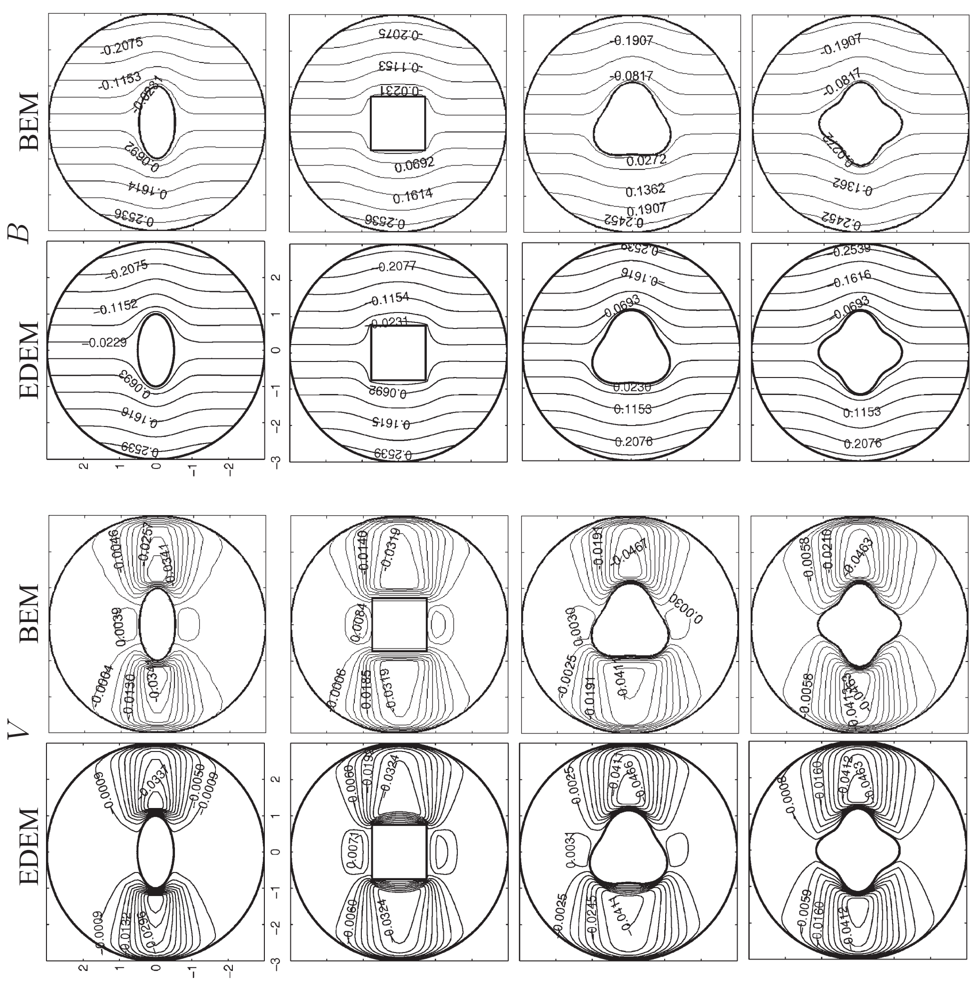

Figure 7. Effect of the shape of the inner pipe wall on V and B when $k = 0, M = 10$ by EDEM and BEM.

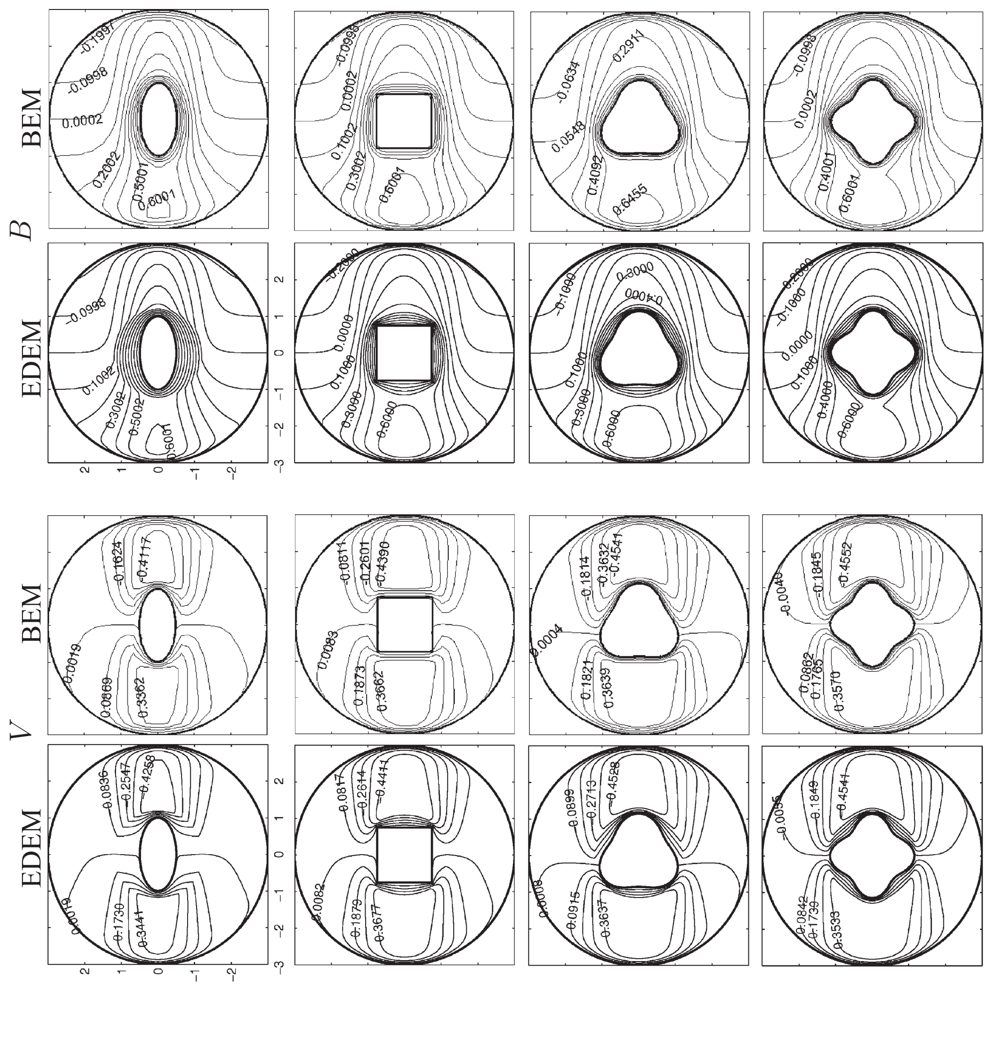

Figure 8. Effect of the shape of the inner pipe wall on V and B when $k = 1, M = 10$ by EDEM and BEM.

larger, fluid is squeezed and the flow extends in the narrowed annular region forming thicker boundary layers along the inner and outer pipe walls in the direction of applied magnetic field. Fluid starts to form secondary flows on the bottom and top of the inner wall as the ellipse gets larger. Moreover, EDEM starts to suffer from computational difficulties due to the need of more collocation points (L) in the discretization of a larger inner wall.

4.2. Problem 2: annular-like domain with square inner pipe wall

In this case, the annular-like domain is taken as the region between the square of side length $\ell = 1.5$ as the inner wall and the circle $r = 3$ as the outer wall. The effect of Hartmann number on the velocity and the induced magnetic field is visualised in Figure 6 when the inner pipe wall is conducting ($k = 1$) using both EDEM and BEM. The velocity and induced magnetic field have similar profiles with very slight alteration in magnitudes when compared to the case of elliptic inner wall (see Figure 3). As it is expected, the most-inner dense current lines are distributed evenly on the square inner pipe wall with an increase in M, which are Hartmann and side layers. Both BEM and EDEM give expected behaviour of MHD flow but EDEM has difficulties for increasing M due to the corners of the square which breaks the smoothness of the boundary. Thus, it can be concluded that the EDEM produces better results for the domains with smooth boundaries as given in Problem 1 in which the inner boundary is elliptical.

4.3. Problem 3: annular-like domain with sinusoidal inner pipe wall

Finally, we consider the MHD pipe flow in annular-like domain with a sinusoidal inner wall defined by $r = r_1 + w\cos(n\theta)$, where r_1 is the radius of base circle, w and n are the amplitude and the number of undulations, respectively. In the calculations, we take $r_1 = 1$, $w = 1.15$, and $n = 3$ and 4. Figures 7 and 8 display the velocity and induced magnetic field lines for different inner pipe walls (ellipse, square, sinusoidal wall with $n = 3$ and 4 from top to bottom) at $M = 10$ and for $k = 0$ and 1, respectively, using both EDEM and BEM. One can see from these figures that, when the inner pipe wall is insulated, velocity develops Hartmann layers for increasing M regardless of the shape of the inner wall. For conducting inner wall, boundary layers in B surround all over the inner wall again regardless of its shape, but velocity boundary layers are more emphasised when inner wall is sinusoidal.

5. Conclusion

The MHD flow in an annular-like pipe under the effect of an externally applied magnetic field is solved numerically by EDEM and BEM. The effects of the Hartmann number, the shape and the conductivity of the inner boundary on the velocity and induced magnetic field are investigated. It is observed that when the intensity of external magnetic field increases, flow is in terms of Hartmann layers around the inner wall regardless of its shape, and it extends in the annular region with the enlargement of the inner wall. Conductivity of the inner boundary forces the current to concantrate near the inner pipe wall. EDEM gives difficulties in obtaining the solution when the inner pipe wall is a square due to the passage from the corners. It is found that the EDEM is computationally less expensive and faster than BEM for small values of M, while it does not give reasonable solutions for large values of M. On the other hand, the BEM is more effective and accurate than EDEM to obtain the solution for higher values of M. The well-known behaviour of MHD flow, namely the flattening tendency of the flow and the boundary layer formation as Hartmann number increases, is captured by both methods.

Disclosure statement

No potential conflict of interest was reported by the authors.

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