Locking in the incompressible limit: pseudo-divergence-free element free Galerkin

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ABSTRACT. *Locking in finite elements has been a major concern since its early developments and has been extensively studied. However, locking in mesh-free methods is still an open topic. Until now the remedies proposed in the literature are extensions of already developed methods for finite elements. Here a new approach is explored and an improved formulation that asymptotically suppresses volumetric locking for the EFG method is proposed. The diffuse divergence converges to the exact divergence. Since the diffuse divergence-free condition can be imposed* a priori, *new interpolation functions are defined that asymptotically verify the incompressibility condition. Modal analysis and numerical results for classical benchmark tests in solids and fluids corroborate this issue.*

RÉSUMÉ. *Depuis la création des éléments finis, les problèmes de verrouillage d'éléments sont au centre de préoccupations de la communauté et ont été largement étudiés. En revanche, les mêmes problèmes dans un contexte sans maillage restent d'actualité. Jusqu'à présent les méthodes proposées dans la littérature sont des extensions des approches déjà développées dans le cadre des éléments finis. Nous proposons une voie exploratrice et une formulation améliorée qui supprime asymptotiquement l'effet de verrouillage volumétrique dans cadre d'EFG. La divergence diffuse converge vers la divergence exacte. Étant donné, que la condition de divergence diffuse nulle peut être imposée a priori, nous proposons dans ce travail des nouvelles fonctions de forme qui vérifient asymptotiquement la condition d'incompressibilité. L'analyse modale et les résultats des cas test classiques pour des solides et fluides confirment cette thèse.*

KEYWORDS: *Locking, Element Free Galerkin, Diffuse derivatives, Moving Least Squares, Incompressible flow, LBB condition*

MOTS-CLÉS : *verrouillage volumétrique, Element Free Galerkin, dérivée diffuse, moindres carrés mobiles, écoulement incompressible, condition inf-sup*

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1. Introduction

Finite element methods require a mesh with a minimum quality. Reasonable meshes may be difficult in certain contexts, for instance, problems involving large deformations, crack propagation, discontinuities and adaptive processes. Recently meshless methods have been developed to overcome these difficulties; among them are the Smooth Particle Hydrodynamics (SPH) (Lucy, 1977; Monaghan, 1988; Randles and Libersky, 1996), the Diffuse Element Method (DEM) (Nayroles, Touzot and Villon, 1992), the Element Free Galerkin Method (EFG) (Belytschko, Krongauz, Organ, Fleming and Krysl, 1996; Belytschko, Lu and Gu, 1994; Belytschko and Tabarra, 1996; Lu, Belytschko and Gu, 1994), the Reproducing Kernel Particle Method (Liu, Chen, Uras and Chang, 1996; Liu, Jun and Zhang, 1995; Liu, Jun, Adee and Belytschko, 1995; Liu, Li and Belytschko, 2000), the HP cloud method (Duarte and Oden, 1995; Duarte and Oden, 1996) and the Partition of Unity Method (Melenk and Babuska, 1996).

The objective of meshless methods is to suppress some of the overhead due to mesh generation by constructing the approximation entirely in terms of nodes without defining *a priori* connectivities between them. Due to the flexibility in constructing the conforming shape functions to meet specific needs for different applications, it has been reported that the meshless methods are paticulary suitable for crack propagation, hp-adaptivity and large deformation problems.

Locking of standard finite elements has been widely studied. It appears because poor numerical interpolation leads to an over-constrained system. It is acknowledged that in a displacement-based finite element method, linear approximations perform poorly for the modeling of incompressible materials. For incompressible, or nearly incompressible, materials an additional constraint appears in the field equations which requires the divergence of the displacement field to be zero in the domain. This constraint is difficult to fulfill for low order elements. Locking is attenuated and can be suppressed for increasing polynomial degrees, in the context of an *hp* adaptive strategy, Babuska and Suri (1992) and Suri (1996) present a review on this issue. Moreover, several techniques are available to alleviate or completely remove the locking phenomena in finite element approximations (see Hughes, 2000).

However, locking in meshless methods is still an open topic. Even recently, Zhu and Atluri (1998) claimed that meshless methods do not exhibit volumetric locking. Now it is clear that this is not true. For instance, Dolbow and Belytschko (1999) analyze the EFG method using the numerical inf-sup condition. Moreover, several authors claim that increasing the dilation parameter locking phenomena in mesh-free methods can be suppressed, or at least attenuated. Their argument is based on numerical examples (Askes, de Borst and Heeres, 1999; Dolbow and Belytschko, 1999) or on the heuristic constraint ratio (Chen, Yoon, Wang and Liu, 2000) proposed by Hughes (2000). In a recent paper by Huerta and Fernández-Méndez (2001) this issue is clarified determining the influence of the dilation parameter on the locking behavior of EFG near the incompressible limit. This is done performing a modal analysis:

studying the fundamental modes (base of the solution space) and their corresponding energy (eigenvalue). In particular EFG behavior is compared with standard finite elements, bilinear and biquadratic. It concludes that an increase of the dilation parameter attenuates, but never suppresses, the volumetric locking and that, as in standard finite elements, an increase in the order of reproducibility reduces the relative number of locking modes.

Thus, large domains of influence alleviate locking but for small domains of influence, however, the direct application of the EFG approximation can result in volumetric locking. In dynamic problems and many non-linear problems, small domains of influence are preferred because they improve the local resolution and enhance the sparsity of the system of equations. Therefore, procedures which avoid locking, even for small domains of influence, are needed. Until now the remedies proposed in the literature are extensions of the methods developed for finite elements.

As noticed before, there are several techniques available to alleviate or remove the locking phenomena in finite element approximations. For example, Suri (1996) shows that locking can be alleviated through the use of higher-order p elements. Alternatively, locking can be removed by mixed methods in which different approximations are implemented for the displacement and pressure fields (see, for instance, Hermann, 1965; Hughes, 2000). However, mixed methods are more expensive due to the need for additional unknowns. Alternatives which do not require additional degrees of freedom are selective reduced integration or strain projection methods. Extensions of these techniques to meshless methods can be found. For example, Dolbow and Belytschko (1999) propose a new formulation of the EFG method using a selective reduced integration and Chen et al. (2000) suggest an improved Reproducing Kernel Particle Method (RKPM) using a pressure projection method.

Here a novel approach is explored. It consists in using interpolation functions that verify approximately the divergence-free constraint. These interpolating functions can be defined *a priori* and are independent of the particle distribution. Moreover, as the density of particles is increased the divergence-free condition is better approximated. This method is based on diffuse derivatives (see Nayroles et al., 1992), which, as proven by Villon (1991), converge to the derivatives of the exact solution when the radius of the support goes to zero (for a fixed dilation parameter).

2. Diffuse derivatives

2.1. *Preliminaries of the EFG method*

This section will not be devoted to develop or discuss mesh-free methods in detail or their relation with moving least squares (MLS) interpolants. There are well known references with excellent presentations of mesh-free methods (see, for instance, Belytschko, Krongauz, Organ, Fleming and Krysl, 1996; Liu, Belytschko and Oden, editors, 1996; Liu, Chen, Jun, Chen, Belytschko, Pan, Uras and Chang, 1996; Liu

et al., 2000; Nayroles et al., 1992). Here some basic notions will be recalled in order to introduce the notation and the approach employed in following sections.

The moving least squares approach is based on the local (*i.e.*, at any point $\boldsymbol{z}$ in the neighborhood of $\boldsymbol{x}$) approximation of the unknown scalar function $u(\boldsymbol{z})$ by u^ρ as

$$u(\boldsymbol{z}) \simeq u^\rho(\boldsymbol{x},\boldsymbol{z}) = \mathbf{P}^{\mathsf{T}}(\boldsymbol{z})\ \mathbf{a}(\boldsymbol{x}) \qquad \text{for } \boldsymbol{z} \text{ near } \boldsymbol{x} \tag{1}$$

where the coefficients $\mathbf{a}(\boldsymbol{x}) = \{\mathrm{a}_0(\boldsymbol{x}), \mathrm{a}_1(\boldsymbol{x}), \ldots, \mathrm{a}_l(\boldsymbol{x})\}^{\mathsf{T}}$ are not constant, they depend on point $\boldsymbol{x}$, and $\mathbf{P}(\boldsymbol{z}) = \{p_0(\boldsymbol{z}), p_1(\boldsymbol{z}), \ldots, p_l(\boldsymbol{z})\}^{\mathsf{T}}$ includes a complete basis of the subspace of polynomials of degree m. In one dimension, it is usual that $p_i(x)$ coincides with the monomials x^i, and, in this particular case, $l = m$. The coefficients $\mathbf{a}$ are obtained by minimization of the functional $J_{\boldsymbol{x}}(\mathbf{a})$ centered in $\boldsymbol{x}$ and defined as

$$J_{\boldsymbol{x}}(\mathbf{a}) = \sum_{i\in I_{\boldsymbol{x}}} \phi(\boldsymbol{x},\boldsymbol{x}_i)\big[u(\boldsymbol{x}_i) - \mathbf{P}(\boldsymbol{x}_i)\ \mathbf{a}(\boldsymbol{x})\big]^2 \tag{2}$$

where $\phi(\boldsymbol{x},\boldsymbol{x}_i)$ is a weighting function (positive, even and with compact support) which characterizes the mesh-free method. For instance, if $\phi(\boldsymbol{x},\boldsymbol{x}_i)$ is continuous together with its first k derivatives, the interpolation is also continuous together with its first k derivatives. The particles cover the computational domain Ω, $\Omega \subset \mathbb{R}^{\mathtt{n_{sd}}}$, and, in particular, a number of particles $\{\boldsymbol{x}_i\}_{i\in I_{\boldsymbol{x}}}$ belong to the support of $\phi(\boldsymbol{x},\boldsymbol{x}_i)$. The minimization of $J_{\boldsymbol{x}}(\mathbf{a})$ induces the standard normal equations in a weighted least-squares problem

$$\mathbf{M}(\boldsymbol{x})\ \mathbf{a}(\boldsymbol{x}) = \sum_{i\in I_{\boldsymbol{x}}} \phi(\boldsymbol{x},\boldsymbol{x}_i)\ u(\boldsymbol{x}_i)\ \mathbf{P}(\boldsymbol{x}_i) \tag{3}$$

where, as usual, the Gram matrix $\mathbf{M}(\boldsymbol{x})$ is the scalar product of the interpolation polynomials:

$$\mathbf{M}(\boldsymbol{x}) = \sum_{i\in I_{\boldsymbol{x}}} \phi(\boldsymbol{x},\boldsymbol{x}_i)\ \mathbf{P}(\boldsymbol{x}_i)^{\mathsf{T}}\mathbf{P}(\boldsymbol{x}_i).$$

That is,

$$< u, v >= \sum_{i\in I_{\boldsymbol{x}}} \phi(\boldsymbol{x},\boldsymbol{x}_i)\ u(\boldsymbol{x}_i)\ v(\boldsymbol{x}_i) \tag{4}$$

must define a discrete scalar product. Thus, several conditions on the particle distribution are implicitly assumed (see, for instance, Huerta and Fernández-Méndez, 2000).

Once the normal equations, Eqs (3), are solved the coefficients $\mathbf{a}$ are substituted in (1). Since the weighting function ϕ usually favors the central point $\boldsymbol{x}$, it seems reasonable to assume that such an approximation is more accurate precisely at $\boldsymbol{z} = \boldsymbol{x}$ and thus the approximation (1) is particularized at $\boldsymbol{x}$, that is,

$$u(\boldsymbol{x}) \simeq u^\rho(\boldsymbol{x}) = \mathbf{P}^{\mathsf{T}}(\boldsymbol{x})\ \mathbf{a}(\boldsymbol{x}) = \mathbf{P}^{\mathsf{T}}(\boldsymbol{x})\ \mathbf{M}^{-1}(\boldsymbol{x}) \sum_{i\in I_{\boldsymbol{x}}} \phi(\boldsymbol{x},\boldsymbol{x}_i)\ u(\boldsymbol{x}_i)\ \mathbf{P}(\boldsymbol{x}_i). \tag{5}$$

This expression can also be written in a standard interpolation form

$$u^\rho(\boldsymbol{x}) = \sum_{i\in I_{\boldsymbol{x}}} N_i^\rho(\boldsymbol{x})\; u(\boldsymbol{x}_i) = \sum_{i\in I_{\boldsymbol{x}}} \underbrace{\Big[\phi(\boldsymbol{x},\boldsymbol{x}_i)\; \mathbf{P}^{\mathsf{T}}(\boldsymbol{x})\; \mathbf{M}^{-1}(\boldsymbol{x})\; \mathbf{P}(\boldsymbol{x}_i)\Big]}_{N_i^\rho(\boldsymbol{x})} u(\boldsymbol{x}_i) \quad (6)$$

2.2. *The diffuse derivative*

The approximation of the derivative of u is the derivative of u^ρ. This requires to derive (5), that is

$$\frac{\partial u}{\partial x_i} \simeq \frac{\partial u^\rho}{\partial x_i} = \frac{\partial \mathbf{P}^{\mathsf{T}}}{\partial x_i}\; \mathbf{a}(\boldsymbol{x}) + \mathbf{P}^{\mathsf{T}}\; \frac{\partial \mathbf{a}}{\partial x_i} \qquad \text{for } i = 1,\ldots,\mathtt{n_{sd}}. \quad (7)$$

Note that the derivative of the polynomials in $\mathbf{P}$ is trivial but the derivative of the coefficients $\mathbf{a}$ requires the resolution of a linear system of equations with the same matrix $\mathbf{M}$ (see Belytschko, Krongauz, Fleming, Organ and Liu, 1996). Moreover, the derivatives of the polynomials can be evaluated *a priori* but the derivatives of the coefficients require the knowledge of the cloud of particles surrounding each point $\boldsymbol{x}$.

Thus the concept of diffuse derivative proposed by Villon (1991) and Nayroles et al. (1992) and defined as

$$\frac{\delta u^\rho}{\delta x_i} = \frac{\partial u^\rho}{\partial z_i}\bigg|_{\boldsymbol{z}=\boldsymbol{x}} = \frac{\partial \mathbf{P}^{\mathsf{T}}}{\partial z_i}\bigg|_{\boldsymbol{z}=\boldsymbol{x}} \mathbf{a}(\boldsymbol{x}) = \frac{\partial \mathbf{P}^{\mathsf{T}}}{\partial x_i}\; \mathbf{a}(\boldsymbol{x}) \qquad \text{for } i = 1,\ldots,\mathtt{n_{sd}}$$

is, from a computational cost point of view, an interesting alternative to (7). Moreover, Villon (1991) shows that the diffuse derivative converges at optimal rate to the derivative of u.

Proposition. *If u^ρ is an approximation to u with an order of consistency m (*i.e. $\mathbf{P}$ *includes a complete basis of the subspace of polynomials of degree m) and ρ/h is constant, then*

$$\left\| \frac{\partial^{|\boldsymbol{k}|} u}{\partial \boldsymbol{x}^{\boldsymbol{k}}} - \frac{\delta^{|\boldsymbol{k}|} u^\rho}{\delta \boldsymbol{x}^{\boldsymbol{k}}} \right\|_\infty \leq C(\boldsymbol{x}) \frac{\rho^{m+1-|\boldsymbol{k}|}}{(m+1)!} \qquad \forall |\boldsymbol{k}| = 0,\ldots,m. \quad (8)$$

where $\boldsymbol{k}$ is a multi-index, $\boldsymbol{k} = (k_1, k_2, \ldots, k_{\mathtt{n_{sd}}})$ and $|\boldsymbol{k}| = k_1 + k_2 + \cdots + k_{\mathtt{n_{sd}}}$.

Proof. Lets assume $u \in \mathcal{C}^{m+1}(\overline{\Omega})$ where $\mathcal{C}^{m+1}$ is the space of $(m+1)$ times continuously differentiable functions. Recall that Taylor's formula of order m can be written as:

$$u(\boldsymbol{x}+\boldsymbol{h}) = \sum_{|\boldsymbol{\alpha}|=0}^{m} \frac{1}{\boldsymbol{\alpha}!} \boldsymbol{h}^{\boldsymbol{\alpha}} \frac{\partial^{|\boldsymbol{\alpha}|} u}{\partial \boldsymbol{x}^{\boldsymbol{\alpha}}}(\boldsymbol{x}) + R_{m+1}(\boldsymbol{x}+\theta\boldsymbol{h}), \quad (9)$$

where $\theta \in]0,1[$, $R_{m+1}(\boldsymbol{x}+\theta\boldsymbol{h})$ is the error term and $\boldsymbol{\alpha}$ is a multi-index such that,

$$\boldsymbol{h}^{\boldsymbol{\alpha}} := h_1^{\alpha_1} h_2^{\alpha_2} \cdots h_{\mathtt{n_{sd}}}^{\alpha_{\mathtt{n_{sd}}}}; \quad \boldsymbol{\alpha}! := \alpha_1!\alpha_2! \cdots \alpha_{\mathtt{n_{sd}}}!; \quad |\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_{\mathtt{n_{sd}}}$$

Equation (9) can be rewritten taking $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{h}$

$$u(\boldsymbol{z}) = \sum_{|\boldsymbol{\alpha}|=0}^{m} \frac{1}{\boldsymbol{\alpha}!} \left(\frac{\boldsymbol{z} - \boldsymbol{x}}{\rho} \right)^{\boldsymbol{\alpha}} \rho^{\boldsymbol{\alpha}} \frac{\partial^{|\boldsymbol{\alpha}|} u}{\partial \boldsymbol{x}^{\boldsymbol{\alpha}}}(\boldsymbol{x}) + R_{m+1}(\boldsymbol{x}, \boldsymbol{z}).$$

Thus, Taylor's formula can also be written as:

$$u(\boldsymbol{z}) = \mathbf{P}^{\mathsf{T}}\Big(\frac{\boldsymbol{z} - \boldsymbol{x}}{\rho} \Big) \mathbf{U}(\boldsymbol{x}) + R_{m+1}(\boldsymbol{x}, \boldsymbol{z}), \tag{10}$$

where

$$\mathbf{P}(\boldsymbol{\xi}) = \left\{ \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} \right\}; \quad \mathbf{U}(\boldsymbol{x}) = \left\{ \rho^{\boldsymbol{\alpha}} \frac{\partial^{|\boldsymbol{\alpha}|} u}{\partial \boldsymbol{x}^{\boldsymbol{\alpha}}} \right\} \qquad |\boldsymbol{\alpha}| = 0, \ldots, m. \tag{11}$$

Observe that $\mathbf{U}(\boldsymbol{x})$ depends on the exact derivatives of u.

The MLS approach is based on the local approximation of the unknown scalar function u by u^{ρ}, see equation (1). Since in equation (10) polynomials $\mathbf{P}(\boldsymbol{\xi})$ are centered and scaled, the MLS interpolant is also centered and scaled,

$$u(\boldsymbol{z}) \simeq u^{\rho}(\boldsymbol{x}, \boldsymbol{z}) = \mathbf{P}^{\mathsf{T}}\Big(\frac{\boldsymbol{z} - \boldsymbol{x}}{\rho} \Big) \, \mathbf{a}(\boldsymbol{x}) \qquad \text{for } \boldsymbol{z} \text{ near } \boldsymbol{x}.$$

Then the MLS approach requires the resolution of the normal equations given by (3), here $u(\boldsymbol{x}_i)$ is substituted by (10)

$$\mathbf{M}(\boldsymbol{x})\mathbf{a}(\boldsymbol{x}) = < \mathbf{P}\Big(\frac{\boldsymbol{z} - \boldsymbol{x}}{\rho} \Big), \mathbf{P}^{\mathsf{T}}\Big(\frac{\boldsymbol{z} - \boldsymbol{x}}{\rho} \Big) \mathbf{U}(\boldsymbol{x}) + R_{m+1}(\boldsymbol{x}, \boldsymbol{z}) >,$$

which can be rearranged as

$$\mathbf{M}(\boldsymbol{x})[\mathbf{a}(\boldsymbol{x}) - \mathbf{U}(\boldsymbol{x})] = \sum_{j \in I_{\boldsymbol{x}}} \phi\Big(\frac{\boldsymbol{x}_j - \boldsymbol{x}}{\rho} \Big) \mathbf{P}\Big(\frac{\boldsymbol{x}_j - \boldsymbol{x}}{\rho} \Big) R_{m+1}(\boldsymbol{x}, \boldsymbol{x}_j) =: \mathbf{b}. \tag{12}$$

Now, lets rewrite the r.h.s. of (12) in a more convenient way. The error term of the Taylor's formula has the form

$$R_{m+1}(\boldsymbol{x}, \boldsymbol{x}_j) = \sum_{|\boldsymbol{\alpha}|=m+1} \frac{(\boldsymbol{x}_j - \boldsymbol{x})^{\boldsymbol{\alpha}}}{(m+1)!} \frac{\partial^{|\boldsymbol{\alpha}|} u}{\partial \boldsymbol{x}^{\boldsymbol{\alpha}}}(\boldsymbol{x}, \boldsymbol{x}_j), \tag{13}$$

substituting (13) in the definition of vector $\mathbf{b}$, see (12), produces

$$\mathbf{b} = \sum_{j \in I_{\boldsymbol{x}}} \phi\Big(\frac{\boldsymbol{x}_j - \boldsymbol{x}}{\rho} \Big) \mathbf{P}\Big(\frac{\boldsymbol{x}_j - \boldsymbol{x}}{\rho} \Big) \sum_{|\boldsymbol{\alpha}|=m+1} \frac{(\boldsymbol{x}_j - \boldsymbol{x})^{\boldsymbol{\alpha}}}{(m+1)!} \frac{\partial^{|\boldsymbol{\alpha}|} u}{\partial \boldsymbol{x}^{\boldsymbol{\alpha}}}(\boldsymbol{x}, \boldsymbol{x}_j).$$

Each component of the previously defined vector $\mathbf{b}$ is associated to the corresponding component of $\mathbf{P}$, namely the polynomial of degree $|\boldsymbol{k}| = 0, \ldots, m$ defined as

$$\boldsymbol{\xi}^{\boldsymbol{k}}/\boldsymbol{k}! = \big(\xi_1^{k_1} \xi_2^{k_2} \cdots \xi_{\mathtt{n_{sd}}}^{k_{\mathtt{n_{sd}}}} \big) / \big(k_1! k_2! \cdots k_{\mathtt{n_{sd}}}! \big).$$

Under these circumstances, each component of $\mathbf{b}$ can be written as

$$\begin{aligned} b_{\boldsymbol{k}} &= \sum_{j\in I_{\boldsymbol{x}}} \phi\Big(\frac{\boldsymbol{x}_j-\boldsymbol{x}}{\rho}\Big)\frac{(\boldsymbol{x}_j-\boldsymbol{x})^{\boldsymbol{k}}}{\rho^{|\boldsymbol{k}|}}\frac{1}{|\boldsymbol{k}|!}\sum_{|\boldsymbol{\alpha}|=m+1}\frac{(\boldsymbol{x}_j-\boldsymbol{x})^{\boldsymbol{\alpha}}}{(m+1)!}\,\frac{\partial^{|\boldsymbol{\alpha}|}u}{\partial\boldsymbol{x}^{\boldsymbol{\alpha}}}(\boldsymbol{x},\boldsymbol{x}_j) \\ &= \frac{\rho^{m+1}}{(m+1)!}\underbrace{\frac{1}{|\boldsymbol{k}|!}\sum_{j\in I_{\boldsymbol{x}}}\phi\Big(\frac{\boldsymbol{x}_j-\boldsymbol{x}}{\rho}\Big)\sum_{|\boldsymbol{\alpha}|=m+1}\Big(\frac{\boldsymbol{x}_j-\boldsymbol{x}}{\rho}\Big)^{\boldsymbol{k}+\boldsymbol{\alpha}}\frac{\partial^{|\boldsymbol{\alpha}|}u}{\partial\boldsymbol{x}^{\boldsymbol{\alpha}}}(\boldsymbol{x},\boldsymbol{x}_j)}_{r_{\boldsymbol{k}}(\boldsymbol{x})} \\ &= \frac{\rho^{m+1}}{(m+1)!}r_{\boldsymbol{k}}(\boldsymbol{x}). \end{aligned} \tag{14}$$

Thus, the r.h.s. of (12) becomes

$$\mathbf{b} = \frac{\rho^{m+1}}{(m+1)!}\,\mathbf{r}(\boldsymbol{x}). \tag{15}$$

Substituting (15) into equation (12) and assuming that $\mathbf{M}$ is regular,

$$\mathbf{a}(\boldsymbol{x}) - \mathbf{U}(\boldsymbol{x}) = \frac{\rho^{m+1}}{(m+1)!}\,\mathbf{M}^{-1}(\boldsymbol{x})\;\mathbf{r}(\boldsymbol{x}).$$

On one hand, $r_{\boldsymbol{k}}$ is bounded for all $|\boldsymbol{k}| = 0,\dots,m$. This can be seen from the definition of $r_{\boldsymbol{k}}$, see (14). Note that for a fixed $\boldsymbol{x}$, if ρ/h is constant, $r_{\boldsymbol{k}}$ is the sum of products of continuous functions in Ω. Thus, it is a continuous function in Ω. Moreover, in every product, there is the weighting function ϕ, which has compact support. Since $r_{\boldsymbol{k}}$ is a continuous function in a compact support it is bounded by a constant that only depends on $\boldsymbol{x}$.

On the other hand, matrix $\mathbf{M}$ is also bounded (see Huerta and Fernández-Méndez, 2000). Then, if both, $\mathbf{M}$ and $r_{\boldsymbol{k}}$, are bounded, a constant $C(\boldsymbol{x})$ can be defined as the bound of $\mathbf{M}^{-1}(\boldsymbol{x})\mathbf{r}(\boldsymbol{x})$ and consequently

$$|\mathbf{a}(\boldsymbol{x}) - \mathbf{U}(\boldsymbol{x})| \le \frac{\rho^{m+1}}{(m+1)!}\,C(\boldsymbol{x})$$

The previous expression can be divided by $\rho^{|\boldsymbol{k}|}$. Then, for each component,

$$\left|\frac{\mathrm{a}_{\boldsymbol{k}}(\boldsymbol{x})}{\rho^{|\boldsymbol{k}|}} - \frac{\mathrm{U}_{\boldsymbol{k}}(\boldsymbol{x})}{\rho^{|\boldsymbol{k}|}}\right| \le \frac{\rho^{m+1-|\boldsymbol{k}|}}{(m+1)!}\,C(\boldsymbol{x}) \qquad \forall\,|\boldsymbol{k}| = 0,\dots,m, \tag{16}$$

where $\mathrm{a}_{\boldsymbol{k}}$ and $\mathrm{U}_{\boldsymbol{k}}$ are the components of $\mathbf{a}$ and $\mathbf{U}$, respectively. Recall that each component of $\mathbf{U}(\boldsymbol{x})$ depends on the corresponding exact derivatives of u, see (11). Now, observe that each component of $\mathbf{a}(\boldsymbol{x})$ shall depend on the corresponding pseudo-derivatives; that is, for $|\boldsymbol{k}| = 0, ..., m$

$$\frac{\delta^{|\boldsymbol{k}|}u^{\rho}}{\delta\boldsymbol{x}^{\boldsymbol{k}}} := \frac{\delta^{|\boldsymbol{k}|}u^{\rho}}{\delta x_1^{\boldsymbol{k}_1}\cdots\delta x_{\mathrm{n_{sd}}}^{\boldsymbol{k}_{\mathrm{n_{sd}}}}} := \left.\frac{\partial^{|\boldsymbol{k}|}u^{\rho}}{\partial z_1^{\boldsymbol{k}_1}\cdots\partial z_{\mathrm{n_{sd}}}^{\boldsymbol{k}_{\mathrm{n_{sd}}}}}\right|_{\boldsymbol{z}=\boldsymbol{x}} = \frac{\mathrm{a}_{\boldsymbol{k}}(\boldsymbol{x})}{\rho^{\boldsymbol{k}_1}\cdots\rho^{\boldsymbol{k}_{\mathrm{n_{sd}}}}}. \tag{17}$$

Finally, replacing the definition of $\mathbf{U}(\boldsymbol{x})$ and $\mathbf{a}(\boldsymbol{x})$ given by (11) and (17), one gets the final expression, which completes the proof,

$$\left\| \frac{\partial^{|\boldsymbol{k}|} u}{\partial \boldsymbol{x}^{\boldsymbol{k}}} - \frac{\delta^{|\boldsymbol{k}|} u^{\rho}}{\delta \boldsymbol{x}^{\boldsymbol{k}}} \right\|_{\infty} \leq C(\boldsymbol{x}) \frac{\rho^{m+1-|\boldsymbol{k}|}}{(m+1)!} \qquad \forall |\boldsymbol{k}| = 0, \ldots, m. \qquad \square$$

3. Pseudo-divergence free condition

3.1. *Diffuse divergence*

In the previous section the diffuse derivative was introduced and its convergence to the actual derivative as $\rho \to 0$ was proved. Incompressible computations require that the approximating field must be divergence free. That is, the solution $\boldsymbol{u}(\boldsymbol{x})$, now a vector $\boldsymbol{u} : \mathbb{R}^{\mathtt{n_{sd}}} \to \mathbb{R}^{\mathtt{n_{sd}}}$, verifies $\nabla \cdot \boldsymbol{u} = 0$, and the approximation $\boldsymbol{u}^{\rho}(\boldsymbol{x})$ should also be divergence-free. This condition however depends on the interpolation space. Here, instead of requiring a divergence-free interpolation, the diffuse divergence of the approximation

$$\begin{aligned} \boldsymbol{u}^{\rho} &= \begin{pmatrix} u_1^{\rho} \\ \vdots \\ u_{\mathtt{n_{sd}}}^{\rho} \end{pmatrix} = \begin{pmatrix} \mathbf{P}^{\mathsf{T}} \mathbf{a}_1 \\ \vdots \\ \mathbf{P}^{\mathsf{T}} \mathbf{a}_{\mathtt{n_{sd}}} \end{pmatrix} \\ &= \begin{pmatrix} p_0(\boldsymbol{x})\, \mathbf{I}_{\mathtt{n_{sd}}} & p_1(\boldsymbol{x})\, \mathbf{I}_{\mathtt{n_{sd}}} & \cdots & p_l(\boldsymbol{x})\, \mathbf{I}_{\mathtt{n_{sd}}} \end{pmatrix} \begin{pmatrix} \mathbf{c}_0(\boldsymbol{x}) \\ \mathbf{c}_1(\boldsymbol{x}) \\ \vdots \\ \mathbf{c}_l(\boldsymbol{x}) \end{pmatrix} \\ &= \mathbf{Q}^{\mathsf{T}} \mathbf{c} \end{aligned}$$

is imposed equal to zero, that is

$$\nabla^{\delta} \cdot \boldsymbol{u}^{\rho} = \sum_{i=1}^{\mathtt{n_{sd}}} \frac{\delta \boldsymbol{u}^{\rho}}{\delta x_i} = \sum_{i=1}^{\mathtt{n_{sd}}} \frac{\partial \mathbf{P}^{\mathsf{T}}}{\partial x_i}\, \mathbf{a}_i(\boldsymbol{x}) = \left(\nabla \cdot \mathbf{Q}^{\mathsf{T}}(\boldsymbol{x}) \right) \mathbf{c}(\boldsymbol{x}) = 0. \tag{18}$$

Note that $\mathbf{I}_{\mathtt{n_{sd}}}$ is the identity matrix of order $\mathtt{n_{sd}}$ and the coefficients have been rearranged as

$$\mathbf{c}^{\mathsf{T}} = \big(\underbrace{\mathrm{a}_{0,1} \ \cdots \ \mathrm{a}_{0,\mathtt{n_{sd}}}}_{\mathbf{c}_0^{\mathsf{T}}(\boldsymbol{x})} \quad \underbrace{\mathrm{a}_{1,1} \ \cdots \ \mathrm{a}_{1,\mathtt{n_{sd}}}}_{\mathbf{c}_1^{\mathsf{T}}(\boldsymbol{x})} \ \cdots \ \underbrace{\mathrm{a}_{l,1} \ \cdots \ \mathrm{a}_{l,\mathtt{n_{sd}}}}_{\mathbf{c}_l^{\mathsf{T}}(\boldsymbol{x})} \big).$$

Equation (18) must hold at each point $\boldsymbol{x}$ and for any approximation. Thus appropriate interpolation functions, $\mathbf{Q}$, must be defined in order to verify (18) and thus ensure asymptotically a divergence-free interpolation (*i.e.,* the divergence-free condition is fulfilled as $\rho \to 0$).

3.2. *A 2D pseudo-divergence free interpolation*

The previous concepts are particularized for a two-dimensional case and in order to define pseudo-divergence-free interpolation functions. Suppose for instance that consistency of order two is desired, then $\mathbf{P}^{\mathsf{T}} = \{1, x_1, x_2, x_1^2, x_1x_2, x_2^2\}$, thus

$$\mathbf{Q}^{\mathsf{T}} = \begin{pmatrix} 1 & 0 & x_1 & 0 & x_2 & 0 & x_1^2/2 & 0 & x_1x_2 & 0 & x_2^2/2 & 0 \\ 0 & 1 & 0 & x_1 & 0 & x_2 & 0 & x_1^2/2 & 0 & x_1x_2 & 0 & x_2^2/2 \end{pmatrix} \tag{19}$$

and

$$\mathbf{c}^{\mathsf{T}} = \begin{pmatrix} \mathrm{a}_{0,1} & \mathrm{a}_{0,2} & \mathrm{a}_{1,1} & \mathrm{a}_{1,2} & \mathrm{a}_{2,1} & \mathrm{a}_{2,2} & \mathrm{a}_{3,1} & \mathrm{a}_{3,2} & \mathrm{a}_{4,1} & \mathrm{a}_{4,2} & \mathrm{a}_{5,1} & \mathrm{a}_{5,2} \end{pmatrix}. \tag{20}$$

The pseudo-divergence-free condition (18) is, in this case, written as

$$\nabla^{\delta} \cdot \boldsymbol{u}^{\rho} = \frac{\partial \mathbf{P}^{\mathsf{T}}}{\partial x_1}\, \mathbf{a}_1 + \frac{\partial \mathbf{P}^{\mathsf{T}}}{\partial x_2}\, \mathbf{a}_2 = 0,$$

which implies

$$(\mathrm{a}_{1,1} + \mathrm{a}_{2,2}) + x_1(\mathrm{a}_{3,1} + \mathrm{a}_{4,2}) + x_2(\mathrm{a}_{4,1} + \mathrm{a}_{5,2}) = 0,$$

and consequently,

$$\mathrm{a}_{1,1} + \mathrm{a}_{2,2} = 0, \quad \mathrm{a}_{3,1} + \mathrm{a}_{4,2} = 0, \quad \mathrm{a}_{4,1} + \mathrm{a}_{5,2} = 0.$$

The influence of these three restrictions in the interpolation functions (19) can be viewed as follows

$$\begin{pmatrix} 1 & 0 & x_1 & 0 & x_2 & 0 & x_1^2/2 & 0 & x_1x_2 & 0 & x_2^2/2 & 0 \\ 0 & 1 & -x_2 & x_1 & 0 & 0 & -x_1x_2 & x_1^2/2 & -x_2^2/2 & 0 & 0 & 0 \end{pmatrix}, \tag{21}$$

where one should note that the coefficients in the x_1 and x_2 directions are now coupled and that the total number of degrees of freedom has decreased.

3.3. *The pseudo-divergence-free EFG method*

Using (21), let $\mathbf{Q}_{\delta}$ be the new interpolation matrix (where obviously the unnecessary columns have been removed). The interpolation is then defined as

$$\boldsymbol{u}(\boldsymbol{z}) \simeq \boldsymbol{u}^{\rho}(\boldsymbol{x}, \boldsymbol{z}) = \begin{pmatrix} u_1^{\rho}(\boldsymbol{x}, \boldsymbol{z}) \\ u_2^{\rho}(\boldsymbol{x}, \boldsymbol{z}) \end{pmatrix} = \mathbf{Q}_{\delta}^{\mathsf{T}}(\boldsymbol{z})\ \mathbf{c}(\boldsymbol{x}). \tag{22}$$

The vector version of the discrete scalar product defined in (4),

$$< \boldsymbol{u}, \boldsymbol{v} >= \sum_{i \in I_{\boldsymbol{x}}} \phi(\boldsymbol{x}, \boldsymbol{x}_i)\ \boldsymbol{u}^{\mathsf{T}}(\boldsymbol{x}_i)\ \boldsymbol{v}(\boldsymbol{x}_i)$$

allows now to reproduce the MLS approximation. Thus at each point $\boldsymbol{x}$ the normal equations should be solved, see (3),

$$\mathbf{M}(\boldsymbol{x})\ \mathbf{c}(\boldsymbol{x}) =< u, \mathbf{Q}_\delta > \qquad \text{with} \quad \mathbf{M}(\boldsymbol{x}) :=< \mathbf{Q}_\delta, \mathbf{Q}_\delta > .$$

Thus, as previously, the coefficients $\mathbf{c}$ are substituted in (22) and the approximation is particularized at $\boldsymbol{z} = \boldsymbol{x}$. Then, equation (5) becomes

$$\boldsymbol{u}(\boldsymbol{x}) \simeq \boldsymbol{u}^\rho(\boldsymbol{x}) = \mathbf{Q}_\delta^{\mathsf{T}}(\boldsymbol{x})\ \mathbf{c}(\boldsymbol{x}) = \mathbf{Q}_\delta^{\mathsf{T}}(\boldsymbol{x})\ \mathbf{M}^{-1}(\boldsymbol{x}) < u, \mathbf{Q}_\delta >,$$

and a final expression similar to (6) can be found as

$$\boldsymbol{u}^\rho(\boldsymbol{x}) = \sum_{i \in I_{\boldsymbol{x}}} \mathbf{N}_i^\rho(\boldsymbol{x})\ \boldsymbol{u}(\boldsymbol{x}_i) = \sum_{i \in I_{\boldsymbol{x}}} \Big[\phi(\boldsymbol{x}, \boldsymbol{x}_i)\ \mathbf{Q}_\delta^{\mathsf{T}}(\boldsymbol{x})\ \mathbf{M}^{-1}(\boldsymbol{x})\ \mathbf{Q}_\delta(\boldsymbol{x}_i)\Big] \boldsymbol{u}(\boldsymbol{x}_i).$$

It is important to note that the matrix of interpolation functions $\mathbf{N}_i^\rho$ is now a full matrix not a diagonal one as standard EFG would induce in this non scalar problem. This is due to the fact that the two components of the solution are linked by the incompressibility restriction.

4. Modal analysis

4.1. *Preliminaries*

The modal analysis presented here follows the same rationale originally presented by Huerta and Fernández-Méndez (2001). It is restricted to small deformations, namely $\boldsymbol{\nabla}^{\mathrm{s}}\boldsymbol{u}$, where $\boldsymbol{u}$ is the displacement and $\boldsymbol{\nabla}^{\mathrm{s}}$ the symmetric gradient, *i.e.* $\boldsymbol{\nabla}^{\mathrm{s}} = \frac{1}{2}\big(\boldsymbol{\nabla}^T + \boldsymbol{\nabla}\big)$. Moreover, linear elastic isotropic materials under plane strain conditions are considered. Dirichlet boundary conditions are imposed on Γ_D, a traction $\boldsymbol{h}$ is prescribed along the Neumann boundary Γ_N and there is a body force $\boldsymbol{f}$. Thus, the problem that needs to be solved may be stated as:

solve for $\boldsymbol{u} \in [H^1_{\Gamma_D}]^2$ such that

$$\frac{E}{1+\nu} \int_\Omega \boldsymbol{\nabla}^{\mathrm{s}}\boldsymbol{v} : \boldsymbol{\nabla}^{\mathrm{s}}\boldsymbol{u} d\Omega + \frac{E\nu}{(1+\nu)(1-2\nu)} \int_\Omega (\boldsymbol{\nabla} \cdot \boldsymbol{v})\, (\boldsymbol{\nabla} \cdot \boldsymbol{u})\, d\Omega$$
$$= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} d\Omega + \int_{\Gamma_N} \boldsymbol{h} \cdot \boldsymbol{v} d\Gamma \qquad \forall \boldsymbol{v} \in [H^1_{0,\Gamma_D}]^2. \quad (23)$$

In this equation, the standard vector subspaces of H^1 are employed for the solution $\boldsymbol{u}$

$$[H^1_{\Gamma_D}]^2 := \big\{\boldsymbol{u} \in [H^1]^2 \mid \boldsymbol{u} = \boldsymbol{u}_D \text{ on } \Gamma_D\big\}$$

(Dirichlet conditions, $\boldsymbol{u}_D$, are automatically satisfied) and for the test functions $\boldsymbol{v}$

$$[H^1_{0,\Gamma_D}]^2 := \big\{\boldsymbol{v} \in [H^1]^2 \mid \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_D\big\}$$

(zero values are imposed along Γ_D).

This equation shows the inherent difficulties of the incompressible limit. The standard *a priori* error estimate emanating from (23) and based on the energy norm, which is induced by the LHS of (23), is

$$\|\boldsymbol{u} - \boldsymbol{u}_h\| \leq \inf_{\boldsymbol{w} \in \mathcal{S}_h} \|\boldsymbol{u} - \boldsymbol{w}\| \leq C_{\boldsymbol{u},\nu,p}\, h^{f(p)} \tag{24}$$

where $\mathcal{S}_h$ is the finite dimensional subspace of $[H^1_{\Gamma_D}]^2$ in which the approximation $\boldsymbol{u}_h$ is sought, $C_{\boldsymbol{u},\nu,p}$ is a constant independent of h (characteristic size of the mesh), and $f(p)$ is a positive monotone function of p (degree of the polynomials used for the interpolation). The subindices of the constant C indicate that it depends on the Poisson ratio, the order of the interpolation and the exact solution itself.

From (23) one can observe the difficulties of the energy norm to produce a small infimum in (24) for values of ν close to 0.5. In fact, in order to have finite values of the energy norm the divergence-free condition must be enforced in the continuum case, *i.e.* $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$ for $\boldsymbol{u} \in [H^1_{\Gamma_D}]^2$, and also in the finite dimensional space, *i.e.* $\boldsymbol{\nabla} \cdot \boldsymbol{u}_h = 0$ for $\boldsymbol{u}_h \in \mathcal{S}_h \subset [H^1_{\Gamma_D}]^2$. In fact, locking will occur when the approximation space $\mathcal{S}_h$ is not rich enough for the approximation to verify the divergence-free condition.

Under these conditions, it is evident that locking may be studied from the LHS of (23). This is the basis for the modal analysis of locking. The discrete eigenfunctions (the eigenvectors) corresponding to the LHS of (23) are computed because they completely describe, in the corresponding space, the behavior of the bilinear operator induced by this LHS.

In computational mechanics it is standard to write the strain, $\boldsymbol{\varepsilon}$, and the stress, $\boldsymbol{\sigma}$, tensors in vector form (Belytschko, Liu and Moran, 2000). Moreover, under the assumptions already discussed, they are related as

$$\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{d}, \qquad \boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}, \qquad \mathbf{C} = \frac{E}{(1+\nu)(1-2\nu)} \begin{pmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{pmatrix}.$$

Where $\mathbf{d}$ is the vector of nodal displacements (the coefficients corresponding to the approximation $\boldsymbol{u}_h$ in the base of $\mathcal{S}_h$), and $\mathbf{B}$ is the standard matrix relating displacements and strains. Then, the stiffness matrix can be computed as usual,

$$\mathbf{K} = \int_\Omega \mathbf{B}^\intercal \mathbf{C} \mathbf{B}\, d\Omega.$$

The modal analysis presented in the following is based on $\mathbf{K}$, which is naturally related to the energy norm in the finite dimensional interpolation space, $\mathcal{S}_h$, defined by the finite elements employed (and characterized by $\mathbf{B}$).

4.2. *Comparing EFG and pseudo-divergence-free EFG*

The incompressible limit is studied by evaluating the eigenvalues associated to each mode as the Poisson ratio, ν, tends to 0.5. Following the procedure proposed by Huerta and Fernández-Méndez (2001) the logarithm of the eigenvalue is plotted as a function of the logarithm of $0.5 - \nu$. Then each mode is classified in three groups:

1) modes that do not present any locking behavior,

2) modes that do have physical locking, that is the eigenvalue goes to infinity because it is a volumetric mode, and

3) modes associated to non-physical locking, that is the eigenvalue goes to infinity but there is no volume variation.

In the last case, the displacement field conserves the total area but suffers from non-physical locking. The interpolation space is not rich enough to ensure the divergence-free condition.

In fact, these last modes do verify that

$$\int_{\square} \boldsymbol{\nabla} \cdot \boldsymbol{u}_h \; d\boldsymbol{x} = 0,$$

but do not comply with the divergence-free condition locally (at each interior point). This is clearly a non-physical locking behavior.

The modal analysis is performed on a distribution of 3×3 particles and for bilinear consistency, that is $\mathbf{P} = \{1, x_1, x_2, x_1\, x_2\}^{\mathsf{T}}$. Figures 1 and 2 show the modes already classified for two different dilation parameters, $\rho/h = 1.2$ and 2.2.

Figure 3 compares the eigenvalues obtained by standard EFG and the pseudo-divergence-free interpolation for two particular non-physical locking modes. Moreover, three values of ratio ρ/h are also compared, namely 1.2, 2.2 and 3.2.

Note that the pseudo-divergence-free interpolation has not suppressed the non-physical locking modes. Thus, for a fixed dilation parameter ρ variations on the ratio ρ/h do not suppress locking. Indeed, the influence of locking is reduced because the eigenvalue is decreased. That is, the energy associated to the locking mode is decreased and this attenuates the volumetric locking. Nevertheless, in the incompressible limit, locking will still be present and it may induce useless numerical results.

This results should be expected. The convergence of the diffuse derivative, see (8), is ensured as ρ approaches zero for a ratio ρ/h kept constant. In other words, convergence is ensured as the interpolation is refined.

This is analyzed in Figure 4 for the non-physical locking mode that presents the largest eigenvalue (the first mode to spoil the approximation). These results are obtained for the "worst" dilation parameter, $\rho/h = 1.2$; that is, the one that induces results more similar to bilinear finite elements.

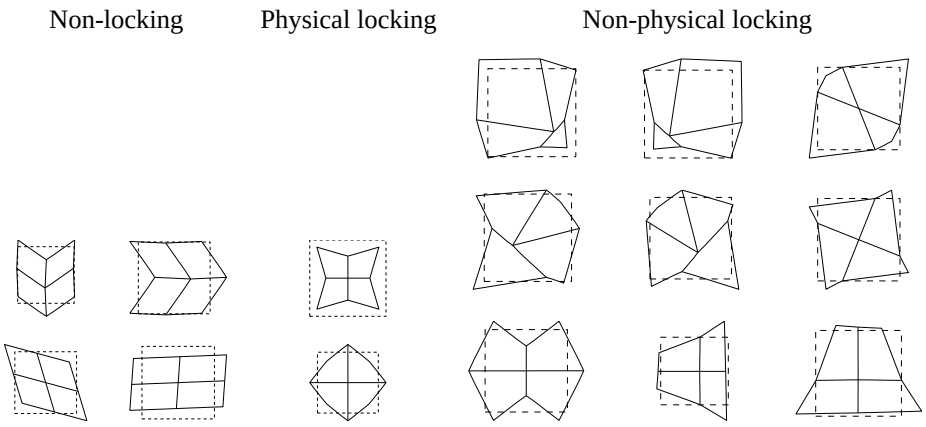

Figure 1. *Modes for a* 3×3 *distribution of particles with bilinear consistency and* $\rho/h = 1.2$.

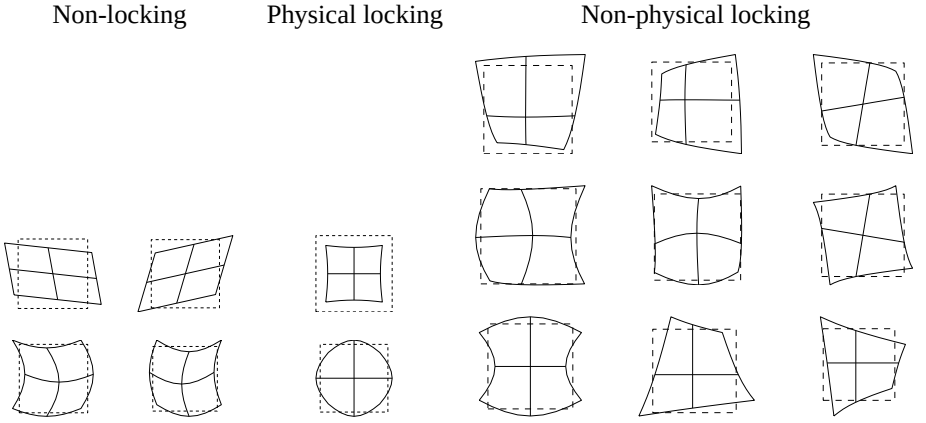

Figure 2. *Modes for a* 3×3 *distribution of particles with bilinear consistency and* $\rho/h = 2.2$.

Four different values of ρ are tested, $\rho = 0.60, 0.24, 0.15$ and 0.05. It is important to note that as ρ decreases the eigenvalue also decreases (and drastically, the scale is logarithmic). Thus, as ρ decreases the influence of locking attenuates.

Moreover, and more importantly, the slope of the curve also decreases as ρ goes to zero (note that for standard EFG the slope remains constant). Thus, in the limit, as expected, the interpolation is divergence-free. Note that for $\rho = 0.05$ and $\nu = 0.5 - 10^{-11}$ the eigenvalue has been reduced in more than three orders of magnitude.

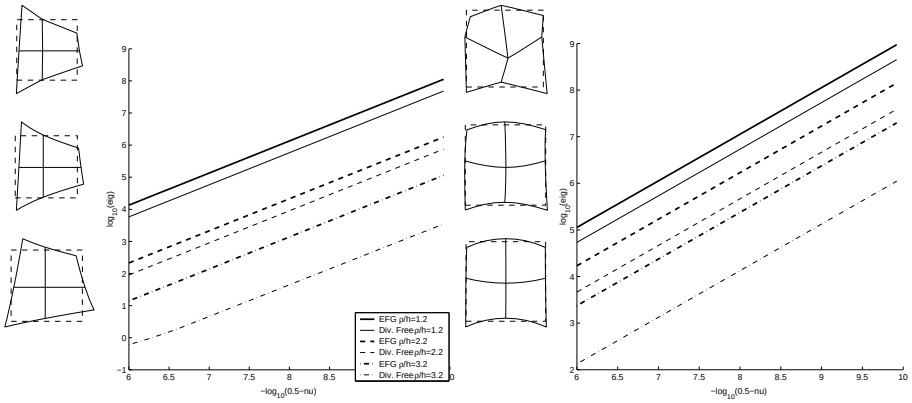

Figure 3. *Comparison between EFG and pseudo-divergence-free interpolations: variation of the eigenvalue as* ν *goes to* 0.5 *for two non-physical locking modes with* $\rho/h = 1.2, 2.2$ *and* 3.2.

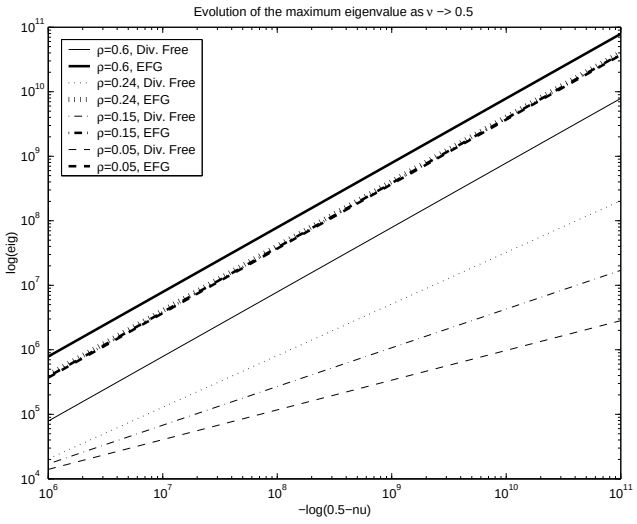

Figure 4. *Comparison between EFG and pseudo-divergence-free interpolations: variation of the maximum eigenvalue as* ν *goes to* 0.5.

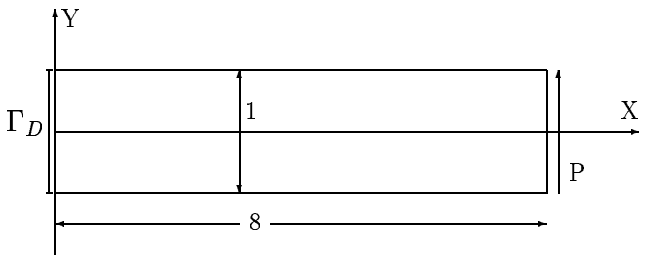

Figure 5. *Cantilever beam problem.*

5. Numerical examples

5.1. *The cantilever beam*

As shown in Figure 5, a beam with linear isotropic material under plane strain conditions and with a parabolic traction applied to the free end is considered. This is a well-known example studied, for instance, by Hughes (2000) and Dolbow and Belytschko (1999). Displacements in both directions are prescribed at Γ_D. The prescribed displacements and the applied traction are such that the solution is known:

$$u_1 = -2\frac{1-\nu^2}{E} y \left[(48 - 3x_1)x_1 + (2 + \frac{\nu}{1-\nu})(x_2^2 - 0.25) \right],$$

$$u_2 = 2\frac{1-\nu^2}{E} \left[3\frac{\nu}{1-\nu} x_2^2 (8 - x_1) + (4 + 5\frac{\nu}{1-\nu})\frac{x_1}{4} + (24 - x_1)x_1^2 \right],$$

$$\sigma_{11} = -12x_2(8 - x_1), \quad \sigma_{22} = 0, \quad \sigma_{12} = 6(0.25 - x_2^2).$$

The problem is solved with uniform distributions of particles. Figure 6 shows the relative L_2 error in displacements for $\nu = 0.3$, $0.5 - 10^{-4}$ and $0.5 - 10^{-6}$. Results are shown for bilinear consistency and $\rho/h = 2.2$. The EFG results are compared with the pseudo-divergence-free interpolation. For EFG the typical convergence rates are obtained when $\nu = 0.3$, but, as expected, results degrade as ν approaches the incompressible limit 0.5. However, the pseudo-divergence free interpolation is able to reproduce the theoretical rate of convergence even for a nearly incompressible case $\nu = 0.5 - 10^{-6}$ and a moderately fine discretization ($h < 0.25$, *i.e.* $\rho < 0.55$).

5.2. *The plate with a hole*

The stress field in an infinite plate with a hole subject to a far-field unit traction in the x direction is (Dolbow and Belytschko, 1999):

$$\sigma_{xx} = 1 - \frac{a^2}{r^2}\left(\frac{3}{2}\cos(2\theta) + \cos(4\theta)\right) + \frac{3a^4}{2r^4}\cos(4\theta)$$

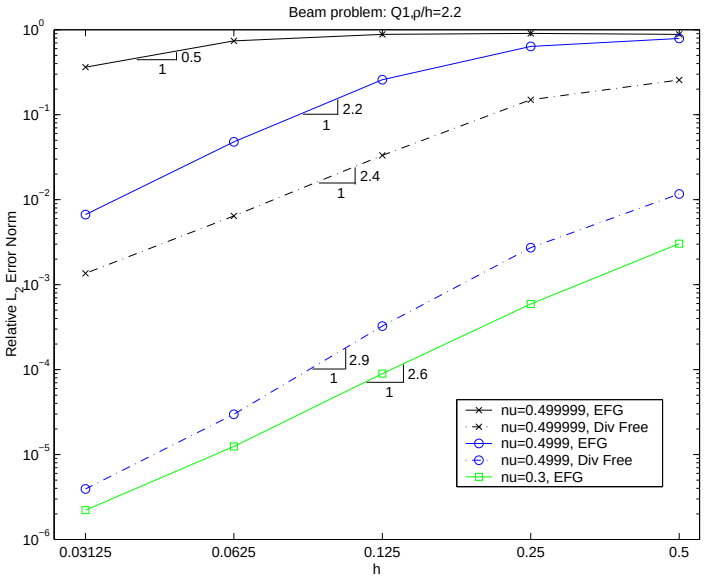

Figure 6. *Cantilever beam with bilinear consistency and* $\rho/h = 2.2$.

$$\sigma_{yy} = -\frac{a^2}{r^2}\left(\frac{1}{2}\cos(2\theta) - \cos(4\theta)\right) - \frac{3a^4}{2r^4}\cos(4\theta)$$

$$\sigma_{xy} = -\frac{a^2}{r^2}\left(\frac{1}{2}\sin(2\theta) + \sin(4\theta)\right) + \frac{3a^4}{2r^4}\sin(4\theta)$$

where $a = 1$ is the hole radius, $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$. The bounded upper quadrant shown in Figure 7 is used to solve the problem. Symmetry conditions are imposed in $x = 0$ and $y = 0$ and the tractions of the exact solution are considered in Γ.

Figure (8) shows the relative L_2 error norm with $\nu = 0.3$, $0.5 - 10^{-4}$ and $0.5 - 10^{-6}$. When $\nu = 0.3$ typical convergence is obtained for EFG but it suffers from locking when the incompressible limit is approached. The improved method maintains the convergence rate even with $\nu = 0.5 - 10^{-6}$. Similar results are obtained using biquadratic consistency. See Figure (9).

6. Stokes problem

It is well known that the study of viscous incompressible flows presents similar difficulties as those found in incompressible solid mechanics. Thus, here, the pseudo-divergence-free method is also used to solve the Stokes problem. Continuous and discrete spaces for Stokes equations are subject to an inf-sup condition (Girault and

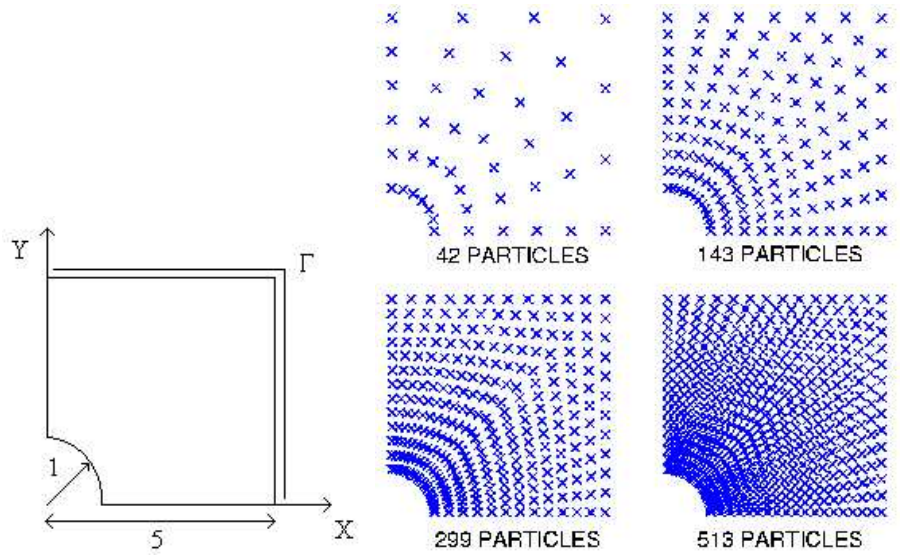

Figure 7. *Problem statement for the plate with hole and discretizations.*

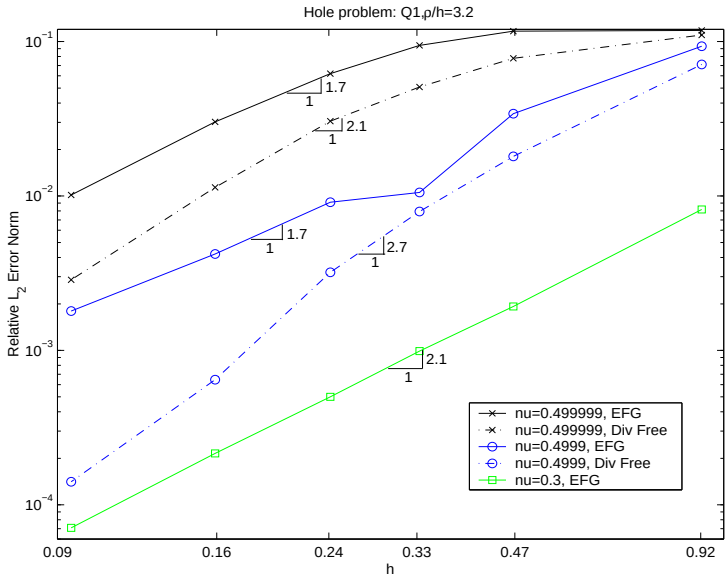

Figure 8. *Hole problem with bilinear consistency and* $\rho/h = 3.2$

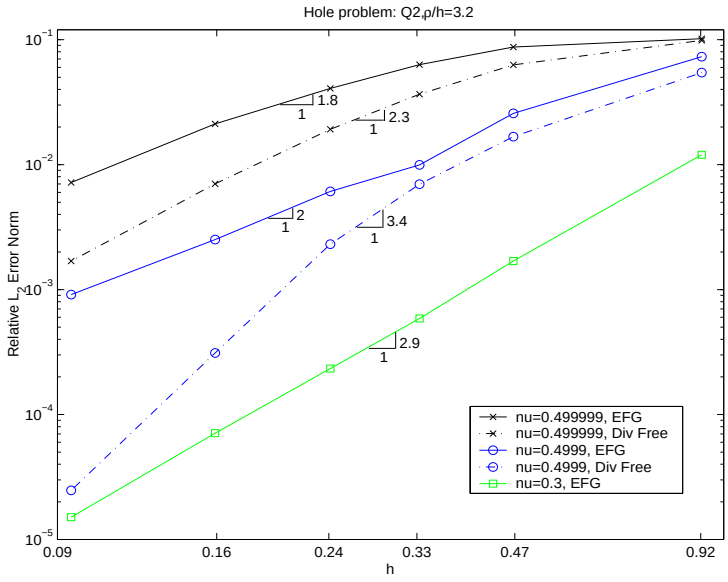

Figure 9. *Hole problem with biquadratic consistency and* $\rho/h = 3.2$

Raviart, 1986). This stability requirement is evidenced in practical computations by the existence of spurious pressure modes. The pseudo-divergence-free velocity field and the pressure field employed here comply the LBB condition asymptotically.

6.1. *Statement of the problem*

Let Ω denote an open bounded region of $\mathbb{R}^2$ with boundary $\partial\Omega$. The 2D Stokes problem in Ω seeks a velocity field $\boldsymbol{u} = (u_1, u_2)$ and a pressure field p such that:

$$\begin{cases} -\nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \partial\Omega, \end{cases} \tag{25}$$

where ν is the viscosity of the fluid and $\boldsymbol{f}$ is the body force (see Donea and Huerta, 2003).

6.2. *Weak form*

Given the problem defined in (25) with $\boldsymbol{u} \in \boldsymbol{\mathcal{V}}$ and $p \in \mathcal{Q}$, where $\boldsymbol{\mathcal{V}} := \left[\mathcal{H}^1(\Omega)\right]^2$ and $\mathcal{Q} := \mathcal{L}_2(\Omega)$, the weak form of the Stokes problem, taking $\boldsymbol{g} = \boldsymbol{0}$, is: find $\boldsymbol{u} \in \boldsymbol{\mathcal{V}}$, $p \in \mathcal{Q}$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + b(\boldsymbol{u}, q) = (\boldsymbol{f}, \boldsymbol{v}) \qquad \forall (\boldsymbol{v}, q) \in \boldsymbol{\mathcal{V}} \times \mathcal{Q},$$

where we define forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ as

$$a(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega \nabla \boldsymbol{v} : \nu \nabla \boldsymbol{u} \, d\Omega = \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}), \quad \text{and}$$

$$b(\boldsymbol{v}, p) := -\int_\Omega p \, \nabla \cdot \boldsymbol{v} \, d\Omega \; = -(p, \nabla \cdot \boldsymbol{v}).$$

Note that $(\cdot, \cdot)$ denotes the standard $\mathcal{L}_2(\Omega)$-scalar product.

We now turn to the consideration of an approximate discrete solution of the problem. Let $\boldsymbol{\mathcal{V}}_\rho$ and $\mathcal{Q}_\rho$ denote finite dimensional subspaces of $\boldsymbol{\mathcal{V}}$ and $\mathcal{Q}$ respectively. The index ρ refers to a characteristic measure of the support of the interpolation functions it is related to the characteristic measure between particles, h. The discrete version of the problem, which in this case uses Nitsche's method (see Arnold, Brezzi, Cockburn and Marini, 2001/02; Babuska, Banerjee and Osborn, 2002; Becker, 2002; Stenberg, 1995) to impose boundary conditions, reads: find $\boldsymbol{u}^\rho \in \boldsymbol{\mathcal{V}}_\rho$, $p^\rho \in \mathcal{Q}_\rho$ such that, $\forall (\boldsymbol{v}^\rho, q^\rho) \in \boldsymbol{\mathcal{V}}_\rho \times \mathcal{Q}_\rho$,

$$\begin{aligned} &a(\boldsymbol{u}^\rho, \boldsymbol{v}^\rho) + b(\boldsymbol{v}^\rho, p^\rho) + b(\boldsymbol{u}^\rho, q^\rho) \\ &\quad - (\nu \partial_{\boldsymbol{n}} \boldsymbol{u}^\rho - p^\rho \boldsymbol{n}, \boldsymbol{v}^\rho)_{\partial\Omega} - (\boldsymbol{u}^\rho, \nu \partial_{\boldsymbol{n}} \boldsymbol{v}^\rho - q^\rho \boldsymbol{n})_{\partial\Omega} + \nu \frac{\gamma}{\rho} (\boldsymbol{u}^\rho, \boldsymbol{v}^\rho)_{\partial\Omega} \\ &\qquad = (\boldsymbol{f}, \boldsymbol{v}^\rho) - (\boldsymbol{g}, \nu \partial_{\boldsymbol{n}} \boldsymbol{v}^\rho - q^\rho \boldsymbol{n})_{\partial\Omega} + \nu \frac{\gamma}{\rho} (\boldsymbol{g}, \boldsymbol{v}^\rho)_{\partial\Omega}. \end{aligned}$$

Now, $(\cdot, \cdot)_{\partial\Omega}$ denotes the $\mathcal{L}_2(\partial\Omega)$-scalar product. Finally, the scalar γ is an arbitrary positive parameter that has to be chosen big enough in order to guarantee stability. Here an eigenvalue problem is solved as proposed by Griebel and Schweitzer (2002).

6.3. *Analytical test*

We consider a test problem with an analytical polynomial solution on the unit square, see Oden and Jacquotte (1984). Homogeneous Dirichlet boundary conditions are imposed on the whole boundary, and the theoretical rates of convergence, recall equation (8), shall be recovered numerically.

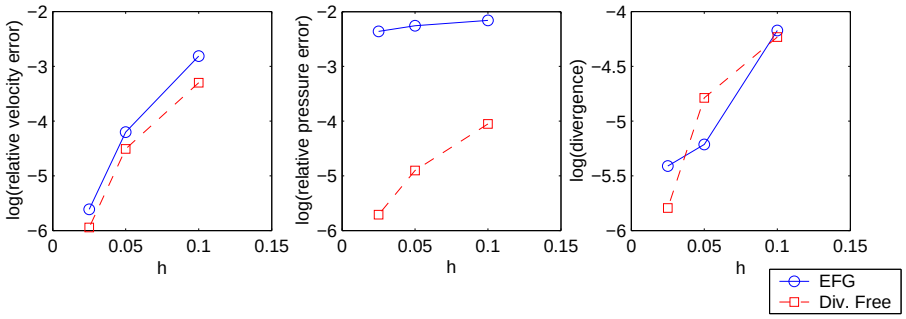

Figure 10. *Convergence results for velocities(left), pressures(middle) and divergence(right).*

We consider the Stokes problem presented in (25) with $\Omega =]0,1[\times]0,1[$ and $\boldsymbol{g} = \mathbf{0}$ on $\partial\Omega$, a polynomial force $\boldsymbol{f}$ is imposed in order to ensure the following solution of the problem:

$$\begin{cases} u_1(x,y) = x^2\,(1-x)^2\,(2\,y - 6\,y^2 + 4\,y^3) \\ u_2(x,y) = (-2\,x + 6\,x^2 - 4\,x^3)\,y^2\,(1-y)^2 \\ p(x,y) = x\,(1-x) \end{cases}$$

We solve this problem with the pseudo-divergence-free MLS method and using $\rho/h = 1.2$ with a bilinear base to approximate both velocity and pressure.

The convergence results are shown in Figure 10. The velocity convergence rates for standard EFG and for the pseudo-divergence-free method are, as expected, similar. However, convergence in pressure is far from optimal in EFG, whereas it presents the theoretical slope in the proposed method. Recall that Eq (8) indicates that diffuse derivatives converge to the actual derivatives as $\rho \to 0$ ($\rho/h = $ cst). Since we use a bilinear base (*i.e.* $m = 1$) the convergence behaves as ρ^1. This means that if we double the number of particles (*i.e.*, if we divide ρ by 2) then the divergence must be at least divided by two. Figure 10 shows exactly this behavior.

6.4. *Driven cavity flow problem (leaky)*

Now we consider Stokes problem, equations (25), with $\Omega =]0,1[\times]0,1[$, $\boldsymbol{f} = (0,0)^\intercal$, $\boldsymbol{g} = (0,0)^\intercal$ on $\partial\Omega \setminus \{y = 1\}$ and $\boldsymbol{g} = (1,0)^\intercal$ on $\partial\Omega \cap \{y = 1\}$. We solve this well-known benchmark problem with the pseudo-divergence-free method using $\rho/h = 2.1$ and a biquadratic base to approximate velocity and pressure. Streamlines, pressure distribution and divergence of $\boldsymbol{u}$ are depicted in Figure 11. Reasonable results are obtained in spite of the equal order interpolation for velocity and pressure. No spurious pressure modes are observed.

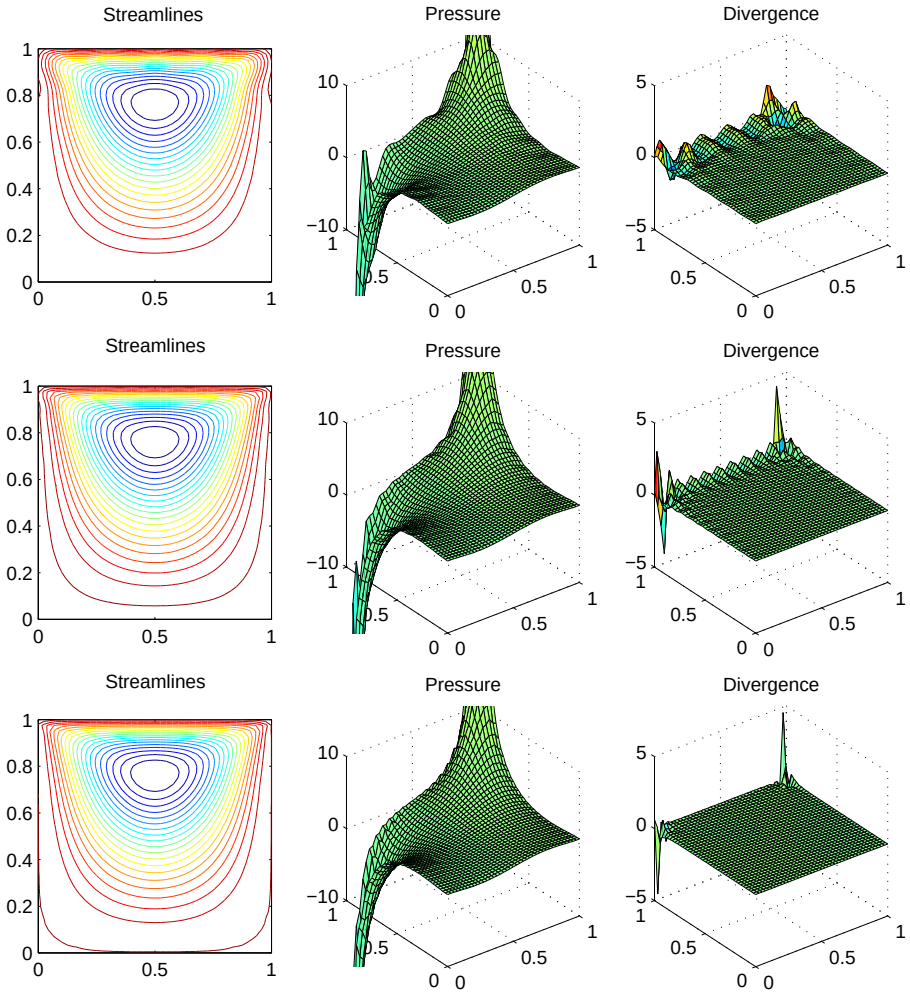

Figure 11. *Pseudo-divergence-free solution for a uniform distribution of 11x11(top), 21x21(middle) and 41x41(bottom) particles.*

7. Conclusions

A novel improved formulation of the Element Free Galerkin method is proposed in order to alleviate volumetric locking. It is based on a pseudo-divergence-free interpolation.Using the concept of diffuse derivatives an a convergence theorem of these derivatives to the ones of the exact solution, the new interpolation proposed is obtained imposing a zero diffuse divergence. In this way is guaranteed that the method verifies asymptotically the incompressibility condition and in addition the imposition can be done *a priori*. This means that the main difference between standard EFG and the im-

proved method is how is chosen the interpolation basis. Modal analysis and numerical results for two classical benchmark tests in solids corroborate that, as expected, diffuse derivatives converge to the derivatives of the exact solution when the discretization is refined (for a fixed dilation parameter) and, of course, that diffuse divergence converges to the exact divergence with the expected theoretical rate. For standard EFG the typical convergence rate is degrade as the incompressible limit is approached but with the improved method good results are obtained even for a nearly incompressible case and a moderately fine discretization. The improved method has also been used to solve the Stokes equations. In this case the LBB condition is not explicitly satisfied because the pseudo-divergence-free interpolation is employed. Reasonable results are obtained in spite of the equal order interpolation for velocity and pressure.

Acknowledgements

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8. References

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