A Special Finite Element for Static and Dynamic Study of Mechanical Systems under Large Motion, Part 1

André Barraco* — Mircea Gh. Munteanu**

** LM2S, CNRS UPRES A 8007*
ENSAM, 151 Bd. De L'Hôpital
F-75013, Paris
barraco@paris.ensam.fr

*** Université Transilvania de Brasov, Roumanie*
29, B-dul Eroilor, 2200 Brasov, Roumanie
mmm@deltanet.ro

ABSTRACT. *In this work the 3D dynamics of mechanical systems, structures and mechanisms, are studied. These systems are divided into so-called macro-elements. Because of large rotations involved by the motion of each macro-element the Euler-Rodrigues parameters are used to describe the global motion of the system. The paper presents in details the study of one macro-element behaviour using a special finite element type for beam system. The stiffness and mass matrices are found starting from the variational formulation of the movement equations expressed only in Euler-Rodrigues parameters. The most important aspect of the proposed approach is that the exact equations, written for the deformed configuration, are solved. Therefore an extremely accurate and very fast convergent method results. This method is non-incremental which means that in static analysis the accuracy does not depend on the number or of the load steps, in many cases only one load step is sufficient.*

RÉSUMÉ. *Dans ce travail le comportement dynamique des systèmes mécaniques, structures et mécanismes est étudié. Ces systèmes sont décomposés en macro-éléments. Les paramètres d'Euler-Rodrigues sont choisis pour décrire les grandes rotations de chaquel élément du maco-élément. Ce papier présente en détail l'étude du comportement d'un macro-élément utilisant un élément fini spécial pour les poutres. Les matrices de rigidité et de masse sont établies à partir d'une formulation variationnelle des équations du mouvement exprimées en utilisant les paramètres d'Euler-Rodrigues. Le point le plus important à souligner dans cette approche est que nous résolvons les équations exactes, écrites dans la configuration déformée actuelle. Ainsi la méthode proposée converge très rapidement vers la solution exacte. Cette méthode est non incrémentale, en statique, en particulier, la précision ne dépend pas du nombre d'incréments de la charge, dans beaucoup de cas un seul pas suffit.*

KEYWORDS: *finite element method, large motion, mechanical systems.*

MOTS-CLÉS : *méthode des éléments finis, grand mouvement, systèmes mécaniques.*

1. Introduction

The elastodynamic problem is generally solved by using the finite element method for space decomposition, an updated Lagrangian technique to write the equilibrium equations and a numerical finite difference scheme for time integration. Real applications, even if the number of degrees of freedom is not very large, would require such a long computational time that the computation might become impossible especially for non-linear problems. On the other hand, very few comprehensive examples and no exact reference solutions are available in the literature. In this paper we address these two issues: we present a way to solve very accurately, in a acceptable computer time, a large scale of non-linear elastodynamic problems and also to provide accurate enough solution, that we call "quasi-exact solution", susceptible to be considered reference solutions. In principle we perform that by solving the exact differential equations written for the actual, deformed, configuration of the system. Obviously the equations are exact in the limits of the accepted hypotheses used to find out the constitutive equations. These equations are written in the global axes system for all bodies belonging to the studied mechanical system. The non-linearity that we consider in this work is geometrical one, but the displacement could be so large that the initial geometry of the system is completely changed. This firstly means that the rotations are very large and they could not be considered as vectors anymore as in small displacement mechanical systems. A material non-linearity could be introduced, too, but this aspect will not be examined in this work.

The basic idea of the method is that the actual configuration of any mechanical system might be uniquely described only by the rotations of some points called nodes, of course with the approximation of a rigid body motion. Because in 3D it is very complicated to work with rotations, the authors decided to define the rotation by using the quaternion or Euler-Rodrigues parameters.

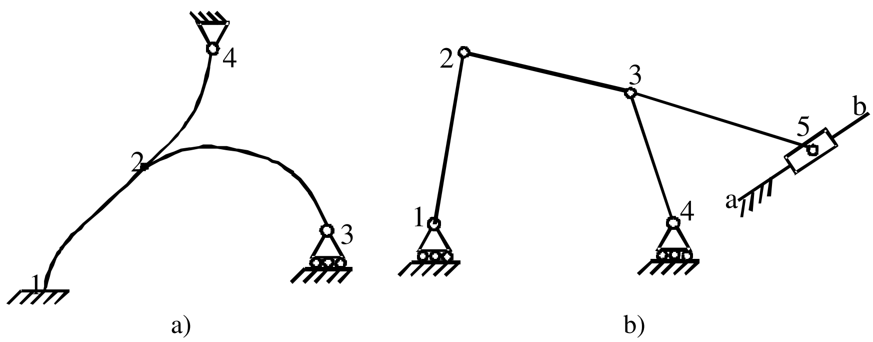


Figure 1. *Two-examples of mechanical systems*

In practice every mechanical system is composed by several bodies. Two examples are given in the Figure 1, a beam structure and a mechanism. In most of the cases it can be considered that the bodies belonging to the system are connected in a finite number of points, see Figure 1. The internal points of connection might be "rigid connection", for instance welded connection as in Figure 1a, and/or might be hinges or other links as in Figure 1b. The method proposed by the authors is based on a finite element method in order to solve any kind of such mechanical systems, in static or dynamic field, in 2D or 3D.

From the point of view of the presented method the mechanical system is decomposed in simple bodies called in this work macro-elements connected in macro-nodes. To simplify the problem we agree to consider that one macro-element is bordered only by two macro-nodes. For instance the example shown in Figure 1a has three macro-elements (1-2, 2-3, 2-4) and four macro-nodes. Each macro-element is divided into finite elements and is studied separately in order to find out its elastic behaviour and mass properties and then the mechanical system is assembled. Two ways may be used for assembling macro-elements.

The first one is based on the so-called closing equations. For instance for the structure shown in the Figure 1a two closing equations sets are needed:

- The three projections of the distance 1-4 are constant and
- The three projections of the distance 1-3 are constant.

Consequently the boundary conditions referring to the translations are imposed. The boundary conditions for rotations (or Euler-Rodrigues parameters in 3D) are imposed in an explicit way. For instance for the clamped macro-nodes we have to impose that the initial direction remains unchanged.

The mechanism represented in Figure 1b is divided in the following macro-elements: 1-2, 2-3, 3-4 and 3-5. The closing equations are (we considered the 2D case to make the explanation simpler):

- The projections of the distance 1-4 are constant and
- The distance 4-5 measured perpendicularly on a-b direction is constant.

The presented method starts from the total potential energy that has to be minimum and the closing equations are introduced by the means of Lagrange multipliers. The multipliers are reactions, for instance the reactions from macro-nodes 3 and 4 for the example shown in Figure 1a.

The second method to assembly the macro-elements into the whole mechanical system consists in considering as unknowns for each macro-element the Euler-Rodrigues parameters of all nodes and the Cartesian coordinates of the macro-nodes. In this case the assembly process of the macro-elements is quite similar to the standard one used for the classical finite element method. The boundary conditions are imposed for both, rotation (clamped ends) and translations (hinges and simple supported ends) using the standard procedure, too. Moreover, the stiffness matrix of the overall mechanical system is sparse, while the first method leads to a full matrix, although the number of unknowns is a little bit larger in the case of this second method. For these reasons we preferred this second method for the assembling process.

The “nature” of each macro-element belongs to one of these three cases:

a) The macro-element is rigid,

b) The macro-element it-self is deformable, but we can apply the small displacement hypothesis, that is we can suppose that its geometry remain unchanged, only the rigid body movements lead to large displacements of the macro-element (this is especially the case of mechanisms),

c) The macro-element is very flexible and therefore it changes its geometry in a very radical manner.

In any above cases each macro-element has large rigid body motion, especially concerning the rotations. In the case a) the shape of the macro-element has no importance, in the dynamic case it is replaced by inertia properties, masses and inertia tensor. In the case b) the macro-element might have any shape, the elastic behaviour and mass properties in a local axis system is obtained using the conventional finite element method and then the stiffness matrix is reduced to few nodes of interest as macro-nodes, points of force application, lumped masses etc.

In this work we will focus on the case c) in which we will consider that each macro-element is a straight or curved very flexible beam, having constant cross-section or not, planar or 3D beam. A Special Finite Element Type (SFET) was elaborated for this purpose.

We divided this paper in two parts. In the first part we will expose the theory of one macro-element as a very elastic 3D curved beam. The general beam theory expressed in Euler-Rodrigues parameters and the corresponding variational formulation will be exposed. In the second part we will present in detail the special finite element used to describe the elastic and inertial properties of one macro-element. Two finite elements will be applied for solving the examples: SFET2 and SFET3, that is two-node and three-node finite element respectively, both of them being curve finite elements. Several static and dynamic examples involving very large elastic and rigid body displacements, that validate this finite element, will be finally presented. The accuracy of the method will be discussed too.

Here two examples are inserted, in order to define more clearly the goal of the paper. Figure 2 shows one 2D example, one straight beam simply supported at its two ends, modelled by two macro-nodes (the force acts in the middle macro-node, the macro-node number 2). The load was applied in 8 steps and the force F had the following values: 0.5, 1.0, 2.0, 3.0, 4.0, 5.0, 10.0, 20.0, respectively.

Figure 3 presents a very highly flexible parallelogram mechanism composed by four macro-elements and 5 macro-nodes. The macro-nodes 1, 2, 4 and 5 are hinges, but the macro-node 3 is a rigid one. In the macro-nodes #5 an elastic rotational element is attached. All the data of the problem are given in the figure. The moment

M_m produces the deformation of the mechanism. If the mechanism is considered rigid the ratio $d\beta/d\alpha$ is constant and equal to 1. But if the mechanism links are very elastic the deformation of the links strongly modifies the kinematics of the mechanism and the ratio $d\beta/d\alpha$ is not longer constant and moreover its value is completely different from 1, see Figure 3b.

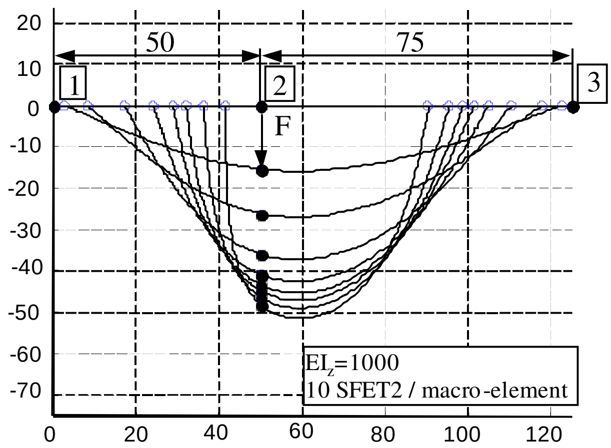
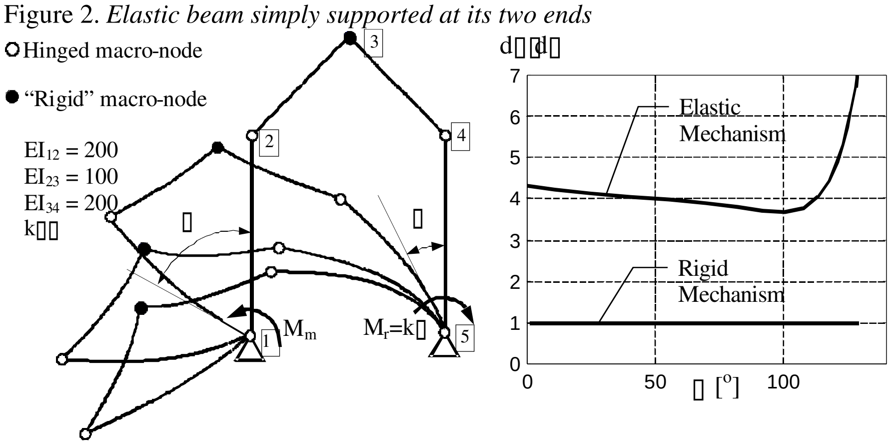


Figure 2. *Elastic beam simply supported at its two ends*



a) b)

Figure 3. *Highly flexible parallelogram mechanism*

2. Beam theory

The exact kinematics of beam theory is now well established. We just recall the first important papers published by Reissner (1973), and later by Hodges (1990) and

Simo (1988). In this paper we use the concept of strain vector and the strain tensor appears as dyadic products of the strain vector and the base vectors. We describe the finite rotation with Euler Rodriguez parameters or quaternion. The beam, in the initial configuration, is composed of the reference line and the reference plane cross sections, which are normal to the reference line. Following the classical assumptions for the bending of the beams, the initial reference line becomes the deformed line and the cross plane sections remain plane and perpendicular to the deformed line (Bernoulli-Euler beam model). A special mention about the torsion must be made: for non-circular cross-section the presented theory remains completely valid if we consider that the warping of the section is freely allowed. For small strain hypothesis, that is adopted here, the bending and torsion are decoupled. The shearing effort effect is neglected, but this introduces negligible errors, as the beam is flexible. For the same reason we can neglect the influence of the axial effort, too, that is the length of the beam remains the same as in the initial configuration. It is possible to add to the model the effect of shearing efforts and axial efforts, but this aspect will be not exposed in this work.

We will study the static and dynamic behaviour of one macro-element, Figure 4, corresponding to a 3D curved beam clamped at point 0, and free at point 1. For static analysis it is simple to add rigid body movement to the macro-element, that is the movement of the point 0, Figure 4. This movement will not affect the strains and stresses in the macro-element.

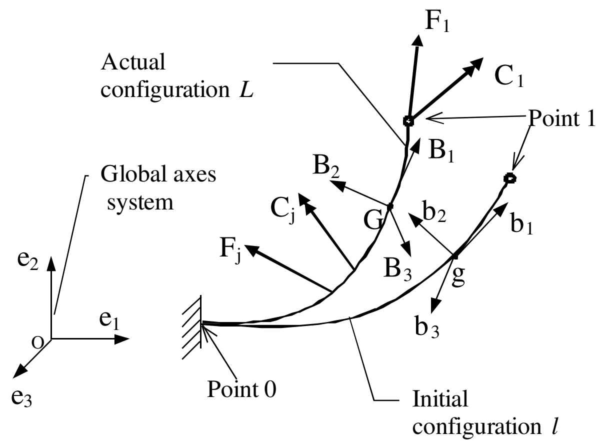


Figure 4. *A macro-element (Euler-Bernoulli beam)*

The position vector of point g, on the reference line, is noted by $\mathbf{r}(s_1)$ and of the corresponding point G, on the actual line, is noted by $\mathbf{R}(S_1,t)$, s_1 and S_1 being the curvilinear coordinate along the reference line for initial configuration and actual configuration respectively and t the time. With the adopted hypothesis it is obvious that $s_1=S_1$. The reference frames attached to each cross section are $R_b(g,\mathbf{b}_i)$, and $R_B(G,\mathbf{B}_i)$, respectively for the initial and the actual positions. We have to point out that the b_2 and b_3, B_2 and B_3 respectively, are central principal axes for the cross section. The kinematics is defined by:

$$\mathrm{R} = \mathrm{r} + \mathrm{U} \qquad [1]$$

$$\mathrm{B}_i = \Lambda_{Bb}\mathrm{b}_i \qquad [2]$$

in which **U** is the displacement vector and Λ_{Bb} the rotation tensor which transforms the initial base into the actual base. To define this rotation we use the quaternion or Euler-Rodrigues parameters. The finite rotation of angle θ around the unit axis **n** is represented by the scalar $l_0 = \cos\frac{\theta}{2}$ and the so-called finite rotation vector $\mathrm{l} = \sin\frac{\theta}{2}\mathrm{n}$. With these parameters the rotation tensor is:

$$\Lambda_{Bb} = (2l_0^2 - 1)\mathrm{I} + 2\mathrm{l} \otimes \mathrm{l} + 2l_0\tilde{\mathrm{l}}$$

where I is the unit tensor, $\otimes$ represents the tensorial product of two vectors and $\tilde{\mathrm{l}}$ the anti-symmetric tensor with the associated matrix components:

$$\tilde{\mathrm{l}} = \begin{bmatrix} 0 & -l_3 & l_2 \\ l_3 & 0 & -l_1 \\ -l_2 & l_1 & 0 \end{bmatrix}$$

We define the matrix corresponding to the quaternion:

$$\hat{l} = (l_0, l_1, l_2, l_3)^{\mathrm{T}} \qquad [3]$$

with l_1, l_2, l_3 the projection of the vector **l** in $R_b(g,\mathbf{b}_i)$ or $R_B(G,\mathbf{B}_i)$. The four Euler-Rodrigues parameters have to verify the relation:

$$\hat{l}^{\mathrm{T}}\hat{l} = 1 \qquad [4]$$

Let consider a point belonging to the current cross-section of the beam, denoted p on the initial configuration and P on the final one, Figure 5. The two position vectors of initial point p and actual point P are given by:

$$\mathrm{r}_p = \mathrm{r} + \mathrm{a} \qquad [5]$$

$$\mathrm{R}_P = \mathrm{r} + \mathrm{U} + \Lambda_{Bb}\mathrm{a} \qquad [6]$$

with $\mathrm{a} = s_\alpha \mathrm{b}_\alpha$, α=2,3 vector which coordinates are constant in $R_b(g,\mathbf{b}_i)$ or $R_B(G,\mathbf{B}_i)$.

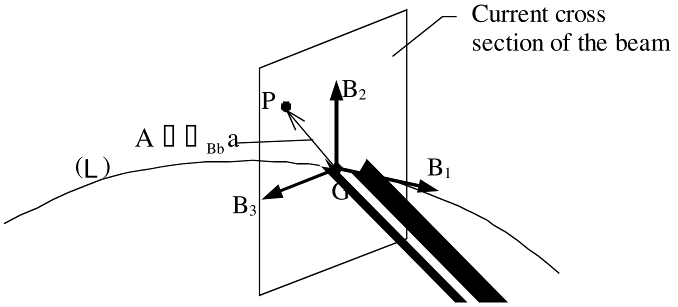


Figure 5. *Current cross section of the beam*

Let us define the strain vector by the following relation:

$$\varepsilon_P = R'_P - \Lambda_{Bb} r'_p \qquad [7]$$

in which the symbol ' denotes the space derivative. This vector is called the spatial or Euler strain vector. Introducing the Relations [5] and [6] in [7] we finally find:

$$\varepsilon_P = \Gamma + \chi \times a \qquad [8]$$

in which Γ is the axial strain vector for the reference line and χ is the bending strain vector defined by:

$$\chi = K - \Lambda_{Bb} K_0$$

that is the difference between the actual curvature K and the initial curvature K_0 pushed forward in the actual reference. But taking into account the assumptions we made, it results that Γ = 0 and therefore:

$$\varepsilon_P = \chi \times a \qquad [8a]$$

The actual curvature vector **K** is related to the finite rotation vector by the following relation:

$$K = 2(l_0 l' - l l_0' + l' \times l)$$

The velocity of a point P of the actual cross section is:

$$V_P = V + \Omega \times A$$

in which **V** is the velocity of point G, Figure 5, of the reference line and Ω is the angular velocity related to the rotations by:

$$\Omega = 2(l_0 \dot{l} - l \dot{l}_0 + \dot{l} \times l)$$

The superscript point denotes the time derivative. Now with these relations the acceleration and the kinetic energy are easy to compute.

We write the movement equations for a slice of beam. If we denote by **R** the stresses resultant and by **M** the resultant moment the equations of motion can be written as (Newton-Euler equations):

$$\rho A \frac{\partial^2 U}{\partial t^2} = \frac{\partial \mathbf{R}}{\partial s_1} + q \qquad [9]$$

$$\frac{\partial}{\partial t}\left(\mathbf{I}\omega\right) = \frac{\partial \mathbf{M}}{\partial s_1} + B_1 \times \mathbf{R} \qquad [10]$$

in which ρA is the mass (A being here the cross section area), **I** the inertia tensor of the slice of beam and $\mathbf{B}_1$ the unit vector belonging to $R_B(G,\mathbf{B}_i)$ tangent to the actual reference line, while **q** is the density of external distributed force. The Equations [9] and [10] form a non-linear system of differential equations allowing to find out the unknowns of the problem, the three displacements **U** and the three rotations. Instead of rotation we will use the four Euler-Rodrigues parameters, that is the equations [9] and [10] have seven unknown functions, and in this case we have to add the Equation [4] to the non-linear system.

We will apply the above theory to solve the macro-element problem represented in the Figure 4, a clamped initial curved beam, loaded with forces and moments. The forces and moments acting on the free end are denoted $\mathbf{F_1}$ and respectively $\mathbf{C_1}$. All the forces $\mathbf{F_j}$ and $\mathbf{C_j}$ are conservative. Once the macro-element problem is solved, based on these results, it will be possible to solve any 3D beam system, statically or dynamically.

One very important remark is that we can define completely the configuration of the beam, the initial or the actual one, only if we know the reference frames attached to each cross section, $R_b(g,\mathbf{b}_i)$; we can write:

$$r(s_1) = r_0 + \int_0^{s_1} b_1 \, ds_1 \qquad [11]$$

where $\mathbf{r}_0$ **is** the position vector of the origin of s_1. If the origin is in the clamped end of the beam $\mathbf{r}_0=\varnothing$, $\varnothing$ being the zero vector. For the actual deformed line we get:

$$R(S_1) = R_0 + \int_0^{s_1} \Lambda_{Bb} b_1 \, ds_1 = R_0 + \int_0^{s_1} B_1 \, ds_1 \qquad [12]$$

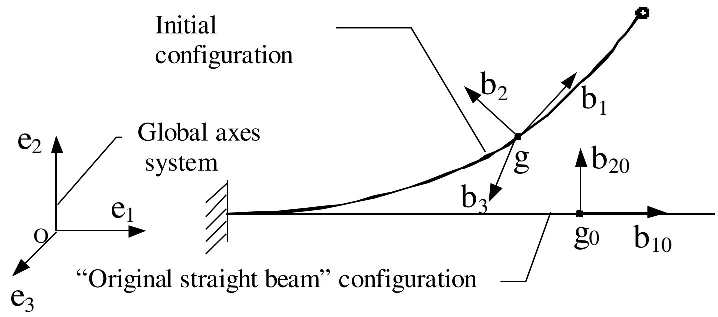


Figure 6. *Initial configuration*

We can consider that the initial configuration of the curved beam is described by the quaternion $\hat{l}_0(s_1)$ that is known and transform an "original straight beam" laying on the e_1 axis, Figure 6, in the initial configuration. The two axes $\mathbf{e}_2$ and $\mathbf{e}_3$ are the principal axes of any section of the original straight beam. Consequently we can write:

$$r(s_1) = r_0 + \int_0^{s_1} b_1 ds_1 = r_0 + \int_0^{s_1} R_{bb_0} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} ds_1 = r_0 + \int_0^{s_1} \begin{Bmatrix} 2(l_{00}^2 + l_{10}^2) - 1 \\ 2(l_{10}l_{20} + l_{00}l_{30}) \\ 2(l_{10}l_{30} - l_{00}l_{20}) \end{Bmatrix} ds_1 \qquad [11a]$$

where in the above relation $l_{00}, l_{10}, l_{20}, l_{30}$ are the components of $\hat{l}_0(s_1)$ and $R_{b0}(g_0, \mathbf{b_{0i}})$ is the reference frame attached to the cross section of the point g_0 of the original straight beam, Figure 6, which we could consider parallel to $R_e(O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. In a similar way we consider that the actual configuration is obtained by transforming the same "original straight beam" by the means of $\hat{l}(S_1)$ and therefore we can write:

$$R(S_1) = R_0 + \int_0^{S_1} B_1 ds_1 = R_0 + \int_0^{S_1} R_{Bb_0} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} dS_1 = R_0 + \int_0^{S_1} \begin{Bmatrix} 2(l_0^2 + l_1^2) - 1 \\ 2(l_1 l_2 + l_0 l_3) \\ 2(l_1 l_3 - l_0 l_2) \end{Bmatrix} dS_1 \qquad [12a]$$

where $\hat{l}(S_1)$ is the quaternion describing the actual configuration, unknown functions. Thus the only unknown of the problem is the quaternion $\hat{l}(s_1)$, Relation [3], that is four unknown functions, instead of seven. Therefore to solve the problem only four equations, the three Equations [10] and the Equation [4], are enough. In the axes system $R_B(G, \mathbf{B}_i)$ the Equation [10] becomes:

$$^{B}IG\frac{\partial^2}{\partial t^2}\hat{l} - D\frac{\partial}{\partial s_1}{}^{B}(\ldots){}^{B}K \ldots C_m{}^{B} \ldots B_1 \ldots {}^{B}R \ldots {}^{B} \ldots I^{B} \ldots \qquad [13]$$

where:

$$ {}^BI = \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix}; \quad G = 2\begin{bmatrix} -l_1 & l_0 & l_3 & -l_2 \\ -l_2 & -l_3 & l_0 & l_1 \\ -l_3 & l_2 & -l_1 & l_0 \end{bmatrix}; \quad D = \begin{bmatrix} GJ & 0 & 0 \\ 0 & EI_2 & 0 \\ 0 & 0 & EI_3 \end{bmatrix} $$

and ${}^BK$ and Ω are the projections of the vectors **K** and Ω. E and G are longitudinal and transversal elasticity moduli, respectively. I_2, I_3 are the principal inertia moments of the current cross section and J is the cross section geometrical property for torsion. We have to point out that in each cross section we have the cohesion force:

$$ \mathbf{R} = \mathbf{R}_F + \mathbf{R}_i + \mathbf{R}_d \qquad [14] $$

where **R** is the stresses resultant in the current cross section situated at curvilinear coordinate S_1, see Equation [9], $\mathbf{R}_F$ the resultant of the forces F_j acting between the current cross section and the free end of the beam, Figure 4, $\mathbf{R}_i$ is the resultant of the inertia forces from the curvilinear coordinate S_1 to the free end of the beam and $\mathbf{R}_d$ is the dumping force that we do not consider in this paper. The inertia force has the expression:

$$ \mathbf{R}_i = \int_{s_1}^{L} \rho A \frac{\partial^2 U}{\partial t^2} ds_1 \qquad [15] $$

computed for the actual configuration of the beam and where:

$$ U(S_1) = R(S_1) - r(s_1) $$

$\mathbf{R}(s_1)$ and $\mathbf{r}(s_1)$ being computed accordingly the Relations [11] and [12]. Obviously:

$$ \frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 R}{\partial t^2} $$

as the $\mathbf{r}(s_1)$ describes the initial configuration of the curved beam which does not depend on time.

Taking again into account that the beam is flexible and therefore the transversal dimensions of the cross section are small comparing to the length of the beam, we could often neglect in the Equation [13] the terms in ${}^BI$, that is the rotational mass inertia, and thus we get a much simpler equation:

$$ D\frac{\partial}{\partial s_1}\left({}^B\Omega - {}^BK\right) + \left[\tilde{C}_m\, {}^B\Omega\right] + \tilde{B}_1\, {}^BR = 0 \qquad [16] $$

In fact, taking into account that the Equations [13] or [16] are moment equations, we replaced the external forces by distributed moments and the translation masses by distributed rotational inertia.

Very important is that the Equations [10], [13] or [16] are exact, obviously in the limits of the adopted hypotheses, and they are written for the actual configuration of the beam.

For the static study of the macro-element, the rigid body movement has no meaning and so the beam behaviour may be described only by the Euler-Rodrigues parameters. But in the dynamic study, as the macro-element could have rigid body movement, the problem has three more unknowns functions, the displacements of the origin of the curvilinear coordinate s_1, $\mathbf{R_0}(t)$ (see Relation [12]). The translations $\mathbf{R_0}(t)$ result during the assembling process of the macro-elements into the mechanical system, structure or mechanism, or they are explicit in some cases, the movement of a free beam system, for instance.

3. Variational formulation

As it is known, the variational formulation is fully equivalent to the constitutive equations. Rayleigh has demonstrated this in 1870. In our case we consider that the movement of the macro-element is described by the Equations [13] to which we add the condition [4]. For instance for the static analysis we could write the first variation of the total potential energy as:

$$\delta\Pi = \int_L \delta\kappa^{B} \cdot D^{B} \cdot \kappa^{B}\, dS_1 - \sum_j \left[\int_0^{s_j} \delta\varphi\, B_1^T\, dS_1 \right] F_j - \sum_j \delta\varphi_j^T C_j = 0 \qquad [17]$$

where L is the length of the macro-element and the two sums are performed for all forces and moments acting on the beam, $\delta\varphi$ is the column matrix of the projections of the vector containing small virtual rotations. The small virtual increment $\delta\varphi$ is similar to the angular velocity ω reported to the global axes system **e**, Figure 4.

The first variation of the total potential energy Π must be zero for any virtual small rotation field $\delta\varphi(S_1)$ which must be geometrically admissible. In the case of the macro-element represented in Figure 4, the virtual rotation field has to fulfil the condition in the clamped end of the beam:

$$\delta\varphi(0) = 0$$

The first term of the Relation [17] represents the variation of the deformation energy, while the last two ones refer to the potential of the external load.

To pass from virtual rotations $\delta\varphi$ to the virtual Euler-Rodrigues parameters $\delta\hat{l}$, we will use the relation:

$$\delta\theta^B = 2\begin{bmatrix} -l_1 & l_0 & l_3 & -l_2 \\ -l_2 & -l_3 & l_0 & l_1 \\ -l_3 & l_2 & -l_1 & l_0 \end{bmatrix}\delta\hat{l} = G\delta\hat{l} \qquad [18]$$

in which $\delta\theta^B$ is written in the current actual reference frame $R_B(G,\mathbf{B}_i)$. To have the virtual rotation vector in the global axes system we apply the rotation matrix Φ_{Bb_0} from global axes system to actual reference frame in the considered current point:

$$\delta\theta = \Phi_{Bb_0} G\delta\hat{l}$$

Also we can write:

$$\delta B_1 = 2\begin{bmatrix} l_0 & l_1 & -l_2 & -l_3 \\ l_3 & l_2 & l_1 & l_0 \\ -l_2 & l_3 & -l_0 & l_1 \end{bmatrix}\delta\hat{l} = G_1\delta\hat{l} \qquad [19]$$

The two matrices, G and G_1, depend on the curvilinear co-ordinate S_1. They are adimensional. Thus the Relation [17] becomes:

$$\delta W = \int_L \delta\varepsilon^{B\,T} D^B \varepsilon \,\mathrm{dS}_1 - \sum_j \int_0^{S_j} \delta\hat{l}^T G_1^{\,T} F_j \mathrm{dS}_1 - \sum_j \delta\hat{l}^T G^T \Phi^T_{j,Bb_0} C_j = 0 \qquad [20]$$

where ε, the strain vector, could be put as below:

$$\varepsilon = {}^B K - {}^b\kappa \qquad [21]$$

where $\mathbf{K}$ contains the projections of the curvature vector of the actual configuration in the reference frame $R_B(G,\mathbf{B}_i)$ and κ contains the projections of the curvature vector of the initial configuration projected into the frame $R_B(g,\mathbf{b}_i)$. The actual curvature matrix is given by:

$${}^B\tilde{K} = G\frac{\mathrm{d}}{\mathrm{ds}}\hat{l}$$

Therefore we can write:

$${}^B K = \begin{Bmatrix} \hat{l}^T A_1 \hat{l}' \\ \hat{l}^T A_2 \hat{l}' \\ \hat{l}^T A_3 \hat{l}' \end{Bmatrix} \quad \text{with} \quad \hat{l}' = \frac{\mathrm{d}}{\mathrm{ds}}\hat{l} \qquad [22]$$

where:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix};\quad A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix};\quad A_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

If the beam is loaded with distributed forces $f(S_1)$ and moments $c(S_1)$ the Equation [20] becomes:

$$\delta\Pi = \int_0^{L} \left[(\varepsilon^{B})^{T} D^{B} \delta\varepsilon - \delta\hat{l}^{T} G^{T} \Phi^{T}_{j,Bb_0} c - \left(\int_0^{S_1} \delta\hat{l}^{T} G_1^{T} d\xi \right) f \right] dS_1 - \sum_j \left(\int_0^{S_j} \delta\hat{l}^{T} G^{T}_{1,j} dS_1 \right) F_j - \sum_j \delta\hat{l}^{T} G_j^{T} \Phi^{T}_{j,Bb_0} C_j = 0 \qquad [23]$$

where the subscript j of $G_{1,j}$ and G_j means that the two matrices are computed in the application point j of the concentrated force or moment, respectively.

The distributed forces might be the inertia or dumping forces. For instance for inertia forces we have:

$$f_i = -\rho A \frac{\partial^2 U}{\partial t^2};\quad c_i = -\rho I \frac{\partial \Omega}{\partial t} \qquad [24]$$

The unknowns of the problem are the four Euler-Rodrigues parameters, four functions, describing the actual configuration at each time $\hat{l}(S_1, t)$. They have to respect the condition $\hat{l}^T \hat{l} = 1$ in any point of the beam, at any time. As the beam is very elastic and therefore transversal dimensions are negligible compared to the length, usually the part of the last terms provided by the second Relation [24] might be eliminated.

It is worth to point out here that the Equation [23] is written in the global axes system $R_e(O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$.

In the approach presented in the paper we try to find a quasi-exact solution of Equations [13] or, what is the same thing, to satisfy the variational principle [23] using the finite element method. An original special curvilinear finite element type (SFET) was elaborated with 4 degrees of freedom per node in 3D, the four component of Euler-Rodrigues quaternion, and only one degree of freedom per node in plane as the vector **n** is known (perpendicular to the plane of the problem). The finite element is a curvilinear one, it has variable cross section area and may have several nodes. The stiffness and mass matrices are found out starting from the variation of the total potential energy written for the actual configuration.

The shape function of the finite element is:

$$l(s) = (1 - \frac{s}{h})l_i + \frac{s}{h}l_{i+1} \quad [25]$$

where l_e represents the column matrix of nodal unknowns having a number of elements equal to four times the number of element nodes. If we substitute the Relations [25] into variational form [23] of the constitutive equations and if we consider only concentrated forces F_j and moments C_j we get for the finite element assembly that replaces the real structure:

$$\delta\Pi = \sum_{n_{el}} \delta\hat{l}_e^T \sum_{i=1}^{3} \left[\int_{L_e} D_i T_i \hat{l}_e \hat{l}_e^T T_i \,dS_1\right] \hat{l}_e - \sum_j \sum_{n_{el,j}} \delta\hat{l}_e^T \left[\int_{L_e} G_1^T dS_1\right] F_j - \sum_j \delta\hat{l}_j^T G_j^T \Lambda^T_{Bb_0,j} C_j = 0 \quad [26]$$

where the matrix T_i has the expression:

$$T_i(S_1) = N(S_1) A_i N'(S_1) - N'(S_1) A_i N(S_1) \quad [27]$$

The Relation [26] provides us a non-linear system in the nodal unknowns $\{l_s\}$, Rodrigues-Euler parameters for the nodes of the finite element assembly. If "n" is the total number of nodes of the finite element assembly, then the number of the nodal unknowns is 4n. The Relation [26] gives 3n non-linear equations, the nodal equilibrium equations in moments. Other n equations are got by applying the Relation [4] for each node.

The numerical method used to solve the differential equations describing the movement of mechanical systems and the special finite element (SFET) are described in details in the second part of the paper. Also several examples (2D and 3D, static and dynamic examples) are presented in order to validate the beam model developed in this paper.

4. Conclusions

In this first part of the paper, we have presented a new and general approach to the dynamics of 2D and 3D mechanical systems which might be either structures or mechanisms. The common point is that in both cases, structures and mechanisms, the displacements are very large, changing completely the initial configuration. As the rotations are large and consequently difficult to handle mainly because they are not vectors, the authors preferred to consider Euler-Rodrigues parameters as unknowns. The exact large movement equations written only in Euler-Rodrigues parameters and the equivalent variational formulation are presented in the paper. Based on this theory, a special finite element type for the dynamics of beam systems might be elaborated. The main advantage of this finite element type, named SFET in the paper, is that it allows without many complications to solve the exact movement equations. It results a non-incremental numerical method: in the static analysis the number or the magnitude of the load steps does not influence the accuracy of the results. The method is very accurate and very rapidly convergent. This might be explained essentially by the following factors: (I) the exact equations are solved, (II) the unknowns are not the linear displacements, but their derivatives, and also (III) the stiffness and mass matrices are polynomials in Euler-Rodrigues parameters, thus avoiding to use trigonometric functions.

The equations describing the static or dynamic behaviour of the beam are written in the global coordinate system, the same for each macro-element in the mechanical system. This approach has several advantages, one of the most important is that a simple expression for inertia forces is obtained.

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