Few remarks on noise control in fluid-structure modelling

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ABSTRACT. *The goal of this paper is to point out few difficulties which occur in fluid-structure modelling. The main point is the bad convergence which appears in the boundary layer between the fluid and the flexible structure when one uses an eigenmode approximation or a finite element with a too coarse mesh. In a first part the mathematical aspects of the coupled model are discussed. Then a control problem which aims at reducing the noise in the coupled system, is considered.*

RSUM. *Le but de cet article est de mettre en évidence quelques difficultés qui peuvent apparaître dans la modélisation fluide-structure. Le point essentiel est la mauvaise convergence qui se développe dans la couche limite entre le fluide et la structure flexible lorsque l'on utilise une approche modale ou même une méthode d'éléments finis avec un maillage trop grossier. Dans une première partie les aspects mathématiques du système couplé, sont discutés. Ensuite le problème du contrôle anti-bruit est posé pour le système couplé.*

KEYWORDS: *fluid-structure, noise control, local surface waves.*

MOTS-CLS : *fluide-structure,contrôle du bruit, ondes de surfaces locales.*

1. Introduction

Fluid-structure models for numerical models have been extensively developed during the last thirty years ([BEL 77], [HOV 78], [JPMRO 84]). The application were mostly for Nuclear technology and Astronautics. There were two different strategies. One is based on a modal representation of each media and the other suggests a global approach by finite element methods.

In the first formulation the eigenvectors of the fluid and the structure are computed separately with independent boundary conditions. When a potential function is used, homogeneous Neumann conditions are mainly used for the boundary of the fluid corresponding to the structure, because they enable to have non vanishing pressure due to the fluid and applied to the structure. It can be interpreted as a rigid wall condition (or homogeneous Dirichlet condition), for the normal displacement. But unfortunately the local effects which occur at the interface between the two media -the fluid and the structure- are eliminated in any finite dimensional approximation. The problem is increased as far as these local effects can be at the origin of the most important phenomena in the interaction. Furthermore in the control of acoustic noise in the fluid from the stucture, these local effects can interact strongly with the control itself and may be at the origin of a transfer of energy from the boundary layer into the fluid. For instance in drag reduction (with a flow), several experiments have proved the efficiency of the phenomenon (see [WSP 88]).

The second formulation is more accurate as far as a very refined mesh is used. Unfortunately this can only be done in elementary problems where the geomery is not too complicate. It is shown for such cases in [EGH 02], that the local effects can be captured and controlled in transcient analysis. But the complexity of real structures suggests to use the first strategy, improved by the theoretical results given here.

2. The model

We start from a priori given equations which are quite familiar for mechanicians. One is for the wave propagation in the fluid and the other is a simple membrane model for the flexible structure. But more advanced models like shells can also be discussed.

2.1. *The acoustic fluid model*

Let Ω be an open set in R^3 with a boundary denoted by $\partial\Omega$. It has a part -say Γ_0- on which the acoustic waves are assumed to be plane, and another one -say Γ_1- occupied by a flexible structure the normal displacement of which is denoted by z and its velocity by $\dfrac{\partial z}{\partial t}$. The velocity of the acoustic waves is represented by a potential function φ. The coordinates of a point in Ω are $x = (x_1, x_2, x_3)$, and on the boundary

Γ_1 they are denoted by $s = (s_1, s_2)$. The derivatives wih respect to s are indicated by the subscript "$._s$". For instance the laplacian on Γ_1 is written Δ_s and the gradient ∇_s. The potential function φ is solution of the following model:

$$\begin{cases} \dfrac{\partial^2 \varphi}{\partial t^2} - c_f^2 \Delta\varphi = 0 \ \ in \ \Omega \times]0, T[\\ \varphi = 0 \ on \ \Gamma_0 \times]0, T[, \\ \dfrac{\partial \varphi}{\partial \nu} = \dfrac{\partial z}{\partial t} \ on \ \Gamma_1 \times]0, T[, \\ \varphi(x, 0) = \varphi_0(x), \ \dfrac{\partial \varphi}{\partial t}(x, 0) = \varphi_1(x) \ in \ \Omega. \end{cases} \qquad [1]$$

Let us point out that the existence and uniqueness of a solution when z is given, are very classical, the reader is refered to J.L. Lions and E. Magenes [JLLEM 68] for details.

2.2. ***The structural model***

Because the structure is assumed to behave like a membrane, the function z is solution of:

$$\begin{cases} \dfrac{\partial^2 z}{\partial t^2} - c_s^2 \Delta_s z = -\dfrac{\varrho_f}{\varrho_s} \dfrac{\partial \varphi}{\partial t} + u \ on \ \Gamma_1 \times]0, T[, \\ z = 0 \ on \ \partial\Gamma_1 \times]0, T[, \\ z(s, 0) = z_0(s), \ \dfrac{\partial z}{\partial t}(s, 0) = z_1(s) \ in \ \Gamma_1 \times]0, T[. \end{cases} \qquad [2]$$

The coefficient $\frac{\varrho_f}{\varrho_s}$ is the ratio between the mass density in the fluid and the surfacic mass density of the membrane. The function u is a control which is applied to the membrane in order to reduce the vibrations in the acoustical open set Ω. The membrane acts the part of a loud speaker excited by a local force u. The support of this force is denoted by Γ_{1c}. Obviously one has: $\Gamma_{1c} \subset \Gamma_1$. In practical applications Γ_{1c} could be a very small part of Γ_1.

Remark 1 *The existence and uniqueness of a solution to the coupled model (1)-(2) can be obtained using a series of eigenmodes. But one can use those of the fluid and of the structure defined separately. Thus the operator is not diagonal in such a basis and the a priori estimates should be obtained globally on the series of the approximate solutions (see [DGE 02]).*

Let us introduce the eigenmodes of the fluid by:

$$\begin{cases} \phi_k \in V = \{\psi \in H^1(\Omega), \ \psi = 0 \ on \ \Gamma_0\}, \ \displaystyle\int_\Omega \phi_k^2(s) ds = 1, \\ -c_f^2 \Delta\phi_k = \lambda_k^f \phi_k \ in \ \Omega, \quad \dfrac{\partial \phi_k}{\partial \nu} = 0 \ on \ \Gamma_1. \end{cases} \qquad [3]$$

Concerning the membrane, the eigenmodes are defined by:

$$\left\{ \begin{array}{l} Z_k \in H_0^1(\Omega), \ \int_{\Gamma_1} Z_k^2(x)dx = 1, \\ -c_s^2 \Delta Z_k = \lambda_k^s Z_k \ on \ \Gamma_1. \end{array} \right. \quad [4]$$

The variational approximation spaces of the coupled model are, for instance, defined by:

$$\left\{ \begin{array}{l} V^N = \{\varphi = \sum_{k=1,N} \alpha_k \phi_k, \ \alpha_k \in R\} \subset V, \\ Z^N = \{z = \sum_{k=1,N} \beta_k Z_k, \ \beta_k \in R\} \subset H_0^1(\Gamma_1). \end{array} \right. \quad [5]$$

Then the approximate model consists in finding $(\varphi^N, z^N) \in V^N \times Z^N$ such that:

$$\left\{ \begin{array}{l} \forall \psi \in V^N, \ \int_\Omega \frac{\partial^2 \varphi^N}{\partial t^2}\psi + c_f^2 \int_\Omega \nabla \varphi^N . \nabla \psi = c_f^2 \int_{\Gamma_1} \frac{\partial z^N}{\partial t}\psi, \\ \forall v \in Z^N, \ \int_{\Gamma_1} \frac{\partial^2 z^N}{\partial t^2} v + c_s^2 \int_{\Gamma_1} \nabla_s z^N . \nabla_s v = -\frac{\varrho_f}{\varrho_s} \int_{\Gamma_1} \frac{\partial \varphi^N}{\partial t} v + \int_{\Gamma_{1c}} uv. \end{array} \right. \quad [6]$$

In order to obtain a unique solution to the previous system, it is necessary to prescribe (for instance), initial conditions. Let us denote by P_φ^N, (respectively P_z^N) the orthogonal projection from $L^2(\Omega)$, (respectively $L^2(\Gamma_1)$), onto V^N, (respectively Z^N); then we set:

$$\left\{ \begin{array}{l} \varphi^N(x,0) = P_\varphi^N \varphi_0(x), \ \frac{\partial \varphi^N}{\partial t}(x,0) = P_\varphi^N \varphi_1(x), \ in \ \Omega, \\ z^N(s,0) = P_z^N z_0(s), \ \frac{\partial z^N}{\partial t}(s,0) = P_z^N z_1(s), \ on \ \Gamma_1. \end{array} \right. \quad [7]$$

The convergence of the series (φ^N, z^N) to the solution of the coupled system, can only be proved in the space:

$$C^0(]0,T[; V \times H_0^1(\Gamma_1)) \cap C^1(]0,T[; L^2(\Omega) \times L^2(\Gamma_1)).$$

Thus no information can be obtained for the normal derivative of φ^N on Γ_1. More precisely from the expression of φ^N one has:

$$\frac{\partial \varphi^N}{\partial \nu} = \sum_{k=1,N} \alpha_k^N(t) \frac{\partial \phi_k}{\partial \nu} = 0 \ on \ \Gamma_1 \times]0,T[.$$

With another respect one could introduce another approximation of the normal derivative of φ by:

$$(\frac{\partial \varphi}{\partial \nu})^N = \frac{\partial z^N}{\partial t} \neq \frac{\partial \varphi^N}{\partial \nu}!$$

which is a consistant one. But it does'nt enable to describe what happens in the fluid near the flexible structure where the variations of $\frac{\partial \varphi}{\partial \nu}$ are very fast. Hence a boundary layer can appears near this boundary Γ_1 and it concerns the normal component of the velocity. But the mechanical behaviour of the fluid near the flexible structure is interesting and can be at the origin of local energetical waves. Their knowledge is important for a good understanding of stationary or transcient analysis. It is admitted now that these local waves can interact with particle instabilities for instance, and can play an important role in micro-vortex shedding from the structure. The analysis of drag reduction or wake knowledge for airfoils, is certainly dependent on the progress which can be done in this modelling. Our goal is to characterize this behaviour for the simplified model (1)-(2), and to give some controllability results of these local waves that we name *"Stoneley waves"*. In fact the Stoneley waves were discovered at the interface between two solids with different waves celerities. Latter on, they were analyzed by I. Cagnard [CAG 62] and Y. C. Fung [FUN 65]. But the mathematical features are the same (almost) that the one we met in fluid-structure interaction. This is why we have adopted this terminology [DGE 02]. It should also be mentioned that up to now, these phenomena were ignored by engineers developing numerical codes in fluid-sructure interaction. Recently L. Dahi has shown in her thesis, a very interesting analysis of these waves but for an infinite media such that there were no reflection and therefore no energetical stationary waves.

3. The local waves at the fluid-structure interface

Let us start with a plane boundary Γ_1. Then we extend the results to curved surfaces.

3.1. *The local waves in cartesian coordinates when* $c_s < c_f$.

Let us consider that Γ_1 is a flat surface as shown on figure 1. The two first coordinates $s = (x_1, x_2)$ describe Γ_1 and x_3 is normal to the boundary. We consider a cylindrical neighbourhood of Γ_1 denoted by B and defined by:

$$B = \{x = (x_1, x_2, x_3),\ s = (x_1, x_2) \in \Gamma_1,\ 0 \leq x_3 \leq H\} \subset \Omega \qquad [8]$$

Let us introduce the eigenmodes of the membrane solution of (4). Then we define a local eigenvalue model by searching a priori stationary solutions such that $(s = (x_1, x_2))$:

$$\begin{cases} (\varphi, z)(x,t) = e^{i\omega t}(A_k(x_3), B_k)Z_k(s), \\ (\lambda_k^s \dfrac{c_f^2}{c_s^2} - \omega^2)A_k - c_f^2 \dfrac{\partial^2 A_k}{\partial x_3^2} = 0, \quad \forall x_3 \in]0, H[, \\ (\omega^2 - \lambda_k^s)B_k = \dfrac{\varrho_f}{\varrho_s} i\omega A_k(0) \\ -\dfrac{\partial A_k}{\partial x_3}(0) = i\omega B_k \ and \ A_k(H) = 0. \end{cases} \qquad [9]$$

A simple computation leads to the following expressions for A_k:

$$A_k(x_3) = B_k i[\frac{\varrho_s c_s}{\varrho_f c_f}\frac{((\frac{c_f}{c_s})^2-1)\lambda_k^s - c_f^2 K_k^2}{\sqrt{\lambda_k^s c_s^2 - K_k^2}}]\frac{sh(K_k(x_3-H))}{sh(K_k H)} \qquad [10]$$

where K_k is solution of the transcendental equation:

$$\frac{\varrho_f H}{\varrho_s}\frac{th(K_k H)}{K_k H} = (\frac{c_s}{c_f})^2[\frac{\lambda_k^s((\frac{c_f}{c_s})^2-1) - c_f^2 K_k^2}{c_s^2 K_k^2 - \lambda_k^s}] \qquad [11]$$

The scalar numbers ω_k are given with respect to k, by:

$$\omega_k = \frac{c_f}{c_s}\sqrt{\lambda_k^s - c_s^2 K_k^2}. \qquad [12]$$

Then the solutions of (11) depend on the fact that K_k is a real or a pure complex number.

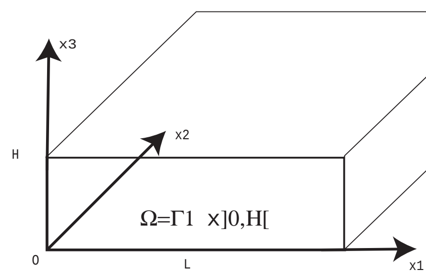

Figure 1. *A particular open set for the analytical computation*

a) Real solutions. If $c_s < c_f$, there is an infinite number of solutions which tend to the infinity with k (because $\lambda_k^s \to \infty$ *when* $k \to \infty$). These solutions are such that:

$$\frac{\sqrt{\lambda_k^s}}{c_s}\sqrt{1-(\frac{c_s}{c_f})^2} \le K_k \le \frac{\sqrt{\lambda_k^s}}{c_s}.$$

Then:

$$\omega_k \simeq \sqrt{\lambda_k^s} \text{ when } k \to \infty.$$

Let us consider one solution for a given k. We set $(x = (s, x_3))$:

$$\begin{cases} \varphi_k^S(x) = \frac{\varrho_s c_s}{\varrho_f c_f}[\frac{((\frac{c_f}{c_s})^2-1)\lambda_k^s - K_k^2}{\sqrt{\lambda_k^s c_f^2 - K_k^2}}]\frac{sh(K_k(x_3-H))}{sh(K_k H)} Z_k(s) \\ z_k^S(s) = Z_k(s). \end{cases} \qquad [13]$$

The exponential decay of φ_k^S in the direction x_3 is increasing with k. As a matter of fact, the series: $\sum_{k=1,\infty} B_k^2$ converges in order to have $\dfrac{\partial z}{\partial t} \in L^2(\Gamma_1)$. Thus one has for instance from a direct computation ($||.||_{0,B}$ is the norm in the space $L^2(B)$):

$$||\varphi_k^S||_{0,B} \simeq \frac{C}{\sqrt{K_k}} \quad and \quad ||\frac{\partial \varphi_k^S}{\partial x_3}||_{0,B} \simeq C_1.$$

It is worth noting that when $k \to \infty$, the function φ_k^S tends to a Dirac measure the support of which is the boundary Γ_1, but the kinematical energy of the normal velocity in the fluid (the $L^2(B)$-norm of $\dfrac{\partial \varphi_k^S}{\partial x_3}$), remains finite. This justifies the existence of a so-called boundary layer at the interface between the fluid and the flexible structure.

b) Purely imaginary solution. Let us set: $K_k = iJ_k$. Then the equation (11) has for each k, a countable infinite number of solutions and the corresponding functions for φ are closer and closer to those solution of (3) when $k \to \infty$.

Remark 2 *If $c_s > c_f$, the local solutions disappear. The roots of the transcendental equation (11) are upper bounded and there is no more local waves.*

Remark 3 *The computations could also been done by substituting the membrane by a shell. But the wave celerity is then infinite for the bending phenomenon. Therefore one has no more a local wave in such a configuration, excepted if the shell is not homogeneous. For instance for a periodical distribution of stiffners it can appear local waves solutions which can induce a boundary layer in the fluid similar to Stoneley waves.*

3.2. *The local waves for a curved boundary*

Let us consider now a tubular neighbourhood B of the curved boundary Γ_1 as shown on figure 2. Let us use a curvilinear system of coordinates $(s, \xi) = (s_1, s_2, \xi)$ and we write the local equations on B. Assuming that $\dfrac{H}{R} << 1$, where R is the minima of the absolute value of the radius of curvature of Γ_1, the solutions of the analogous of (9) are very close to the one we obtained in the previous section. The exponential decay is still valid and even the estimates on φ_k. But the functions Z_k are now the eigenmodes of a curved membrane.

3.3. *Local waves for an arbitrarly shaped domain.*

Let us come back to the full open set Ω. Furthermore we assume, in a first step, that the control function $u(s,t)$ is zero. The solution of the coupled fluid-structure model is still denoted by (φ, z). The function $z(s,t)$ can be split into the eigenvector basis $\{Z_k\}$. We set:

$$z(s,t) = \sum_{k=1,\infty} B_k(t) Z_k(s).$$

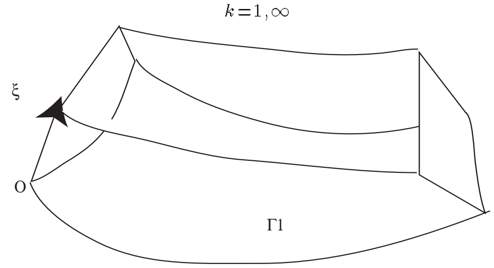

Figure 2. *Curved neighbourhood of the structure*

Then we associate the function defined on B by:

$$\varphi^S(x,t) = \sum_{k=1,\infty} A_k(t) \varphi_k^S(x). \qquad [14]$$

The function φ_k^S are those defined in section (3.1) and (3.2) for a curved boundary Γ_1. It is extended to Ω by zero. Then the coefficients A_k are solution of:

$$\frac{\partial^2 A_k}{\partial t^2} + \omega_k^2 A_k = 0.$$

and therefore:

$$A_k(t) = A_k(0) cos(\omega_k t) + \frac{1}{\omega_k} \frac{\partial A_k}{\partial t}(0) sin(\omega_k t)$$

and from the continuity of the normal velocity along Γ_1:

$$\frac{\partial B_k}{\partial t}(t) = \omega_k A_k(t) = \omega_k A_k(0) cos(\omega_k t) + \frac{\partial A_k}{\partial t}(0) sin(\omega_k t).$$

Finally the couple (φ^S, z^S) is a local Stoneley wave extended to the whole domain Ω. It is characterized by the initial values of $z(s,0)$ on Γ_1, (ie. $B_k(0)$ and $\frac{\partial B_k}{\partial t}(0)$). One can check if necessary that all the equations of the coupled model are satisfied. But the initial conditions for φ^S are restricted. If the control function u is no more zero, then the expressions of $B_k(t)$ are modified in order to take into account this new term. But the mathematical behaviour from the boundary Γ_1 is not changed (exponential decay).

3.4. *Coupling between the Stoneley waves and the cavity waves in* Ω

Let us imagine that at t=0, the initial conditions can be represented by a local Stoneley waves. Because the open set Ω is different from B, there is no orthogonality between interior waves (ie. those corresponding to a rigid structure on Γ_1), and the Stoneley waves. In fact this orthogonality can be proved [DGE 02], if $\Omega = B$. In this section we discuss the growth of the coupling between the local Stoneley waves and the interior waves with respect to time and we discuss the influence of the geometry (ie. the boundary of Ω). Let us set on $B\times]0,T[$ (because the function z^S span the space $H_0^1(\Gamma_1)$):

$$\left\{ \begin{array}{l} (\varphi, z) = (\varphi^S, z^S) + (\delta\varphi, 0), \\ and\ let\ us\ assume\ that\ \delta\varphi(x,0) = \dfrac{\partial \delta\varphi}{\partial t}(x,0) = 0. \end{array} \right. \quad [15]$$

On $\Omega\times]0,T[$ and outside of $B\times]0,T[$, the function φ^S is prolongated by zero. Thus $\delta\varphi$ is solution of:

$$\left\{ \begin{array}{l} \dfrac{\partial^2 \delta\varphi}{\partial t^2} - c_f^2 \Delta\delta\varphi = -\dfrac{\partial^2 \varphi^S}{\partial t^2} + c_f^2 \Delta\varphi^S \ \ in\ \Omega\times]0,T[\\ \delta\varphi = 0\ on\ \Gamma_0\times]0,T[,\ \ \dfrac{\partial \delta\varphi}{\partial \nu} = 0\ on\ \Gamma_1\times]0,T[, \\ \delta\varphi(x,0) = 0,\ \dfrac{\partial\delta\varphi}{\partial t}(x,0) = 0\ in\ \Omega. \end{array} \right. \quad [16]$$

It is possible to obtain an energy estimate on $\delta\varphi$ by multiplying the system (15) by $\dfrac{\partial\delta\varphi}{\partial t}$ and by integrating over the open set Ω. But one can note that the right hand side of the first equation (16) is a Dirac distribution, the support of which is the boundary of B, different from Γ_1. Thus we obtain, formally:

$$\frac{\partial}{\partial t}\{\frac{1}{2}\int_\Omega |\frac{\partial\delta\varphi}{\partial t}|^2 + \frac{c_f^2}{2}\int_\Omega |\nabla\delta\varphi|^2\} = c_f^2 \int_{\partial B-\Gamma_1} \frac{\partial\varphi^S}{\partial\nu}\frac{\partial\delta\varphi}{\partial t} \quad [17]$$

(ν is the outwards unit normal along ∂B). Let us now integrate the preceding relation from 0 to t. We obtain:

$$\left\{ \begin{array}{l} \delta E(t) = \dfrac{1}{2}\{\displaystyle\int_\Omega |\frac{\partial\delta\varphi}{\partial t}|^2 + c_f^2|\nabla\delta\varphi|^2\} - c_f^2\int_{\partial B-\Gamma_1}\frac{\partial\varphi^S}{\partial\nu}\delta\varphi \\ \qquad\quad = -c_f^2\displaystyle\int_0^t\int_{\partial B-\Gamma_1}\frac{\partial^2\varphi^S}{\partial t\partial\nu}\delta\varphi \\ (because\ at\ t=0\ one\ has\ \delta E(0) = 0). \end{array} \right. \quad [18]$$

Let us consider that the open set Ω and the subset B are those drawn on figure 3. Then the integrals over $\partial B - \Gamma_1$ are restricted to the boundary Γ_2 shown on figure 3 ($\delta\varphi = 0$ on the remaining parts of $\partial B - \Gamma_1$). Because the function φ^S and its derivatives (with respect to x_3 and t are exponentially decreasing with respect to x_3, one can deduce from standard estimates that for a given set of smooth enough initial conditions which are purely Stoneley waves as mentioned above, one has:

$$\left\{ \begin{array}{l} E(t) = \dfrac{1}{2}\{\displaystyle\int_\Omega |\frac{\partial\delta\varphi}{\partial t}|^2 + c_f^2|\nabla\delta\varphi|^2\}(t) \le \epsilon(H)t, \\ where\ \epsilon(H) \to 0\ when\ H \to H_{max}\ (see\ figure\ 3). \end{array} \right. \quad [19]$$

For instance for $H = H_{max}$ one can prove easily that: $\epsilon(H_{max}) = 0$, furthermore $\epsilon(H)$ decreases exponentially with H. The inequality (19) proves that for t small enough, the energy of the system remains mainly on the Stoneley waves. It is possible to improve the estimate (19) which is global. In particular one can obtain informations contained in $\delta\varphi$, using a trick introduced by C. Morawetz [MRS 77]. Let us first state the result.

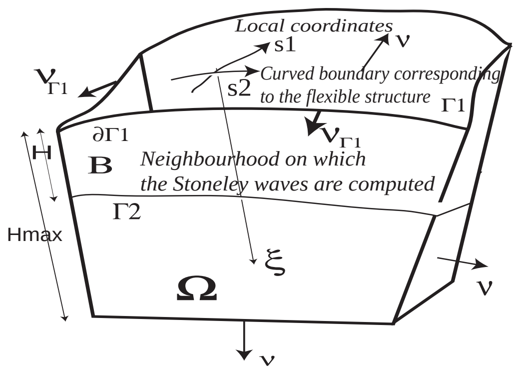

Figure 3. *The open sets* Ω *and* B

Theorem 1 *Let $\delta\varphi$ be the solution of (15) and φ^S the Stoneley wave defined at (14) with smooth enough initial conditions. The open set Ω is assumed t be shaped as on figure 3. Then there exists a constant C_0 which neither depends on the initial conditions of φ^S nor on t, and such that:*

$$| \int_0^t \int_{\Gamma_1} \frac{\partial \delta\varphi}{\partial t}^2 - c_f^2 |\nabla \delta\varphi|^2 | \le C_0 \epsilon(H) t E^S(0) \qquad [20]$$

where $E^S(0)$ is the initial energy of the Stoneley wave defined by:

$$E^S(0) = \frac{1}{2}\{\int_\Omega \frac{\partial \varphi^S}{\partial t}^2 + c_f |\nabla \varphi^S|\} \qquad [21]$$

Remark 4 *The previous result is not obvious and requires a lot of computations that we also mention in the following for a slightly different goal (the inverse inequality for the controllability). It is worth noting that there is a hidden regularity on the boundary terms. The functional spaces in which the existence and uniqueness are proved does'nt enable to make sense to the boundary terms involved in (19). But as it has been underlined by J.L. Lions [JLL 88], the equilibrium equations satisfied by the solution (φ, z) give a sufficient additional information. These results are obtained from a particular choice of the test functions.*

4. The optimal control problem

Let us consider an initial perturbation of the fluid-structure model which corresponds to a pure Stoneley wave. Let us denote by (φ_0, z_0) and (φ_1, z_1) these initial conditions. Then for a given function u which is the control applied to the structure, we define the following criterion for each time delay T and any marginal costs of the control ϵ:

$$\left\{ \begin{array}{l} J^\epsilon(u) = 1/2\{a \int_\Omega (\frac{\partial \varphi}{\partial t})^2(x,T)dx + b\int_\Omega |\nabla\varphi|^2(x,T)dx \\ \qquad +c\int_{\Gamma_1}(\frac{\partial z}{\partial t})^2(s,T)ds + d\int_{\Gamma_1}|\nabla_s z|^2(s,T)ds \\ \qquad +\epsilon\int_0^T\int_{\Gamma_{1c}} u(s,t)^2 dsdt\}. \end{array} \right. \quad [22]$$

The four coefficients a, b, c, d have to be adjusted from engineering considerations. But we show in the following that some choice are more judicious than other in order to obtain a simple version of the gradient of the criterion J^ϵ. The first point concerns the existence of a unique solution to the next optimization problem:

$$\min_{u \in L^2(\Gamma_{1c}\times]0,T[)} J^\epsilon(u) \quad [23]$$

Theorem 2 *For any smooth enough initial condition and $\epsilon > 0,\ T > 0$, the problem (22) has a unique solution.*

In order to define a numerical method for computing the optimal control solution of (22), it is convenient to define the gradient of J^ϵ. It is quite well-known that the easiest way is to introduce the adjoint state. It is obtained from the transpose of the operator. But the final conditions at time T are chosen such that they enable to express simply the gradient of the criterion J^ϵ. Let us first introduce (ψ, d) such that:

$$\left\{ \begin{array}{l} \frac{\partial^2\psi}{\partial t^2} - c_f^2\Delta\psi = 0 \;\; in\ \Omega\times]0,T[\\ \psi = 0 \ on\ \Gamma_0\times]0,T[, \\ \frac{\partial\psi}{\partial\nu} = \frac{\varrho_f}{\varrho_s c_f^2}\frac{\partial d}{\partial t}\ on\ \Gamma_1\times]0,T[, \\ \frac{\partial^2 d}{\partial t^2} - c_s^2\Delta_s d = -c_f^2\frac{\partial\psi}{\partial t}\ on\ \Gamma_1\times]0,T[, \\ d = 0\ on\ \partial\Gamma_1\times]0,T[. \end{array} \right. \quad [24]$$

But the preceding equations should be understood in a distribution sense, on Ω and Γ_1 separately. Furthermore, final conditions should be prescribed on ψ and d. This is the purpose of the following. Let us now define the weak derivative of (φ, z) with respect to the control variable u in the direction v setting:

$$(\varphi^1, z^1) = \lim_{\eta\to 0}\frac{(\varphi,z)(u+\eta v) - (\varphi,z)(u)}{\eta} = (\varphi,z)(v).$$

Then one can check that (φ^1, z^1) is the unique solution of:

$$\begin{cases} \dfrac{\partial^2 \varphi^1}{\partial t^2} - c_f^2 \Delta \varphi^1 = 0 \ \ in \ \Omega \times]0,T[\\ \varphi^1 = 0 \ on \ \Gamma_0 \times]0,T[, \\ \dfrac{\partial \varphi^1}{\partial \nu} = \dfrac{\partial z^1}{\partial t} \ on \ \Gamma_1 \times]0,T[, \\ \varphi^1(x,0) = 0, \ \dfrac{\partial \varphi^1}{\partial t}(x,0) = 0 \ in \ \Omega \\ \dfrac{\partial^2 z^1}{\partial t^2} - c_s^2 \Delta_s z^1 = -\dfrac{\varrho_f}{\varrho_s} \dfrac{\partial \varphi^1}{\partial t} + v \ on \ \Gamma_1 \times]0,T[, \\ z^1(s,t) = 0 \ on \ \partial\Gamma_1 \times]0,T[, \\ z^1(x,0) = 0, \ \dfrac{\partial z^1}{\partial t}(x,0) = 0 \ on \ \Gamma_1. \end{cases} \qquad [25]$$

Then the gradient of the criterion $J^\epsilon(u)$ in the direction v is defined by:

$$\frac{\partial J^\epsilon}{\partial u}(u)(v) = a \int_\Omega \frac{\partial \varphi}{\partial t} \frac{\partial \varphi^1}{\partial t}(x,T)dx + b \int_\Omega \nabla\varphi . \nabla\varphi^1(x,T)dx$$
$$+ c \int_{\Gamma_1} \frac{\partial z}{\partial t} \frac{\partial z^1}{\partial t}(s,T)ds + d \int_{\Gamma_1} \nabla_s z . \nabla_s z^1(s,T))ds + \epsilon \int_0^T \int_{\Gamma_{1c}} uv(s,t)dsdt.$$

The interest of the adjoint state is mainly to give a practical local expression of the gradient of J^ϵ. Let us multiply the system (25) formally by the adjoint state (ψ, d) and let us integrate by parts. If $< .,. >$ denotes the duality between the spaces V and V', and $<< .,. >>$ the duality between $H_0^1(\Gamma_1)$ and $H^{-1}(\Gamma_1)$, one obtains (assuming that what is written has a mathematical meaning):

$$\begin{cases} - < \dfrac{\partial \psi}{\partial t}, \varphi^1 > (T) + \displaystyle\int_\Omega \psi \dfrac{\partial \varphi^1}{\partial t}(x,T)dx \\ + \dfrac{\varrho_f}{\varrho_s} \displaystyle\int_0^T \int_{\Gamma_1} \varphi^1 \dfrac{\partial d}{\partial t}(s,t)(s)dsdt - c_f^2 \int_0^T \int_{\Gamma_1} \dfrac{\partial z^1}{\partial t} \psi(s,t)dsdt = 0, \\ - << \dfrac{\partial d}{\partial t}, z^1 >> (T) + \displaystyle\int_{\Gamma_1} \dfrac{\partial z^1}{\partial t} d(s,T)ds + \dfrac{\varrho_f}{\varrho_s} \int_{\Gamma_1} \varphi^1 d(s,T)ds \\ - \dfrac{\varrho_f}{\varrho_s} \displaystyle\int_0^T \int_{\Gamma_1} \varphi^1 \dfrac{\partial d}{\partial t}(s,t)dsdt - c_f^2 \int_{\Gamma_1} z^1 \psi(s,T)ds \\ + c_f^2 \displaystyle\int_0^T \int_{\Gamma_1} \psi \dfrac{\partial z^1}{\partial t}(s,t)dsdt = \int_0^T \int_{\Gamma_{1c}} dv(s,t)dsdt. \end{cases} \qquad [26]$$

Then the final conditions on the adjoint state variables are chosen such that:

$$\frac{\partial J^\epsilon}{\partial u}(u)(v) = \int_0^T \int_{\Gamma_{1c}} (d + \epsilon u)dsdt$$

and therefore the optimality condition is:

$$(d + \epsilon u)(s,t)v(s,t) = 0, \ \ \forall (s,t) \in \Gamma_{1c} \times]0,T[. \qquad [27]$$

This suggests to define these final conditions by:

$$\left\{\begin{array}{l} \forall \xi \in L^2(\Omega), \;\; \int_\Omega \psi(x,T)\xi(x)dx = a\int_\Omega \dfrac{\partial \varphi}{\partial t}(x,T)\xi(x)dx, \\ \forall \xi \in V, \;\; -< \dfrac{\partial \psi}{\partial t}, \xi > (T) = b\displaystyle\int_\Omega \nabla\varphi(x,T).\nabla\xi(x)dx \\ \qquad\qquad\qquad\qquad\qquad\qquad -\dfrac{\varrho_f}{\varrho_s}\displaystyle\int_{\Gamma_1} d(s,T)\xi(s)ds, \\ \forall v \in L^2(\Gamma_1), \;\; \displaystyle\int_{\Gamma_1} d(s,T)v(s)ds = c\int_{\Gamma_1}\dfrac{\partial z}{\partial t}(s,T)v(s)ds, \\ \forall v \in H_0^1(\Gamma_1), \;\; -<< \dfrac{\partial d}{\partial t}, v >> (T) = d\displaystyle\int_{\Gamma_1}\nabla_s z(s,T).\nabla_s v(s)ds \\ \qquad\qquad\qquad\qquad\qquad\qquad +c_f^2\displaystyle\int_{\Gamma_1}\psi(s,T)v(s)ds. \end{array}\right. \quad [28]$$

From these relations we can deduce the following ones:

$$\left\{\begin{array}{l} \psi(x,T) = a\dfrac{\partial\varphi}{\partial t}(x,T), \\ d(s,T) = c\dfrac{\partial z}{\partial t}(s,T), \\ \dfrac{\partial d}{\partial t}(s,T) = d\Delta_s z(s,T) - c_f^2\psi(s,T). \end{array}\right. \quad [29]$$

The second relation in (27) is more difficult to interpret. One could say that:

$$\left\{\begin{array}{l} \psi(x,T) = b\Delta\varphi(x,T), \;\; in \;\; \Omega, \\ and \\ \dfrac{\partial\psi}{\partial t}(s,T) = \dfrac{\varrho_f}{\varrho_s}d(s,T) - b\dfrac{\partial z}{\partial t}(s,T), \; on \; \Gamma_1 \\ or \; else: \\ \dfrac{\partial\psi}{\partial t}(s,T) = b\Delta\varphi(x,T) + (c\dfrac{\varrho_f}{\varrho_s} - b)\dfrac{\partial z}{\partial t}(s,T)\delta_{\Gamma_1}(x). \end{array}\right.$$

But the mathematical meaning of this relations which are not necessarily compatible, is not obvious at all and must be defined more accurately.

4.1. *The difficulty in characterizing the adjoint state at time* T

The relation (25) which has been obtained for $\dfrac{\partial\psi}{\partial t}(x,T)$ has to be read carrefully. First of all we recall that this term is defined as an element of the dual space V'. Thus the first point is to characterize the space V'. The result is known but we recall it for sake of clarity in the explanations. Then we discuss both the strategy in order to simplify the computation and the importance of the local terms contained in $\dfrac{\partial\psi}{\partial t}(x,T)$ which could be interpreted by a local measure on Γ_1, for the adjoint state.

Theorem 3 *Let V be the functional space defined at (3). Let l be a linear and continuous form on V (ie. an element of V'). Then there exists a unique couple (p, r) in the space $H_0^1(\Omega) \times (H_{00}^{1/2}(\Gamma_1))'$ such that:*

$$\forall v \in V, \quad l(v) = \int_\Omega \nabla p . \nabla v(x) dx + << r, v >>$$

where $<< .,. >>$ is the duality between $(H_{00}^{1/2}(\Gamma_1))'$ and $(H_{00}^{1/2}(\Gamma_1))$). The space $H_{00}^{1/2}(\Gamma_1)$ is the trace of functions of V on the boundary Γ_1. The element p is the unique solution of:

$$\forall v \in H_0^1(\Omega), \quad \int_\Omega \nabla p . \nabla v(x) dx = l(v).$$

Then we characterize r as the remaining term.

The proof is very standard. But it is interesting to define a practical strategy in order to construct r. A convenient way is to use the Steklov eigenvalue problem. Let us set:

$$\begin{cases} find\ (\xi_k, S_k) \in R^{+*} \times V \quad such\ that: \\ -\Delta S_k = 0, \ \ in\ \Omega, \quad \dfrac{\partial S_k}{\partial \nu} = \xi_k S_k, \ \ on\ \ \Gamma_1, \quad \displaystyle\int_{\Gamma_1} S_k^2(s) ds = 1. \end{cases} \qquad [30]$$

The eigenvectors $\{\frac{1}{\sqrt{\xi_k}} S_k\}$ generate a basis in the space $H_{00}^{1/2}(\Gamma_1)$. Let us now introduce the closed subspace of V defined by:

$$V_1 = \{v \in V, \ -\Delta v = 0 \ in\ \Omega\}.$$

We can define a scalar product on this space using a prolongation operator -say P- from $H_{00}^{1/2}(\Gamma_1)$ into V_1, which is equiped with the scalar product:

$$\forall v_1, \ v_2 \ \in V, \quad ((v_1, v_2)) = \int_\Omega \nabla v_1 . \nabla v_2(x) dx.$$

Thus we set:

$$\forall r_1, \ r_2 \ \in H_{00}^{1/2}(\Gamma_1), \ \ ((r_1, r_2)) = ((P r_1, P r_2)).$$

Finally the elements of the dual space $(H_{00}^{1/2}(\Gamma_1))'$ can be associated with elements of V_1 by this scalar product. The vectors $\{S_k\}$ span a Hilbert basis in $L^2(\Gamma_1)$. Hence for any element v in V_1, one has:

$$\forall v \in V_1, \ \ v = \sum_{k=1,\infty} \alpha_k S_k.$$

If l is an element of V' one can write:

$$\forall v \in V_1, \ \ l(v) = \sum_{k=1,\infty} \alpha_k l(S_k) = << r, v >>, \ \ (because \int_\Omega \nabla p . \nabla v(x) dx = 0).$$

Thus r can be identified with the summ of the series in $(H_{00}^{1/2}(\Gamma_1))'$ defined by:

$$r = \sum_{k=1,\infty} l(S_k) S_{k|\Gamma_1}.$$

But we can also associate to r the element of V_1 defined by (Lax-Milgram):

$$R \in V_1,\ \forall v \in V_1,\ \ ((R, v)) =<< r, v >> .$$

or else using the Steklov basis:

$$R = \sum_{k=1,\infty} \frac{l(S_k)}{\xi_k} S_k.$$

Finally we proved the following result:

Theorem 4 *Let l be an element of V'. Then there exists a unique couple $(p, r) \in H_0^1(\Omega) \times (H_{00}^{1/2}(\Gamma_1))'$ such that :*

$$\begin{cases} p \in H_0^1,\ \ \forall v \in H_0^1(\Omega),\ \ \displaystyle\int_\Omega \nabla p \nabla v(x) dx = l(v), \\ r = \sum_{k=1,\infty} l(S_k) S_{k|\Gamma_1}, \end{cases}$$

and one has:

$$\forall v \in V,\ \ l(v) = \int_\Omega \nabla p.\nabla v(x) dx + \sum_{k=1,\infty} \frac{l(S_k)}{\xi_k} \int_\Omega \nabla S_k.\nabla v\ (x) dx$$

or else:

$$\forall v \in V,\ \ l(v) = \int_\Omega \nabla p.\nabla v(x) dx + \sum_{k=1,\infty} l(S_k) \int_{\Gamma_1} S_k v\ (s) ds.$$

Remark 5 *Theorem 4 gives a numerical scheme for computing a representant of l in the space V. We set:*

$$L = p + R = p + \sum_{k=1,\infty} \frac{l(S_k)}{\xi_k} S_k\ \ \in V, \tag{31}$$

and thus:

$$\forall v \in V,\ \ l(v) = \int_\Omega \nabla L.\nabla v(x) dx. \tag{32}$$

From a numerical point of view, the formula (31) can be used as soon as finite element approximations of the eigenmodes S_k and of the function p, have been computed. But obviously one should use a truncation of the series (31).

4.2. *The final condition for the adjoint state variables*

Let us recall that we obtained the following expression for the gradient of the criterion J^ϵ:

$$\frac{\partial J^\epsilon}{\partial u}(u).(v) = \int_0^T \int_{\Gamma_{1c}} (d^\epsilon + \epsilon u^\epsilon)(s,t)v(s,t)dsdt.$$

But the the final condition satisfied by $\psi(x,T)$ is such that ($< .,. >$ denotes the duality between V and V'):

$$\forall v \in V, - < \frac{\partial \psi}{\partial t}(x,T), v >= b\int_\Omega \nabla\varphi(x,T).\nabla v(x)dx - \frac{\varrho_f}{\varrho_s}\int_{\Gamma_1} d(s,T)v(s)ds,$$

or else because of the conditions prescribed on $d(s,T)$, (see (29)):

$$\forall v \in V, - < \frac{\partial \psi}{\partial t}(x,T), v >= b\int_\Omega \nabla\varphi(x,T).\nabla v(x)dx - c\frac{\varrho_f}{\varrho_s}\int_{\Gamma_1} \frac{\partial z}{\partial t}(s,T)v(s)ds.$$

following the Remark 5, we suggest to define an element L in V by:

$$L = p + \sum_{k=1,\infty} \frac{[b\int_\Omega \nabla\varphi(x,T)\nabla S_k(x)dx - c\frac{\varrho_f}{\varrho_s}\int_{\Gamma_1}\frac{\partial z}{\partial t}(s,T)S_k(s)ds]}{\xi_k} S_k,$$

where p is the unique solution of the Poisson problem:

$$p \in H^1_0(\Omega),\ s.t. : \forall v \in H^1_0(\Omega),\ \int_\Omega \nabla p(x).\nabla v(x)dx = b\int_\Omega \nabla\varphi(x,T).\nabla v(x)dx.$$

Then we set:

$$\frac{\partial \psi}{\partial t}(x,T) = (-\Delta L, \frac{\partial L}{\partial \nu})\ \in V'.$$

There is an important case where the expression of $\frac{\partial \psi}{\partial t}(x,T)$ can be drastically simplified. Let us assume that $b = c\frac{\varrho_f}{\varrho_s}$. Then one can check directly that: $\frac{\partial L}{\partial \nu} = 0,\ on\ \Gamma_1$. Therefore, we can set: $\frac{\partial \psi}{\partial t}(x,T) = -\Delta L(x) = c_f^2\Delta\varphi(x,T)$. Obviously this is much more convenient for the numerical implementation. But it is also true that this requires a restriction on the definition of the criterion in the definition of the criterion of the control problem.

5. Exact controllability of the Stoneley waves

The method that we use in this section is an extension of the one inroduced and developed by J.L. Lions [JLL 88]. But the complexity is very much increased because of the interaction between the fluid and the structure. Furthermore the results that we obtain are not always positive. Several restrictions on the geometry are necessary in our formulation. Finally the exact controllabillity is only proved for a restricted subspace of the initial conditions corresponding to the Stoneley waves. Nevertheless this is the most important point because the control of Stoneley local waves is our initial goal. The first step is to define a formal expansion of the optimal control solution defined in section 4. In the second step, we use a multiplier method in oder to derive both Lagrange and Euler energy invariants. Then the last step consists in discussing the exact controllability on the basis of the previous results.

5.1. *The aymptotic expansion*

Let us set:

$$\left\{ \begin{array}{l} (\varphi^\epsilon, z^\epsilon) = (\varphi^0, z^0) + \epsilon(\varphi^1, z^1) + \dots \\ (\psi^\epsilon, d^\epsilon) = (\psi^0, d^0) + \epsilon(\psi^1, d^1) + \dots u^\epsilon = u^0 + \epsilon u^1 + \dots \end{array} \right. \quad [33]$$

Then by introducing these expressions into the equations satisfied by the optimal solution $\varphi^\epsilon, z^\epsilon, \psi^\epsilon, d^\epsilon, u^\epsilon$, one obtains:
1) Order zero

$$\begin{array}{l} \left\{ \begin{array}{l} \dfrac{\partial^2 \psi^0}{\partial t^2} - c_f^2 \Delta \psi^0 = 0 \ \ in\ \Omega \times]0,T[, \\ \dfrac{\partial^2 d^0}{\partial t^2} - c_s^2 \Delta_s d^0 = -c_f^2 \dfrac{\partial \psi^0}{\partial t}\ in\ \Gamma_1 \times]0,T[, \\ \psi^0 = 0\ \ on\ \Gamma_0 \times]0,T[,\ \ c_f^2 \dfrac{\partial \psi^0}{\partial t} = \dfrac{\varrho_f}{\varrho_s} \dfrac{\partial d^0}{\partial t}\ \ on\ \Gamma_1 \times]0,T[, \\ \psi^0(s,t) = 0\ \ \forall (s,t) \in \Gamma_{1c} \times]0,T[\end{array} \right. \\ \left\{ \begin{array}{l} \dfrac{\partial^2 \varphi^0}{\partial t^2} - c_f^2 \Delta \varphi^0 = 0,\ \ in\ \Omega \times]0,T[, \\ \varphi^0 = 0\ on\ \Gamma_0 \times]0,T[,\ \dfrac{\partial \varphi^0}{\partial \nu} = \dfrac{\partial z^0}{\partial t}\ on\ \Gamma_1 \times]0,T[, \\ \\ \dfrac{\partial^2 z^0}{\partial t^2} - c_s^2 \Delta_s z^0 = -\dfrac{\varrho_f}{\varrho_s} \dfrac{\partial \varphi^0}{\partial t} + u^0,\ \ on\ \Gamma_1 \times]0,T[\\ z^0 = 0\ \ on\ \partial\Gamma_1 \times]0,T[, \end{array} \right. \\ \left\{ \begin{array}{l} d^0(s,T) = c \dfrac{\partial z^0}{\partial t}(s,T),\ \ \dfrac{\partial d^0}{\partial t}(s,T) = d\Delta_s z^0(s,T) - c_f^2 \psi^0(s,T), \\ \psi^0(x,T) = \varphi^{0\,\prime}(x,T), \\ \dfrac{\partial \psi^0}{\partial t}(x,T) = \varphi^0(x,T) - \dfrac{\varrho_f}{\varrho_s} d^0(s,T) \chi_{\Gamma_{1c}}(s) \end{array} \right. \end{array} \quad [34]$$

1) Order one (we don't write everything but only what is needed for our purpose)

$$\left\{\begin{array}{l}\dfrac{\partial^2\psi^1}{\partial t^2}-c_f^2\Delta\psi^1=0,\quad in\ \Omega\times]0,T[,\\ c_f^2\dfrac{\partial\psi^1}{\partial\nu}=\dfrac{\varrho_f}{\varrho_s}\dfrac{\partial d^1}{\partial t}\ \ on\ \Gamma_1\times]0,T[,\\ \dfrac{\partial^2 d^1}{\partial t^2}-c_s^2\Delta_s d^1=-c_f^2\dfrac{\partial\psi^1}{\partial t}\ on\ \Gamma_1\times]0,T[\\ d^1=0\ \ on\ \partial\Gamma_1\times]0,T[\\ d^1(s,t)+u^0(s,t)=0\ \ \forall(s,t)\ on\Gamma_{1c}\times]0,T[.\end{array}\right. \quad [35]$$

There are two basic steps in our analysis. One consists in proving that $(\psi^0, d^0) = 0$. In fact this proves the exact controllability and the main point is the homogeneous condition satisfied by d^0 on $\Gamma_{1c}\times]0,T[$. The second step gives a way to construct the optimal control u^0 from (ψ^1, d^1) using (35) with ad'hoc initial conditions. A last point, that we do not develop here is the convergence of $(\psi^\epsilon, z^\epsilon, u^\epsilon)$ to (ψ^0, z^0, u^0) when ϵ tends to zero.

5.2. *A priori estimates on* (ψ^0, d^0)

Let us consider a couple (ψ^0, d^0) solution of (34). The energy is defined by:

$$\left\{\begin{array}{l}E(0)=\dfrac{1}{2}\{\displaystyle\int_\Omega(\dfrac{\partial\psi^0}{\partial t})^2(x,0)dx+c_f^2\int_\Omega(|\nabla\psi|)^2(x,0)dx\}\\ +\dfrac{c_f^2\varrho_s}{2\varrho_f}\{\displaystyle\int_{\Gamma_1}(\dfrac{\partial d^0}{\partial t})^2(s,0)ds+c_s^2\int_{\Gamma_1}(|\nabla_s d^0|)^2(s,0)ds\}\end{array}\right. \quad [36]$$

Let H be the largest distance between a point of Ω and the boundary Γ_1 (see figure 3), and let us assume that there exists a point x_0 in the space R^3 and another one x_1 such that:

$$\left\{\begin{array}{l}\forall\ x\in\Gamma_0,\ \ (x-x_0).\nu\le 0,\ \forall x\in\Gamma_1,\ (x-x_0).\nu\ge H\ge 0,\\ \forall x\in\partial\Gamma_1,\ (x-x_1).\nu_{\Gamma_1}\ge 0\end{array}\right. \quad [37]$$

where ν_{Γ_1} is the unit outwards (and tangential) normal to Γ_1 along $\partial\Gamma_1$. Then one can prove the following result using the multiplier method developed by J.L. Lions [JLL 88].

Theorem 5 *The inverse of the smallest eigenvalue of the Steklov problem (29) is denoted by* $c_0^2 = \dfrac{1}{\xi_0}$. *The vectors* $q = x - x_0$ *and* $q_1 = x - x_1$ *are chosen such that the inequalities (37) are satisfied. Finally we assume that the surface* Γ_1 *is flat for sake of brevety. The diameter of* Ω *is D and the one of* Γ_1 *is E. Let us set:*

$$T_1 = 2max(\frac{E}{c_s}, \frac{\sqrt{D^2+\dfrac{Hc_0^2}{2}}}{c_f}),\ \ T_2 = \frac{2Ec_0}{c_f}max(1, \frac{\varrho_f}{\varrho_s c_f^2 c_s^2})$$

$$\xi = 1 - \frac{c_0^2}{2(\sqrt{H^2+\dfrac{\varrho_s}{\varrho_f}c_0^2}-H)}\ \ T_0 = \frac{T_1+T_2}{\xi}$$

and let us assume that $\xi > 0$ *which can be easily satisfied using an a priori estimate on* c_0^2. *Then one has the following estimate:*

$$\left\{ \begin{array}{ll} (T-T_0)E(0)\xi \leq & \frac{1}{2}\int_0^T \int_{\Gamma_1} [(\frac{\partial \psi^0}{\partial t})^2 - c_f^2(|\nabla_s \psi^0|)^2] q.\nu(s,t) ds dt \\ & \quad + c_f^2 \frac{\varrho_s}{2\varrho_f} \int_0^T \int_{\partial\Gamma_1} (\frac{\partial d^0}{\partial \nu})^2 q_1.\nu_{\partial\Gamma_1} ds dt \\ & \quad + \frac{c_f^2}{2} \int_0^T \int_{\Gamma_0} (\frac{\partial \psi^0}{\partial \nu})^2 q.\nu(s,t) ds dt \\ & \leq c_1 E(0)(T+T_0') \\ & \\ & where\ c_1\ and\ T_0',\ are\ large\ enough\ constants. \end{array} \right.$$

The same result can be obtained for a curved surface Γ_1 *but with a restriction on the curvature.*

Remark 6 *Because* $q.\nu \leq 0$ *on* Γ_0, *the last term of the first inequality can be omitted.*

Remark 7 *If* ψ^0 *is a Stoneley wave, then one has in a "close" neighbourhood of* Γ_1*:*

$$\psi^S(x,t) = \sum_{k=1,\infty} A_k(t)\varphi_k^S(x).$$

with the notation $(x = (s,\xi))$*:*

$$\begin{aligned} \varphi_k^S(x) &= \frac{\varrho_s c_s}{\varrho_f c_f} \frac{((\frac{c_f}{c_s})^2 - 1)\lambda_k^s - K_k^2}{\sqrt{\lambda_k^s c_f^2 - K_k^2}} \frac{sh(K_k(\xi - H))}{sh(K_k H)} Z_k(s) \\ &= D_k \frac{sh(K_k(\xi - H))}{sh(K_k H)} Z_k(s). \end{aligned}$$

Then because of the orthogonality of the eigenvectors Z_k *in the space* $L^2(\Gamma_1)$, *one has (observing that on* Γ_1, $q.\nu = constant\ +\ o(\frac{1}{|R|})$ *where* R *is the minimal radius of curvature of the boundary* Γ_1*):*

$$\begin{aligned} \int_0^T \int_{\Gamma_1} [(\frac{\partial \psi^0}{\partial t})^2 - c_f^2 \quad & (|\nabla_s \psi^0|)^2](s,t) ds dt \\ & = \sum_{k=1}^{\infty} D_k^2 \int_0^T [(\frac{\partial A_k}{\partial t})^2(t) - c_f^2 \lambda_k^s A_k^2(t)] dt. \end{aligned}$$

Then from the exact expression of the coefficient A_k *given at section (3.3), we deduce that:* $(\omega_k^2 \leq c_f^2 \lambda_k^s)$,

$$\int_0^T \int_{\Gamma_1} [(\frac{\partial \psi^0}{\partial t})^2 - c_f^2(|\nabla_s \psi^0|)^2] q.\nu(s,t) ds dt \leq 0$$

From a compilation of the previous results, we deduce the following inequality where the notations have been defined above:

$$(T - T_0)E(0)\xi \leq c_f^2 \frac{\varrho_s}{2\varrho_f} \int_0^T \int_{\partial\Gamma_1} (\frac{\partial d^0}{\partial \nu})^2 q.\nu ds dt \qquad [38]$$

Theorem 6 *Assuming the hypothesis of Theorem 5, and if $\partial\Gamma_1$ is contained into $\bar{\Gamma}_c$, for $T > T_0$, one has:*

$$E(0) = 0.$$

Because of the final conditions satisfied by (ψ^0, d^0), one can prove that

$$\varphi^0(x,T) = 0,\ \frac{\partial \varphi^0}{\partial t}(x,T) = 0,\ z^0(s,T) = 0,\ \frac{\partial z^0}{\partial t}(s,T) = 0.$$

In other words, the optimal control u^0, if we can compute it, is such that the control is exact at time T for smooth enough initial conditions. Then the result can be extended to more general initial data (ie. finite energy), by a density argument.

5.3. *Determination of an exact control*

Let us now consider the system (35) which characterizes (ψ^1, d^1), as soon as the initial conditions are prescribed. Thus we define arbitrary initial conditions on Ω and on Γ_1:

$$\Psi = (\Psi_0, \Psi_1),\ and\ D = (D_0, D_1),\ X = (\psi, d)$$

and we associate the element (ψ^1, d^1) solution of (35) and such that:

$$\psi^1(x,0) = \Psi_0(x),\ \frac{\partial \psi^1}{\partial t}(x,0) = \Psi_1(x),\ d^1(s,0) = D_0(s),\ \frac{\partial d^1}{\partial t}(s,0) = D_1(s).$$

Then multiplying the equations, (φ^0, z^0) is solution of which, and from several integrations by parts, taking into account that at time T, the functions φ^0, z^0 and their time derivatives are zero, one deduces that for any $(\delta\Psi, \delta D)$ (the solution of (35) asssociated to this initial condition is denoted by $(\delta\psi, \delta d)$ and we set: $\delta X = (\delta\Psi, \delta D)$)

$$\forall \delta X \in V^*,\ \ \Lambda(X, \delta X) = L(\delta X) \qquad [39]$$

where the bilinear form $\Lambda(.,.)$ and the linear form $L(.)$ are defined by:

$$\left\{ \begin{array}{l} \Lambda(X,\delta X) = \displaystyle\int_0^T \int_{\Gamma_{1c}} d^1(s,t)\delta d^1(s,t) ds dt, \\ L(\delta X) = \displaystyle\int_\Omega \varphi_1(x)\delta\Psi_0(x) - \int_\Omega \varphi_0(x)\delta\Psi_1(x)dx + \int_{\Gamma_1} z_1(s)\delta D_0(s)ds \\ \qquad - \displaystyle\int_{\Gamma_1} z_0(s)\delta D_0(s)ds + \frac{\varrho_f}{\varrho_s}\int_{\Gamma_1} \varphi_0 \delta D_0(s)ds - c_f^2 \int_{\gamma_1} z_0 \delta\Psi_0(s)ds. \end{array} \right. \qquad [40]$$

The solution X of this variational equation should be in V^* which is the completed space of $V \times L^2(\Omega) \times H_0^1(\Gamma_1) \times L^2(\Gamma_1)$ with respect to the norm; $\sqrt{\Lambda(X,X)}$. It is a norm under suitable assumptions, because of Theorem 6. But the explicit characterization of the space V^* is not easy. In fact, as far as finite dimensional approximations are considered, the question doesn't make sense. This is the case for an eigenmode approximation as we discussed it in the previous sections. But for stability analysis , it is usefull to have an explicit expression for V^*. The linear form $L(.)$ must be in the dual space of V^* denoted $V^{*\prime}$. This enables one, from the expression of the linear form $L(.)$, to characterize the space for the initial conditions which can be exacly controlled. This problem is not fully solved presently. But there is another possibility which is more mathematically satisfying. Let us mention a discussion which is similar to the one given in J.L. Lions [JLL 88]. The goal is to define a strategy (ie. a control law) which enables to control exactly any initial conditions which is a Stoneley wave such that $(z_0, z_1) \in H_0^1(\Gamma_1) \times L^2(\Gamma_1)$. It consists in changing a little bit the optimal control problem we started from. There are two new points. First of all, the control variable u is chosen in the space $H_0^1(]0,T[; L^2(\omega))$. The new cost function J^ϵ is now:

$$\left\{ \begin{aligned} J^\epsilon(u) = \quad & 1/2\{a \int_\Omega (\frac{\partial \varphi}{\partial t})^2(x,T)dx + b\int_\Omega |\nabla\varphi|^2(x,T)dx \\ & +c\int_{\Gamma_1}(\frac{\partial z}{\partial t})^2(s,T)ds + d\int_{\Gamma_1}|\nabla_s z|^2(s,T)ds \\ & +\epsilon\int_0^T\int_{\Gamma_{1c}} u^2(s,t)dsdt + \epsilon^2\int_0^T\int_{\Gamma_{1c}}(\frac{\partial u}{\partial t})^2(s,t)dsdt\} \end{aligned} \right. \quad [41]$$

The control is also modified at the right handside of the state equations. The structural model is changed into the following one:

$$\frac{\partial^2 z}{\partial t^2} - c_s^2\Delta_s z = -\frac{\varrho_f}{\varrho_s}\frac{\partial\varphi}{\partial t} + u - \frac{\partial^2 u}{\partial t^2} \;\; on \; \Gamma_1\times]0,T[, \; z=0 \; on \; \partial\Gamma_1\times]0,T[, \quad [42]$$

Then we can apply the asymptotic method with respect to the small parameter ϵ. The limit control is then defined from the new variational formulation analogous to (39) but with the new expression of the bilinear form $\Lambda(.,.)$:

$$\Lambda(X,\delta X) = \int_0^T\int_{\Gamma_{1c}}\{d^1\delta d^1(s,t) + \frac{\partial d^1}{\partial t}\frac{\partial\delta d^1}{\partial t}(s,t)\}dsdt.$$

The linear form $L(.)$ is unchanged. The new point is that one can prove that there exists a strictly positive constant c_2 small enough, such that for T large enough, one can prove using ad'hoc test functions that if Γ_{1c} is an arbitrary neighbourhood of a part of $\partial\Gamma_1$ and if (ψ^1, d^1) is a Stoneley wave, then for a large enough constant c_5 (see J.L. Lions [JLL 88]):

$$\begin{aligned} (T-T_0)c_2E(0) \quad & \leq \int_0^T\int_{\Gamma_{1c}}\{(\psi^1)^2 + (\frac{\partial\psi^1}{\partial t})^2\}(s,t)dsdt \\ & \leq c_5\int_0^T\int_{\Gamma_{1c}}(\frac{\partial\psi^1}{\partial t})^2(s,t)dsdt, \end{aligned}$$

where $E(0)$ is the energy of the initial Stoneley wave defined at (36). Thus once (39) is solved with this new expression of Λ, we set formally:

$$u^{00} = -\{d^1(s,t) - \frac{\partial^2 d^1}{\partial t^2} + \frac{\partial d^1}{\partial t}(s,T)\delta_T(t) - \frac{\partial d^1}{\partial t}(s,0)\delta_0(t)\}. \qquad [43]$$

It is worth noting that the control law contains impulses and the structural model is now:

$$\frac{\partial^2 z}{\partial t^2} - c_s^2 \Delta_s z = -\frac{\varrho_f}{\varrho_s}\frac{\partial \varphi}{\partial t} + u^{00} \;\; on \; \Gamma_1 \times]0,T[, \; z = 0 \; on \; \partial\Gamma_1 \times]0,T[. \qquad [44]$$

Here again the boundary layer which appear on the control at time t=T (and t=0), can be modelled using smooth functions compared to the Dirac distributions (because for $\epsilon \neq 0$, one should have: $u^\epsilon(s,0) = u^\epsilon(s,T) = 0$). But the best strategy is certainly to keep the optimal control problem (ie. $\epsilon \neq 0$), for which the control u^ϵ is more regular ($C^0([0,T];L^2(\Gamma_{1c}))$).

Remark 8 *It can be proved from quite classical methods in singular perturbation analysis that when $\epsilon \to 0$ the sequence $(\varphi^\epsilon, z^\epsilon, u^\epsilon)$ converges to the term (φ^0, z^0, u^0) in an ad'hoc space for the first model (in which V^* is not identified but the control law u^ϵ is stable in $L^2(]0,T[\times\Gamma_{1c})$ ie. $u^\epsilon \to u^0$ in this space), and to (φ^0, z^0, u^{00}) in the second case where V^* is known but the open set Γ_{1c} should be a neighbourhood of $\partial\Gamma_1$, and the control is an element of a distribution space with impulses which are not very satisfying from the mechanical point of view. Therefore it can also be interesting to construct a more regular control law (ie. $\in L^2(]0,T[\times\Gamma_{1c})$) using the method described in J.L. Lions [JLL 88], p.420.*

6. Conclusion

Our analysis has been restricted to a very particular fluid-structure modelling for which the acoustic component is dominant. Our goal was to point out a basic difficulty which occurs when one tries to take into account the local effect of the coupling at the interface between the fluid and the flexible structure. Local waves, (which can be stationnary waves), similar to those described by Stoneley [STO 24] and analyzed by Cagnard [CAG 62], can be the predominant mechanical phenomenon when the wave celerity of the structure is smaller than the one in the fluid. This is an important case in aerodynamical applications. Therefore we have suggested a numerical method in order to improve the classical strategies based on eigenmode approximations and which are not very efficient for this problem. Then an optimum control has been introduced in order to control (exactly) the local behaviour of the coupled model. Obviously geometrical and mechanical restrictions have been necessary. But once again the local waves can be at the origin of a very *"sharp"* phenomenon (boundary layers) near the fluid-structure interface. In fact the exact control (HUM method of J.L. Lions [JLL 88]), that we have suggested is exact but the initial conditions which are exactly controlled can only be characterized if the support is a neighbourhood of the lateral boundary of the structure. A large number of improvements are still necessary. One of them is the coupling between active and passive controls. Another one is the modelling of the actuators used. In this case one idea is to use piezo-devices stacked in several layers and with different voltages and sticked on a plate-like structure in order to have a bending phenomenon for which the control delay is as small as we wish because of the infinite wave celerity in such a structure (see [ZUA 87]).

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