Hybrid frictional contact particles-in elements

Hachmi Ben Dhia — Malek Zarroug

École Centrale Paris - MSSMat
92295 Châtenay-Malabry
{bendhia,zarroug}@mssmat.ecp.fr

ABSTRACT. *By analyzing functional, geometrical and numerical integration aspects, hybrid frictional contact particles-in-elements are derived from the continuous hybrid formulation of 3D large transformation frictional contact problems given in [BEN 99] [BEN 00a]. The suggested approach is illustrated by some academic and industrial contact tests.*

RÉSUMÉ. *Dans ce travail, on propose des éléments hybrides pour les problèmes de contact entre solides 2D et 3D en grandes transformations. Permettant, entre autres, d'éviter les oscillations parasites des pressions de contact, ces éléments sont dérivés à partir de discrétisations "élémentéro-particulaires" d'une formulation hybride continue [BEN 99], [BEN 00a]. Des résultats numériques, pour des cas académiques et d'autres industriels éclairent notre démarche.*

KEYWORDS: *contact, friction, hybrid, contact elements.*

MOTS-CLÉS : *contact, frottement, hybride, éléments contact.*

1. Introduction

During the last years, significant improvements have been realized for the treatment of frictional contact between two deformable bodies undergoing large transformations. Let us mention in particular the attempt for deriving discrete F.E.M. formulations from continuous ones in a Lagrangian framework ([LAU 93], [KLA 95], [PIE 99], [BEN 99], [BEN 00a]). However, some theoretical and numerical aspects are still to be investigated more deeply. In particular, the fundamental *contact element* concept is a question which has not yet been completely clarified. This paper aims to adress this issue.

Beginning with a continuous hybrid formulation developed in [BEN 99, BEN 00a], we try in this paper to enlighten the notion of *contact element* by carrying out some numerical approximations and by using some mathematical elements. An outline of the paper is the following : the continuous mixed formulation is recalled in section 2. To derive discrete mixed contact problems from the continuous one, the target application, the functional spaces choice (respecting the *Inf-Sup* condition), the numerical integration scheme (leading to finite particules-in-elements methods [BEN 00b]) and the treatment of surface irregularities by an interpolation technique are discussed in the section 3. The numerical aspects discussed in section 3 are illustrated by numerical tests given in the final section.

2. Mixed continuous formulation

Let $\mathcal{B}^i$, for $i = 1, 2$, be two deformable bodies occupying $\overline{\Omega}^i$ in their initial configurations and $\overline{\Omega}_t^i$ in their current ones ; Ω^i and Ω_t^i being two domains of $\mathbb{R}^3$. We assume that these two bodies can come into contact at any time t during their movement. Each boundary $\partial\Omega^i$ and $\partial\Omega_t^i$ is assumed to be as regular as necessary. We denote by Γ_c^i and γ_c^i the potential contact surfaces of $\mathcal{B}^i$ in their initial and current configurations, respectively (whith no explicit reference to time to simplify the notations).

By using *Signorini* and *Coulomb* laws of unilateral contact and friction, the *Virtual Work* and the generalized *Action* and *Reaction Principles*, by neglecting inertia terms and using an upward Euler scheme to approximate tangent velocities, it is established in [BEN 99], [BEN 00a] that the quasi-static frictional contact problem consists in finding, at a fictive time t_k, a couple of displacement fields $\boldsymbol{u} = (\boldsymbol{u}^1, \boldsymbol{u}^2)$, a normal pressure field λ and a vector-valued semi-lagrangian friction multiplier $\boldsymbol{\Lambda}$ satisfying the following system : (where the reference to the fictive time t_k has been omitted to simplify the notations)

$$\left\{\sum_{i=1}^{2}\int_{\Omega_p^i} Tr(\mathbf{\Pi}^i(\nabla_p(\boldsymbol{w}^i))^T)d\Omega\right\} - \int_{\Gamma_c}\chi(g_n)g_n[[\boldsymbol{w}]]_n d\Gamma$$

$$-\int_{\Gamma_c}\mu\chi(g_n)\lambda P_{B(0,1)}(\boldsymbol{g}_\tau)\cdot[[\boldsymbol{w}]]_\tau d\Gamma = L_{\boldsymbol{u}}(\boldsymbol{w}) \ \ \forall \ \boldsymbol{w} = (\boldsymbol{w}^1, \boldsymbol{w}^2) \in \boldsymbol{CA} \quad [1]$$

$$\int_{\Gamma_c}\frac{-1}{\rho_n}\left\{\lambda - \chi(g_n)g_n\right\}\lambda^* d\Gamma = 0, \qquad \forall \ \lambda^* \in H \quad [2]$$

$$\int_{\Gamma_c}\frac{-\mu\chi(g_n)\lambda}{\underline{\rho}_\tau}\left\{\mathbf{\Lambda} - P_{B(0,1)}(\boldsymbol{g}_\tau)\right\}.\mathbf{\Lambda}^* d\Gamma$$

$$+\int_{\Gamma_c}(\chi - 1)\mathbf{\Lambda}\mathbf{\Lambda}^* d\Gamma = 0, \forall\, \mathbf{\Lambda}^* \in \boldsymbol{H} \quad [3]$$

where we have used the following notations :

$Tr(A) = A_{ii}$ is the trace of the second-order tensor A [4]

Π^i the first Piola-Kirchhoff stress tensor given by a *ad hoc* behaviour law

$\nabla_p(\boldsymbol{w}^i)$ the gradient with respect to reference coordinates $\boldsymbol{p}$ of $\boldsymbol{w}^i$

$\boldsymbol{x} = \boldsymbol{p} + \boldsymbol{u}(\boldsymbol{p}, t)$ is the current position, $\boldsymbol{p}$ being the initial one [5]

$g_n = \lambda - \rho_n d_n$ $(\rho_n > 0)$ is the augmented lagrangian contact multiplier [6]

$d_n = [[\boldsymbol{x}]]_n$ is the normal gap [7]

$\boldsymbol{g}_\tau = \mathbf{\Lambda} + \rho_\tau \boldsymbol{v}_\tau$ $(\rho_\tau > 0)$ is the augmented semi-lagrangian friction multiplier

$\boldsymbol{v}_\tau = [[\Delta\boldsymbol{x}]]_\tau$ is the increment of the relative tangential displacement [8]

χ the negative half-axis indicator function

$P_{B(0,1)}$ the projection on the unit ball

$L_{\boldsymbol{u}}(\boldsymbol{w})$ the virtual work of external loads (which may depend on $\boldsymbol{u}$)

$\boldsymbol{CA}$ the space of admissible displacement fields

H the space of virtual contact semi-multiplier fields

$\boldsymbol{H}$ the space of virtual friction multiplier fields

Moreover for a couple of vector-valued fields $\boldsymbol{w} = (\boldsymbol{w}^1, \boldsymbol{w}^2)$ defined on $\Gamma_c^1 \times \Gamma_c^2$, the quantities $[[\boldsymbol{w}]]_n$ and $[[\boldsymbol{w}]]_\tau$ refer to the normal and tangential "jump"-like functions. More precisely, for $\boldsymbol{w} = (\boldsymbol{w}^1, \boldsymbol{w}^2)$,

$$[[\boldsymbol{w}]]_n = (\boldsymbol{w}^1(\boldsymbol{p}^1) - \boldsymbol{w}^2(\mathcal{A}_p^k(\boldsymbol{p}^1)).\boldsymbol{n}(\mathcal{A}_p^k(\boldsymbol{p}^1)) = (\boldsymbol{w}^1(\boldsymbol{p}^1) - \boldsymbol{w}^2(\overline{\boldsymbol{p}}^1)).\boldsymbol{n}(\overline{\boldsymbol{p}}^1)$$

$$[[\boldsymbol{w}]]_\tau = [Id - \boldsymbol{n}(\mathcal{A}_p^k(\boldsymbol{p}^1) \otimes \boldsymbol{n}(\mathcal{A}_p^k(\boldsymbol{p}^1))]\{\boldsymbol{w}^1(\boldsymbol{p}^1) - \boldsymbol{w}^2(\mathcal{A}_p^k(\boldsymbol{p}^1))\} \quad [9]$$

where $\boldsymbol{n}$ refers to the inward unit normal to Γ_c^2 and where $\mathcal{A}_p^k$ is the reference target application, defined from $\Gamma_c = \Gamma_c^1$ into Γ_c^2 at time t_k with an appropriate contact-pairing procedure (cf. subsection 3.1 for details).

Equations [1], [2] and [3] stand for the *Virtual Work Principle* taking into account the contact friction efforts and using the *Action and Reaction Principle*, the weak *Signorini* contact laws formulation and the weak *Coulomb* friction laws formulation, respectively. The last term of equation [3] ensures that no friction can take place if there is no contact. This is a slight modification of the formulation given in [BEN 99]-[BEN 00a] where it was implicitly assumed that the friction density of forces is equal to zero where the contact is not effective. However, this modification seems for us of nice practical interest, at least as far as frictional pressures and not forces are involved in the problem.

To solve numerically this mixed highly nonlinear problem, two tools are needed. First we have to define a discretization method. Secondly a solution strategy has to be specified. The first point is detailed in the sequel. For the second we refer to [BEN 99] and [BEN 00a].

3. Discretizations

Formally, for the discretization by the finite element method of the problem defined by [1]-[3], one has to define finite element spaces denoted by $(\boldsymbol{CA}_{h_u})$, (H_{h_λ}) and $(\boldsymbol{H}_{h_\Lambda})$ for the approximation of the three fields $\boldsymbol{u}$, λ and $\boldsymbol{\Lambda}$. The discrete problem then reads:

Find $(\boldsymbol{u}_{h_u}, \lambda_{h_\lambda}, \boldsymbol{\Lambda}_{h_\Lambda}) \in \boldsymbol{CA}_{h_u} \times H_{h_\lambda} \times \boldsymbol{H}_{h_\Lambda}$;

$$\left\{\sum_{i=1}^{2}\int_{\Omega^i_{h_u}} Tr(\Pi^i_{h_u}(\nabla_p(\boldsymbol{w}^i_{h_u}))^T)d\Omega\right\} - \int_{\Gamma_{c_h}} \chi((g_h)_n)(g_h)_n[[\boldsymbol{w}_{h_u}]]_n d\Gamma$$
$$-\int_{\Gamma_{c_h}} \mu\chi((g_h)_n)\lambda P_{B(0,1)}((\boldsymbol{g}_h)_\tau)\cdot[[\boldsymbol{w}_{h_u}]]_\tau d\Gamma = 0, \ \ \forall\, \boldsymbol{w}_{h_u} \in \mathbf{CA}_{h_u} \quad [10]$$

$$\int_{\Gamma_{c_h}} \frac{-1}{\rho_n}\left\{\lambda_{h_\lambda} - \chi((g_h)_n)(g_h)_n\right\}\lambda^*_{h_\lambda} d\Gamma = 0, \qquad \forall\ \lambda^*_{h_\lambda} \in H_{h_\lambda}, \quad [11]$$

$$\int_{\Gamma_{c_h}} \frac{-\mu\chi((g_h)_n)\lambda_{h_\lambda}}{\underline{\rho}_\tau}\left\{\boldsymbol{\Lambda}_{h_\Lambda} - P_{B(0,1)}((\boldsymbol{g}_h)_\tau)\right\}.\boldsymbol{\Lambda}^*_{h_\Lambda} d\Gamma$$
$$+\int_{\Gamma_{c_h}} (\chi - 1)\boldsymbol{\Lambda}_{h_\Lambda}\boldsymbol{\Lambda}^*_{h_\Lambda} d\Gamma = 0, \qquad \forall\ \boldsymbol{\Lambda}^*_{h_\Lambda} \in \mathbf{H}_{h_\Lambda} \quad [12]$$

where

$$(g_h)_n = \lambda_{h_\lambda} - \rho_n[[\boldsymbol{x}_{h_u}]]_n \qquad [13]$$

$$(\boldsymbol{g}_h)_\tau = \Lambda_{h_\Lambda} + \rho_\tau[[\Delta\boldsymbol{x}_{h_u}]]_\tau \qquad [14]$$

and where the domains Ω^i and the interface Γ_c are replaced by (generally) approximate ones denoted by $\Omega^i_{h_u}$ and Γ_{c_h}, respectively. This is due to the triangulation of the domains.

Many questions and technical points have now to be discussed, namely:

$\boldsymbol{i}$) the construction of the target application,

$\boldsymbol{ii}$) the choice of functional spaces,

$\boldsymbol{iii}$) the numerical integration scheme,

$\boldsymbol{iv}$) the treatise of the contact surface irregularity introduced by the triangulation.

A “contact element” may be a reasonable answer to these issues !

3.1. *The target application*

Classically the following procedure is used: each point $\boldsymbol{x}$ of γ_c is paired with the nearest one located on γ_c^2 and denoted by $\overline{\boldsymbol{x}}$. A target application $\mathcal{A}_t$ can then be defined as:

$$\mathcal{A}_t(\boldsymbol{x}) = \overline{\boldsymbol{x}} = \underset{\boldsymbol{y}\in\gamma_c^2}{ArgMin}\frac{1}{2} \parallel \boldsymbol{x} - \boldsymbol{y} \parallel^2 \qquad [15]$$

where $\parallel . \parallel$ is the euclidian norm. This application can be pulled back to the initial configuration and then a reference (evolutive) target application $\mathcal{A}_p$ can also be defined easily. But notice that the reference target application can be defined by introducing an admissible physical seek direction $\boldsymbol{d}$ (see figure 1, [BEN 95b] and [BEN 00a] for details). The influence of such a procedure on numerical results is clarified here by a test reported in subsection 4.1.

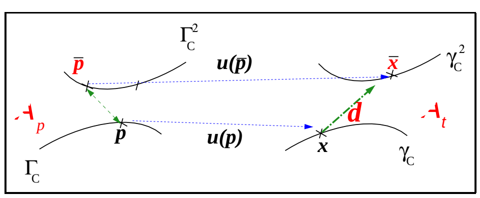

Figure 1. *Target application with a seek direction* $\boldsymbol{d}$

REMARK. — In some contact formulations derived directly in the discrete level the “non local” caracter of contact (expressed by 15) is approximated : the target point is looked for in a pre-defined collection of target "master" elements associated to the "slave" point. This procedure (with the definition of discrete contact forces) has suggested the (probably incomplete) notion of "contact element" as shown in figure 2.

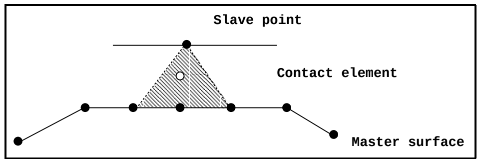

Figure 2. *A representation of contact elements*

3.2. *Compatible functional spaces*

As the formulation is mixed, its numerical approximation requires some care (cf. e.g. [BRE 78]). In order to avoid some pathological numerical behaviours, the approximation spaces have to be related by an abstract condition, called the *Inf-sup* condition. Very few results concerning this condition, even in the small perturbation contact framework, are available in the literature. Let us recall the one given by Haslinger, Hlavaček and Nečas [HAS 96] (which is quite similar to the historical one formulated by Babuška [BAB 73] for a non homogeneous Dirichlet Poisson problem treated by the lagrangian approach). This condition is stated as follows [HAS 96] [BAB 73]. Let P_1 and P_0 finite element spaces be associated with regular meshes of the interior of the domains and the contact boundary. Let h_u and h_λ denote the respective mesh parameters. Then the *Inf-Sup* condition is satisfied if:

$$\exists c > 0\,;\, h_\lambda \geq c h_u \qquad [16]$$

Unfortunately this condition still leaves practical difficulties since the positive constant c depends on the domains Ω^i in a complex manner and the only way to define it is still a numerical one which is not quite satisfactory. In a recent work of Bathe and Brezzi other choices are discussed: by considering the case of plane interfaces, it is shown for a model problem that a uniform *Inf-Sup* condition is satisfied for continuous piecewise linear displacement fields and Lagrange contact multipliers without any restrictive condition of the type [16]. However the proof does not seem to be extendible to the case where the contact interface is not plane. We shall however keep in mind this theoretical result especially when dealing with the numerical integration of contact terms in the following section.

3.3. *Contact integration scheme*

Even when the discrete fields are elementwise polynomial, the contact terms of the discrete problem [10]-[14] involve the integration of functions which are not elemen-

twise polynomial. This is mainly due to:
$i)$ the incompatibility of approximation spaces (see figure 2),
$ii)$ the heterogeneous character of contact: within an element belonging to the contact interface : one can find quite different status, namely contacting and noncontacting points, the first being either sticking or sliding (see figure 3). These status are mathematically expressed by the irregular function χ and the projection on the unit ball which is also irregular.

Keeping all this in mind, the question is now how to evaluate contact terms ? For this purpose, let us consider the following representative contact coupling term (see equation [1])

$$I_c^{ij} = \int_{\Gamma_{c_h}} \chi((g_{h_u})_n) N_{h_\lambda}^{1,i}(\boldsymbol{p}) \Big(\boldsymbol{N}_{h_u}^{2,j}(\overline{\boldsymbol{p}}) \Big)_n d\Gamma \qquad [17]$$

where $N_{h_\lambda}^{1,i}$ and $\boldsymbol{N}_{h_u}^{2,j}$ stand for basis functions in H_{h_λ} and $\boldsymbol{H}_{h_{u^2}}$, respectively.
Let us assume that the mesh of Γ_c is the trace of the mesh of Ω^1 on Γ_c and that continuous piecewise linear finite elements are used for the approximation of the three fields. Clearly, the integral I_c^{ij} cannot be evaluated exactly and one can only choose, in some sense, the points on which the impenetrability is enforced. The choice of the nodes located on the boundary Γ_{c_h} allows the recovering of the well-known node-on-facets strategy. But at this stage, one can also use either classical Gauss or Gauss-Lobato or Simpson integration schemes. Whatever the integration scheme used, it leads to approximate contact terms and the theoretical question of *Inf-sup* condition has to be reformulated, at this level. We refer here to the work of Oden and Kikuchi [KIK 88] for a Simpson integration scheme in a particular situation. Here an attempt is made to measure the influence of the integration scheme. It is believed that if the integration scheme is developed with sufficient precision, the theoretical compatibility results recalled in the previous sub-section will be recovered (for plane contact interfaces when equal-order interpolation spaces are used). We suggest then to evaluate the contact terms by using sub-elements Simpson integration schemes, rather than high-order Gauss ones. By using several sub-elements, we try to evaluate accurately the irregular contact terms in order to recover the theoretical *Inf-Sup* condition mentioned in the previous subsection (at least when this condition is satisfied). The influence of our finite-elements/particules high level integration strategy on numerical results is shown in subsection (5.2).

REMARK. — The method we describe above is costly. The reader is referred to [BEN 01] for another more economic approach which is also ¨quasi-symmetric¨.

Now, without going into details, let us also stress the fact that the element level is not sufficient for the numerical evaluation of I_c^{ij}. Rather one has to carry out loops based on both elements $e_i \subset \Gamma_{c_h}$ and sampling contact points $(G_k)_{k=1,nb1}$ belonging to e_i (defined by the numerical scheme). For each point G_k of e_i, one has to find

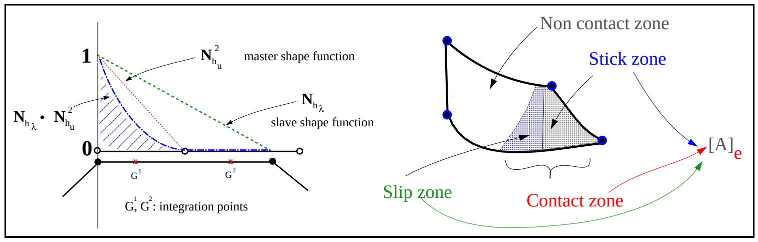

Figure 3. *Incompatibility- Element heterogeneity of contact*

out the target element e^2 (belonging to the trace of the mesh of Ω^2 on Γ^2_c) and the target point $G^2_k \in e^2$, by solving local nonlinear problems (this is the price to pay for large transformations contact problems and for flexibility needs). Once this is done, element/particle coupling contact terms can be evaluated and assembled directly in the global contact coupling term in a rather classical manner. Hence, particle contributions are substituted for element ones. This is the reason for which this discrete method was called finite element/particle method [BEN 00b].

3.4. *Normal field regularization*

The approximation of contact problems by means of the finite element method introduces generally polygonal (*2D*) or polyhedral (*3D*) approximation of the domains. This leads to only piecewise regular normal fields. The discontinuity of the normal field may create some pathologies such as numerical shocks [KRS 00]. In order to get rid of these pathologies, classical regularization procedures were suggested in the literature. These are based mainly in spline curves or surfaces [KRS 00]. The main drawbacks of this regularization is $i)$ the hight cost for industrial problems, $ii)$ the perturbation of the contact surfaces (see figure 4) which has been analyzed in [BEN 95a]

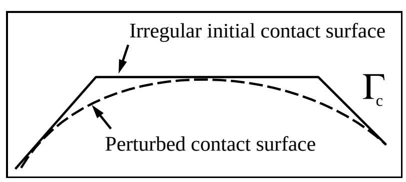

Figure 4. *Spline regularization*

An alternative regularization method is studied here. First by using a rather classical averaging technique taking into account the measure of the elements, the normal field $\boldsymbol{n}$ can be made single-valued everywhere on $\gamma^2_{c_h}$. Secondly, by using only the values of the normals at the nodes of $\gamma^2_{c_h}$, an interpolated continuous piecewise linear

vector-valued normal field $\boldsymbol{n}_{interp}$ is defined on $\gamma^2_{c_h}$ (see figure 5). Notice here the (continuous-based) weighting procedure used to create the regularized normal field.
The importance of this regularization method is shown by the numerical test (4.4).

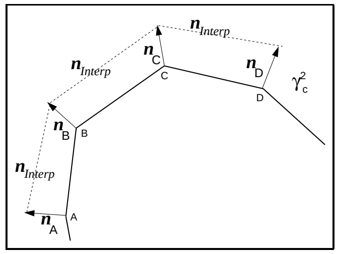

Figure 5. *Normal-vector interpolation*

4. Numerical results

The following numerical results aim to show the effectiveness of each of the previously discussed points.

4.1. *Target applications*

To measure the influence of the target application, the classical symmetric *Hertz* problem is considered (see figure 6). This problem is solved under an axisymmetry condition properties. The material proprieties of the two half-spheres are: $E = 20.000\ MPa$ and $\nu = 0.2$. The meshes take into account the symmetry of the problem. Each one has 205 nodes with 15 linear triangular elements and 162 isoparametric bilinear quadrilateral ones.

We have tested two “target” applications one based on the “proximity” notion and the other based on the (oy) seek direction, respecting precisely the physical symmetry of the problem. Due to the symmetry of the problem (same geometry, same meshes, symmetrical prescribed displacement on each half-sphere), the vertical displacement ’DY’ of the point A has to be equal to zero and the friction load has to be equal to zero. By the proximity pairing procedure we find a non null vertical displacement ’DY’, while by taking (oy) as a seek direction the correct result is retrieved. Moreover, by activating friction phenomena, we have found that the friction forces are not equal to zero when the “proximity” method is used. The fact that these friction loads should be equal to zero (as expected physically) is also retrieved by taking the physical seek direction (oy).

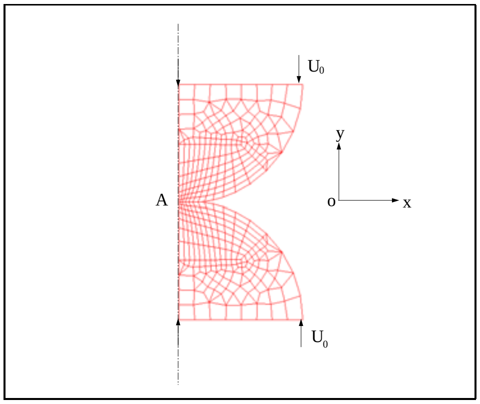

Figure 6. *Hertz problem: Deformed meshes*

Method	Displacement 'DY' of A
Proximity approach	-5.05E-04
(oy) seek direction	3.88375E-16

Table 1. *Influence of the pairing method*

4.2. *Numerical integration test*

To illustrate the problem of numerical integration, we have chosen a frictionless contact between two blocs with nonmatching contact interfaces (the benchmark of Taylor and Papadopoulos). A vertical displacement $\boldsymbol{U}_0$ is imposed to the upper-bloc on its upper-edge, while the other is clamped on its lower edge as shown in figure 7. For this problem we have used $m-$Simpson schemes.

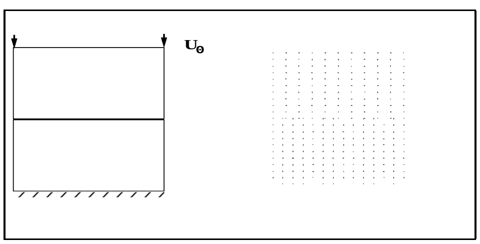

Figure 7. *Two block problem - Nonmatching contact interfaces*

By $m-$Simpson scheme for the evaluation of the integral of a function over the reference 1-D element $[0, 1]$, we mean an integration method in which the classical reference element $[0, 1]$ is divided into 2^m equal sub-elements and the use of the Simpson integration scheme on each sub-element.Three integration schemes are tested ($m = 0$, $m = 1$ and $m = 2$). The integration points coordinates and the weights are given in the following table.

0−Simpson integration scheme

Integration Point	*Abscissae*	*Weight*
1	*0.0*	*1/6*
2	*0.5*	*2/3*
3	*1.0*	*1/6*

1−Simpson integration scheme

Integration Point	*Abscissae*	*Weight*
1	*0.0*	*1/12*
2	*0.25*	*1/3*
3	*0.5*	*1/6*
4	*0.75*	*1/3*
5	*1.0*	*1/12*

2−Simpson integration scheme

Integration Point	*Abscissae*	*Weight*
1	*0.0*	*1/24*
2	*0.125*	*1/6*
3	*0.25*	*1/12*
4	*0.375*	*1/6*
5	*0.5*	*1/12*
6	*0.625*	*1/6*
7	*0.75*	*1/12*
8	*0.875*	*1/6*
9	*1.0*	*1/24*

Figure 8. *Sub-element technique using $m-$Simpson integration scheme*

The normal contact multiplier profiles are integration scheme dependent as illustrated in figure 9.

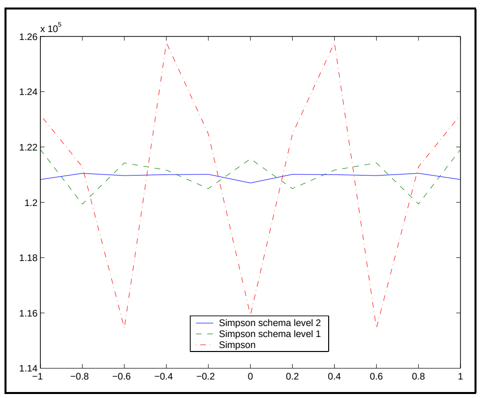

Figure 9. *Normal contact multiplier λ on contact interface*

The spurious oscillation amplitude decreases by increasing the number of integration points. It is noted that with increasing integration points, we obtain more accurate integral approximations. Notice also that an alternative approach consists in creating a virtual interface compatible to both nonmatching interfaces by using the *Arlequin* method. We have tested the effectiveness of this approach. The result is not reported here but it shows that an “precise” integration scheme gives the correct contact solution. This exact integration can be achieved by introducing (rather heavy in the $3D$ case) geometrical tools to define a compatible integration mesh, namely a mesh for which the behaviour of the different fields involved in the problem (10)-(14) are elementwise regular.

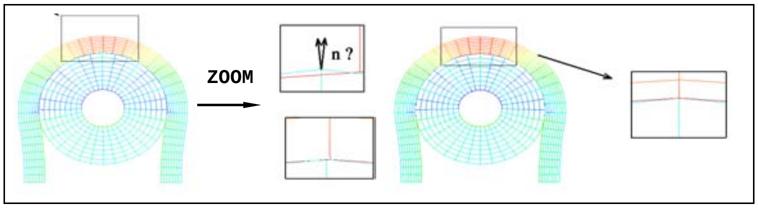

Figure 10. *Influence of the regularization of the normal field (a)– without regularization (b)– with regularization*

Let us here stress the fact that this procedure is an appropriate answer for mortar like elements used to handle equality constraints [BER 94]. But this is still an approximation for general unilateral contact interfaces.

4.3. *Regularization of the normal field*

To illustrate this geometrical aspect we have chosen a simple *2D* example of non-conforming contact interfaces. We consider an elastic cord of thickness 1, wrapped on an elastic cylinder whose inner and outer radius are $= 1$ and 3, respectively. A prescribed vertical displacement $u_0 = 1$ is applied to the two ends of the cord whereas the interior surface of the cylinder is clamped. We have chosen an isotropic elastic constitutive material law with $E = 147.6E + 05$ and $\nu = 0.47$, following the data given by [PAP 98]. In addition we have treated this example under large transformations and under frictionless and plane strain hypotheses. A classical treatment (without regularization of the normal vector) introduces an artificial dissymmetry (cf. figure 11-a). With the use of the regularization the symmetry is recovered (cf. figure 11-b). Another expected benefit behaviour of our solution strategy, especially when friction is involved.

4.4. *Application to stretching problem*

We treat here the benchmark test given by Wagoner et al [WAG 88]. It consists in stretching an elestoplasic sheet with a hemispherical punch. (see figure 11): The hardening law is the following :

$$\boldsymbol{\sigma}_0 = 539(10^{-4} + \boldsymbol{\epsilon}_p)^{0.216} \qquad [18]$$

The Young modulus E and the Poisson ration ν are equal to $E = 69004N/mm^2$ and $\nu = 0.3$. The friction coefficient is equal to $\mu = 0.2$. In figure 12, we have plotted the deformed structure. This test shows the effectiveness of our solution strategy even when many other nonlinearities (elasto-plasticity, large deformations) are involved in the frictional contact mecanical problem.

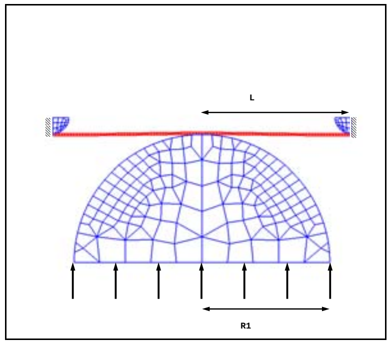

Figure 11. *The Wagoner Benchmark*

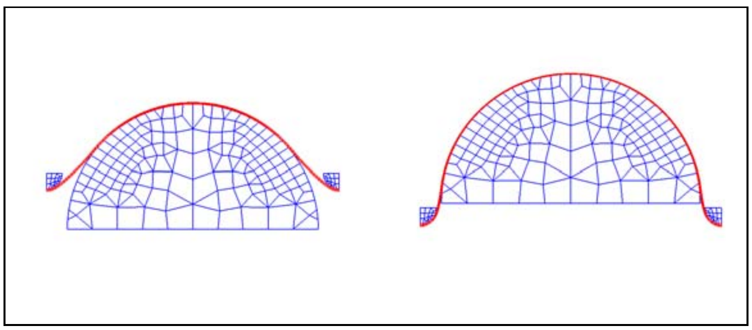

Figure 12. *Deformed structures for two different positions of the punch*

Other industrial applications are now being tested. The results will be given in a forthcoming work.

Acknowledgements

The support of Electricité de France is gratefully acknowledged.

5. References

[BAB 73] BABUSKA I., "The finite element method with lagrangian multipliers", *Numer. Math.*, vol. 20, 1973, p. 179-192.

[BEN 95a] BEN DHIA H., *Numerical analysis of perturbed contact problems in Contact Mecanics, edited by M. Raous et al.,* Plenum Press, New York, 1995.

[BEN 95b] BEN DHIA H., DURVILLE D. , "Calembour: An implicit method based on enriched kinematical thin plate model for sheet metal forming simulation", *J. of Materials Processing Technology*, vol. 50, 1995, p. 70-80.

[BEN 99] BEN DHIA H., VAUTIER I., "Une formulation pour traiter le contact frottement en 3D dans le $Code\_Aster$", *Rapport EDF HI-75/99/007/A*, , 1999.

[BEN 00a] BEN DHIA H., VAUTIER I., ZARROUG M., "Problèmes de contact frottant en grandes transformations: du continu au discret", *Revue Européenne des Eléments Finis*, vol. 9, 2000, p. 243-261.

[BEN 00b] BEN DHIA H., ZARROUG M., MASSIN P., VAUTIER I., "Finite elements/points for a hybrid formulation of contact-friction problems in large transformations", *Chicago, The 20th ICTAM Congress*, , 2000.

[BEN 01] BEN DHIA H., ZARROUG M., "Contact in the Arlequin framework", *Proceedings of the third Contact Mechanics International Symposium (eds) Martins & Marques*, Kluwer, 2001, p. 403-411.

[BER 94] BERNARDI C., MADAY Y., PATRA A.T., "A new nonconfirming approach to domain decomposition : the mortar element method", *Collège de France Seminar*, vol. H. Brezis, J.L. Lions, num. Pitman, 1994, p. 13-51.

[BRE 78] BREZZI, F., HAGER, W., RAVIART, P.A., "Error estimates for the finite element solution of variational inequalities", *Numer. Mat.*, vol. 31, 1978, p. 1-16.

[HAS 96] HASLINGER J., HLAVAČEK I., NEČS J., "Numerical methods for unilateral problems in solid mechanics", *In Handbook of Numerical Ana lysis*, vol. IV, num. part2, 1996, p. 313-485.

[KIK 88] KIKUCHI N., ODEN J.T., "Contact Problems in Elasticity: a Study of Variational Inequalities and Finite Element Methods", *SIAM, Philadelphia*, , 1988.

[KLA 95] KLARBRING A., "Large displacement frictional contact : a continuum frameworke for finite element discretization", *Eur.J.Mech., A/Solids*, vol. 14, num. 2, 1995, p. 237-253.

[KRS 00] KRSTULOVIC-OPERA L., WRIGGERS P. AND KORELEC J., "Symbolically generated 3D smooth polynomial frictional contact element based on the quadratic Bézier surfaces", *Barcelone, ECCOMAS*, , 2000.

[LAU 93] LAURSEN T.A., SIMO J.C., "A continuum-based finite element formulation for the implicit solution of multibodies, large deformation frictional contact problems", *Int. J. Numer. Meth. Engrg*, vol. 36, 1993, p. 3451-3485.

[PAP 98] PAPADOPOULOS. P, SOLBERG. J.M, "A Lagrange Multiplier Method for the Finite Element Solution of Frictionless Contact Problems", *Mathl.Comput. Modelling*, vol. 28, num. 4-8, 1998, p. 373-384.

[PIE 99] PIETRZAK G., CURNIER A., "Large deformation frictional contact mechanics: continuum formulation and augmented Lagrangian treatment", *Comput. Methods Appl. Mech. Engrg.*, vol. 177, 1999, p. 351-381.

[WAG 88] WAGONER R. NAKAMACHI E. L. C., "A benchmark test for sheet metal forming analysis", *Technical Report*, , num. ERC/NSM-S-90-22, 1988, Page Ohio Satate University.