A shell finite element for viscoelastically damped sandwich structures

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ABSTRACT. *In this paper, a shell finite element is proposed for viscoelastically damped sandwich structures, in which a thin viscoelastic layer is sandwiched between identical elastic layers. The sandwich finite element is obtained by assembling three elements throughout the thickness of the sandwich structure. Using specific assumptions and displacement continuity at the interfaces, one reduces to eight the number of degrees of freedom per node that are the longitudinal displacements of the elastic layers, the deflection and three rotations. The finite element computations have been compared with known analytical, numerical and experimental data concerning the vibrations of sandwich beams, plates and shells.*

RÉSUMÉ. *Dans cet article, un élément fini de coque est proposé pour les structures sandwich viscoélastiques, constituées d'une fine couche viscoélastique intercalée entre deux parements élastiques. L'élément fini sandwich est obtenu par l'assemblage de trois éléments suivant l'épaisseur de la structure sandwich. L'utilisation des hypothèses spécifiques et les conditions de continuité du déplacement aux interfaces permettent de réduire à huit le nombre de degrés de liberté par nœuds : les déplacements dans le plan des parements élastiques, la flèche et trois rotations. Les résultats des simulations numériques pour des vibrations de poutres, plaques et coques sandwich ont été comparés à des solutions analytiques, numériques et à des données expérimentales.*

KEYWORDS: *Finite element, plate, vibrations, viscoelasticity, sandwich, damping.*

MOTS-CLÉS : *Elément fini, plaque, vibrations, viscoélasticité, sandwich, amortissement.*

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1. Introduction

Typical viscoelastically damped structures are sandwich structures in which a thin viscoelastic layer is sandwiched between identical elastic layers (Fig. 1). This kind of structures is used in many areas to reduce vibrations and noise. Here, the damping is introduced by an important transverse shear in the viscoelastic layer. It is due to the difference between in-plane displacements of the elastic layers and also to the low stiffness of the central layer. The numerical simulation of these structures requires first the use of a thin shell model to obtain a reasonable computational cost, second a proper account of the transverse shear in the viscoelastic layer.

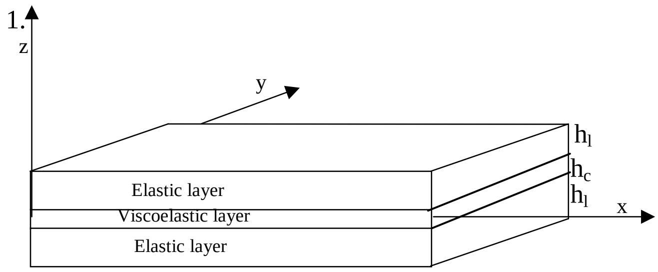

Figure 1. *Sandwich structure with a viscoelastic middle layer*

Initially, analytical methods were developed to yield approximate loss factors and natural frequencies of sandwich beams or plates with simple geometries and boundary conditions [KER 59], [DIT 67], [MEA 69], [YAN 72], [ORA 74], [RAO 74], [SAD 84], [CUP 95], [HE 96], [HU 00]. In practice, it is necessary to design sandwich structures with a complex geometry and generic boundary conditions by using finite element simulations. Finite elements are still being used for analysis of damped sandwich structures [BAB 98], [LU 79], [JOH 81], [SON 81], [ALA 84], [MA 92], [RIK 93], [RAM 94], [SAI 99] with various numerical algorithms.

A simple idea presented in [SON 81], [JOH 81], [RIK 93] consists in assembling three elements throughout the thickness of the sandwich structure. The most relevant choice is to model the elastic layers by shell element and the core by a three dimensional one. These shell/volume/shell elements have permitted to predict reasonable approximations of the damping of the structure. With account of the continuity of the displacement, this yields sandwich elements involving at least twelve degrees of freedom per node. This number can be reduced to ten in the case of flat plate, because the in-plane rotation can be neglected. Another difficulty in such approaches is the compatibility of the approximation of the displacement field between core and faces. Specific finite elements have been developed for shells of revolution [ALA 84], for conical shells [RAM 94] or for beams [BAB 98], [SAI 99].

A plate/volume/plate finite element has been presented by Ma and He [MA 92]. They account for the assumption of common transverse displacement and rotations in the faces as in most analytical studies. Furthermore, they limit their analysis to the bending motions of plates. All these hypotheses permit them to reduce to five the number of degrees of freedom per nodes : the transverse displacement, two rotations and two additional rotations in the core.

Generally, the stiffness matrix of damped sandwich structures depends on the vibration frequency. Then, the natural vibration study of sandwich structures introduces a non-linear complex eigenvalues problem. This problem can be solved by various numerical methods: the complex eigenvalues method, the modal strain energy method (MSEM), the direct frequency response method, the asymptotic approach, the order-reduction-iteration technique and the asymptotic numerical method (ANM). Only the two last methods are able to solve directly and exactly the non-linear complex eigenvalues problem. A review of these numerical methods can be found in reference [DAY 01].

In this paper, we present a class of shell/volume/shell finite element assemblies. The number of degrees of freedom is reduced by assuming that the transverse displacement and the rotations are the same in the elastic layers. Unlike in [MA 92], this element can be applied to shell structures with any geometry. The approach is validated by comparison with analytical, numerical and experimental results.

2. Finite element formulation

In this section a finite element for sandwich shells is presented. This element is the assembly of one volume and two plate elements throughout the thickness of the sandwich structure. The shell curvature is accounted for by the usual plane facets technique [BAT 90].

2.1. *Basic hypotheses for the displacements*

In order to evaluate accurately damping in the sandwich structures, one has to take into account the shear deformation in the viscoelastic layer. This shear results from the difference between the in-plane displacements of the elastic layers. So, we consider the following assumptions, which were also used in many studies [CUP 95], [MA 92], [RAO 78]:

– All points on a normal to the plate have the same transverse displacement w(x,y,t)

– No slip occurs at the interfaces between central and elastic layers.

– All points of the elastic layers on a normal have the same rotations.

– The core material is homogeneous, isotropic and viscoelastic. So, the Young's modulus is complex and depends on the vibration frequency. As in most analyses, the Poisson's ratio is assumed constant.

– The elastic layers have the same thickness, which is denoted by h_l.

Using the classical plate theory, one can write the displacement in the elastic layers as follows:

$$\begin{cases} U^s(x,y,z,t) = u^s(x,y,t) + (z - z_0)R_x(x,y,t) \\ V^s(x,y,z,t) = v^s(x,y,t) + (z - z_0)R_y(x,y,t) \\ W^s(x,y,z,t) = w(x,y,t) \end{cases} \quad \text{In the superior layer [1]}$$

$$\begin{cases} U^i(x,y,z,t) = u^i(x,y,t) + (z + z_0)R_x(x,y,t) \\ V^i(x,y,z,t) = v^i(x,y,t) + (z + z_0)R_y(x,y,t) \\ W^i(x,y,z,t) = w(x,y,t) \end{cases} \quad \text{In the inferior layer [2]}$$

where $z_0 = (h_l+h_c) / 2$, h_c is the thickness of the core. u^s (x, y, t), v^s(x, y, t), u^i(x, y, t), v^i(x, y, t) are the components of the mid-plane displacements of the faces. R_x and R_y are the rotations of the normal to the mid-planes of the face layers.

Like in reference [HE 96], we suppose that the displacement of the central layer can be written in the following form:

$$\begin{cases} U^c(x,y,z,t) = u^c(x,y,t) + zR_x^c(x,y,t) \\ V^c(x,y,z,t) = v^c(x,y,t) + zR_y^c(x,y,t) \\ W^c(x,y,z,t) = w(x,y,t) \end{cases} \quad [3]$$

Considering the continuity conditions of displacements at the interfaces between the central and the face layers, the displacement in the viscoelastic layer can be rewritten in the following form:

$$\begin{cases} U^c(x,y,z,t) = \dfrac{u^s(x,y,t) + u^i(x,y,t)}{2} + z(\dfrac{u^s(x,y,t) - u^i(x,y,t)}{h_c} - \dfrac{h_l}{h_c}R_x(x,y,t)) \\ V^c(x,y,z,t) = \dfrac{v^s(x,y,t) + v^i(x,y,t)}{2} + z(\dfrac{v^s(x,y,t) - v^i(x,y,t)}{h_c} - \dfrac{h_l}{h_c}R_y(x,y,t)) \\ W^c(x,y,z,t) = w(x,y,t) \end{cases} \quad [4]$$

This reduces the number of independent generalized displacements to seven, namely u^s, v^s, u^i,. v^i, w, R_x and R_y.

Many plate elements can be applied to interpolate the displacement in the elastic layers [BAT 90]. Generally, one can write the displacement of the sandwich volume element in the following form:

$$\begin{cases} \left\{\tilde{U}^s\right\} = [N^s]\{q\}^e \;,\; \left\{\tilde{U}^i\right\} = [N^i]\{q\}^e \;,\; \left\{\tilde{U}^c\right\} = [N^c]\{q\}^e \\ \left\{\tilde{U}\right\}^t = \{U, V, W\} \\ \{q\}^{e^t} = \left\{u_j^s, u_j^i, v_j^s, v_j^i, w_j, R_{xj}, R_{yj}, R_{zj}\right\} \; j = 1, NE \end{cases} \quad [5]$$

where NE is the number of nodes in the considered shell element. $[N^s]$, $[N^c]$ and $[N^i]$ are matrices that only depend on the shape functions of the plate finite element in the elastic layer. When dealing with shells, a fictive rotation R_z must be introduced, as explained in [BAT 90], to connect the rotation vector between neighbor elements.

An important point in this approach is the exact continuity between the faces and the core. In what follows, we only discretize displacements and rotations of the faces. The displacement field in the core will be deduced from the formula [4], which avoids any incompatibility between the discretization in the core and faces.

2.2. *Expressions of the strains*

With the above formulae for the displacements and assuming that the deformation of the elastic layers obeys classical thin plate theory, the strains in these layers are in the following form:

$$\begin{cases} \varepsilon_{xx}^l(x, y, z, t) = \dfrac{\partial U^l}{\partial x}, \quad \varepsilon_{yy}^l(x, y, z, t) = \dfrac{\partial V^l}{\partial y} \\ 2\varepsilon_{xy}^l(x, y, z, t) = \dfrac{\partial U^l}{\partial y} + \dfrac{\partial V^l}{\partial x} \\ 2\varepsilon_{xz}^l(x, y, z, t) = \dfrac{\partial U^l}{\partial z} + \dfrac{\partial w}{\partial x} \\ 2\varepsilon_{zy}^l(x, y, z, t) = \dfrac{\partial V^l}{\partial z} + \dfrac{\partial w}{\partial y} \end{cases} \quad [6]$$

where l=s or i denotes the superior and inferior layers respectively. The strains in the viscoelastic core can be expressed in the following form:

$$\begin{cases} \varepsilon^c_{xx}(x,y,z,t) = \left[A_1(z)\right]\left[\dfrac{\partial u^s}{\partial x}, \dfrac{\partial u^i}{\partial x}, \dfrac{\partial R_x}{\partial x}\right]^t \\ \varepsilon^c_{yy}(x,y,z,t) = \left[A_1(z)\right]\left[\dfrac{\partial v^s}{\partial y}, \dfrac{\partial v^i}{\partial y}, \dfrac{\partial R_y}{\partial y}\right]^t \\ \varepsilon^c_{zz}(x,y,z,t) = 0 \\ 2\varepsilon^c_{xy}(x,y,z,t) = \left[A_1(z)\right]\left[(\dfrac{\partial u^s}{\partial y} + \dfrac{\partial v^s}{\partial x}), (\dfrac{\partial u^i}{\partial y} + \dfrac{\partial v^i}{\partial x}), (\dfrac{\partial R_x}{\partial y} + \dfrac{\partial R_y}{\partial x})\right]^t \\ 2\varepsilon^c_{xz}(x,y,z,t) = \left[A_2\right]\left[u^s, u^i, R_x\right]^t \\ 2\varepsilon^c_{zy}(x,y,z,t) = \left[A_2\right]\left[v^s, v^i, R_y\right]^t \end{cases} \qquad [7]$$

where $\left[A_1(z)\right] = \left[(\frac{1}{2} + \frac{z}{h_c}), (\frac{1}{2} - \frac{z}{h_c}), (-z\frac{h_l}{h_c})\right]$, $\left[A_2\right] = \left[\frac{1}{h_c}, -\frac{1}{h_c}, (-1 - \frac{h_l}{h_c})\right]$

As in many works, the plane stress hypothesis is assumed for the elastic layers. Especially, the shear strains (i.e. $\varepsilon^l_{xz} = \varepsilon^l_{yz} = 0$) are neglected, because this not necessary for the considered applications. But such shear strains could be included without any difficulty. A discussion of this point can be found in [CUP 95]. Unlike in [CUP 95],[He 96],[MA 92], the present model takes into account all the strain components in the viscoelastic layer, except the transverse one.

2.3. *Finite element formulation for the vibrations of sandwich shells*

In this section, we recall the modeling of natural vibrations of damped sandwich structures [SON 81]. The formulation is obtained from the virtual work equation:

$$\int_V (\{\delta\varepsilon\}^t.\{\sigma\} + \rho\{\delta\mathbf{U}\}^t.\left\{\frac{\partial^2 \mathbf{U}}{\partial t^2}\right\})dV = 0 \qquad [8]$$

where σ, ε and **U** are respectively the stress and strain tensors, and the generalized displacement at a point within the body V. The mass density of the material is denoted by ρ. The stress and strain tensors and the displacement U can be expressed as harmonic time functions:

$$\mathbf{U}(x,y,z,t) = u(x,y,z)\, e^{i\omega t}$$

$$\sigma(x,y,z,t) = \sigma(x,y,z)\, e^{i\omega t}$$
$$\varepsilon(x,y,z,t) = \varepsilon(x,y,z)\, e^{i\omega t} \quad [9]$$

where $i = \sqrt{-1}$ and ω is the vibration frequency. In the presence of damping, the frequency ω is a complex number.

The viscoelastic damping behavior is accounted for through the stress-strain law of the viscoelastic layer, which can be written in the form :

$$\{\sigma^c\} = [C^c(\omega)]\{\varepsilon^c\} \quad [10]$$

where the real and imaginary part of $C^c(\omega)$ characterizes respectively energy storage and dissipative behavior of the viscoelastic material. In the elastic layers, the classical plane stress elastic law is written in similar way. It depends on the Young's modulus E_l and Poisson ratio ν_l.

From equations [8],[9],[10] and finite element discretization, the discrete equations for viscoelastic sandwich structures can be written in the following form:

$$[K(\omega) - \omega^2 M]\{U\} = \{0\} \quad [11]$$

where $[K(\omega)]$ is the complex stiffness matrix, which depends on the frequency. The mass matrix $[M]$ is assumed to be constant in this analysis. The complex nodal vibration eigenmode is denoted by $\{U\}$.

The element stiffness and mass matrices are respectively of the form:

$$\begin{cases} [K^e] = [K_e^s] + [K_e^c] + [K_e^i] \\ [K_e^\alpha] = \int_{\alpha\ \text{layer}} [B^\alpha]^t [C^\alpha][B^\alpha] dv^\alpha \end{cases} \quad [12]$$

$$\begin{cases} [M^e] = [M_e^s] + [M_e^c] + [M_e^i] \\ [M_e^\alpha] = \int_{\alpha\ \text{layer}} \rho^\alpha [N^\alpha]^t [N^\alpha] dv^\alpha \end{cases} \quad [13]$$

where α = i or s or c. The matrices $[B^\alpha]$ depend only on the shape functions of the considered finite element and $[C^\alpha]$ are the matrices obtained from the strain stress laws. In this way, we finally obtain a class of sandwich finite element with only eight degrees of freedom per node, that are for instance u^s, v^s, u^i,. v^i, w, R_x, R_y and R_z.

One can note that the matrices $[K_e^s]$ and $[K_e^i]$ are the element stiffness matrices of the elastic layers, while $[M_e^s]$ and $[M_e^i]$ are the corresponding mass matrices. The strain-displacement matrix in the core $[B^c]$ is easily formed from the shape functions of the latter plate element and from equations [5] and [7]. Some terms of this matrix depend linearly on z, so that two integration points are considered through the core thickness. In this way, one can form a finite element for sandwich structures from any plate element in the elastic layers. The so obtained sandwich element is a shell/volume/shell element. But the two shells are required to be parallel and the volume element is quite different from classical ones, because the core matrix $[B^c]$ is build up from shell shape functions and not from classical volume shape functions as in [SON 81].

In this paper, the basic shell element is a three node flat triangle according to the discrete Kirchhoff theory (DKT) [BAT 82], [BAT 90]. Curved shells are discretized by connecting flat DKT elements what is quite usual. Often, this element is referred as DKT18, because it includes 18 degrees of freedom [BAT 90]. When extended to sandwich shells, this number passes to 24, to describe correctly the shear in the core. A similar element had been introduced in [MA 92], that had 15 degrees of freedom, but that was limited to bending motions of plates.

As said previously, there are numerical difficulties in solving equation [11], because $K(\omega)$ is a complex matrix and depends on frequency. Several different ways have been proposed to solve more or less exactly the nonlinear eigenvalues problem [11]: the complex eigenvalues method [WIL 65], the modal strain energy method (MSEM) [JOH 79],[SON 81], the direct frequency response method [SON 81], the asymptotic approach [MA 92], [HE 96], the order-reduction-iteration technique [CHE 99] and the asymptotic-numerical method (ANM) [DAY 01]. In this paper, we limit ourselves to an approximated and cheap method (MSEM) and to ANM, that solves exactly the nonlinear eigenvalues problem.

After solving equation [11], the loss factor and natural frequency are calculated from the complex eigenvalue ω as follows:

$$\omega^2 = \omega_n^2(1+i\eta) \qquad [14]$$

when ω_n is the natural frequency and η is the loss factor.

3. Numerical examples

In the following section, we present five numerical examples in which the viscoelastic material is linear, homogeneous and isotropic. The stress-strain law of the central layer is written in the following form:

$$\sigma=2\mu\varepsilon+\lambda I_3 tr(\varepsilon) \quad , \quad \mu=\frac{E(\omega)}{2(1+\upsilon_c)} \quad , \quad \lambda=\frac{\upsilon_c E(\omega)}{(1+\upsilon_c)(1-2\upsilon_c)} \qquad [15]$$

where I_3 is the identity matrix. As in most analyses, the Poisson's ratio υ_c is assumed constant.

3.1. *Sandwich beams*

The present finite element procedure has been applied to the vibration analysis of a 3-layer cantilever sandwich beam (elastic/viscoelastic/elastic). In order to assess the accuracy of the assumptions for the displacement (Section 2.1), we have also used a two-dimensional eight nodes quadrilateral elements (Q16) to discretize the sandwich beam as in [DAY 01]. Because the latter element does not imply any plate kinematic assumptions, the so obtained results can be considered as reference solutions of the cantilever beam problem. The latter computations have been performed with a sufficiently fine mesh (six elements through the thickness). Several tests are presented, that mainly differ from one another by the beam dimensions and the material properties. For these simulations, the nonlinear eigenvalue problem is solved "exactly" by ANM [DAY 01].

3.1.1. *Beam 1*

The first test has been designed to analyze viscoelastic structures, in which the modulus depends strongly on the frequency. The beam dimensions and material properties are presented in Table 1. The chosen viscoelastic model is rather intricate and it has been proposed in [FER 80] for fitting the stress relaxation master curve to any experimental results. The viscoelastic material is represented by a generalized Maxwell model that can be written in the following form:

$$E(\omega)=E_c+\eta_0 i\omega+\sum_{j=1}^{Nmax}\frac{i\omega}{\left(\frac{i\omega}{E_j}+\frac{1}{\eta_j}\right)}$$

where the coefficients E_j and η_j of the model are obtained as described in [LAN 93] from experimental tests on a pure polymer. Nmax is the number of Maxwell elements. The steel company USINOR has provided these data. Note that the relaxation experiments are performed at a given temperature and the modeling extended to any temperature using the WLF law [FER 80]. We consider the law of this polymer at 60°C. For this, the variation of the shear modulus with frequency is strongly nonlinear. The corresponding curves have been presented in [DAY 01]. So, the ANM is used to calculate the loss factor and natural frequency. In table 2, the numerical results are presented and compared with experimental results from an actual sandwich beam.

Elastic layers	
Young's modulus :	$E_l = 2.1\ 10^{11}$ N/m^2
Poisson's ratio :	$\upsilon_l = 0.3$
Mass density :	$\rho_l = 7800$ kg/m^3
Thickness :	$h_l = 0.6$ mm
Viscoelastic layer	
Elastic modulus :	$E_c = 27216\ 10^2$ N/m^2
Poisson's ratio :	$\upsilon_c = 0.44$
Mass density :	$\rho_c = 1200$ kg/m^3
Thickness :	$h_c = 0.045$ mm
Whole beam	
Length	L = 178 mm
Width	l = 10 mm

Table 1. *Material properties and dimensions for the beam 1*

Present sandwich shell element		2D-discretisation (reference) [DAY 01]		Experimental results	
Frequency	Loss factor	Frequency	Loss factor	frequency	Loss factor
30.70	$3.58\ 10^{-2}$	30.60	$3.58\ 10^{-2}$	34	$2.75\ 10^{-2}$
156.57	$7.54\ 10^{-1}$	156.27	$7.56\ 10^{-1}$	147	$5.03\ 10^{-1}$
382.98	$7.54\ 10^{-1}$	382.77	$7.52\ 10^{-1}$	357	$6.68\ 10^{-1}$

Table 2. *Natural frequencies and loss factors of the beam 1 at 60°C. The solver is ANM*

One observes that the presented shell/volume/shell element gives almost the same results as the reference two-dimensional analysis. This validates all the assumptions of our sandwich shell element, especially the basic formulae [4] for the displacement in the core. Some differences exist between the two numerical simulations and experimental results. They are mainly due to the identification of the viscoelastic properties of the central layer. Indeed, it is not obvious that the behavior of the pure polymer is the same as in the sandwich. The identification by the WLF [FER 80] law is also questionable.

3.1.2. *Beam 2*

The second test has been studied quite extensively in [SON 81] and [JOH 81] from theoretical and experimental points of view. The viscoelastic properties of the central layer are introduced by a complex modulus that is assumed to be constant. This modulus is written in the form:

$$E(\omega) = E_c(1+i\eta_c)$$

where E_c is the real elastic modulus and η_c is the core loss factor. The beam dimensions and material properties are presented in Table 3.

Elastic layers	
Young's modulus :	$E_l = 6.9\ 10^{10}$ N/m^2
Poisson's ratio :	$\upsilon_l = 0.3$
Mass density :	$\rho_l = 2766$ kg/m^3
Thickness :	$h_l = 1.524$ mm
Viscoelastic layer	
Elastic modulus :	$E_c = 1794\ 10^3$ N/m^2
Poisson's ratio :	$\upsilon_c = 0.3$
Mass density :	$\rho_c = 968.1$ kg/m^3
Thickness :	$h_c = 0.127$ mm
Whole beam	
Length	L = 178 mm
Width	l = 12.7 mm

Table 3. *Material properties and dimensions for the beam 2*

We have computed the "exact" values of the six first complex eigenfrequencies, *i.e.* of the natural frequency and of the loss factor, the solver being the Asymptotic Numerical Method [DAY 01]. As for the previous example, we have compared the results given by our plate/volume/plate element with those obtained with a 2D-analysis and a very fine mesh : 6 quadratic elements through the thickness and 30 along the length of the beam. The plate/volume/plate computations have been performed with a mesh involving 40 elements along the length. As compared with the reference 2D-analysis, our new element gives very good results for the loss factor and a very good approximation for the natural frequencies (the difference is less than 1% for the two first ones, less than 2% for the four other ones, see Table 4). Thus, one sees the advantage of the present element, because one recovers about the same results with much fewer degrees of freedom through the thickness (8 in our approach as compared with 13x3=39 degrees of freedom for six quadratic volume elements or with 7x3=21 degrees of freedom for three volume elements). Nevertheless, the DKT elements are not so as accurate as the quadratic volume element because the membrane problem is poorly discretized by constant strain triangles and the interpolation in the mass matrix is linear. To improve the performance of the sandwich shell element, it would be interesting to replace the DKT by a quadratic shell element : this approach will be presented elsewhere. Note that the numerical results are also consistent with those of the literature [SON 81].

Present sandwich shell element		2D-discretisation results (reference) [DAY 01]		Results from [SON 81]	
Frequency	Loss factor	Frequency	Loss factor	Frequency	Loss factor
65.0	$8.160\ 10^{-2}$	64.5	$8.160\ 10^{-2}$	64.7	$8.250\ 10^{-2}$
300.0	$7.170\ 10^{-2}$	297.5	$7.170\ 10^{-2}$	298.0	$7.150\ 10^{-2}$
753.3	$4.560\ 10^{-2}$	745.2	$4.590\ 10^{-2}$	748.2	$4.599\ 10^{-2}$
1416.0	$2.634\ 10^{-2}$	1398.7	$2.640\ 10^{-2}$	1409.5	$2.646\ 10^{-2}$
2306.0	$1.680\ 10^{-2}$	2271.2	$1.680\ 10^{-2}$	2305.0	$1.680\ 10^{-2}$
3428.3	$1.140\ 10^{-2}$	3363.5	$1.140\ 10^{-2}$	3447.0	$1.140\ 10^{-2}$

Table 4. *Natural frequencies and loss factors of the beam 2 when η_c =0.3*

3.1.3. *About the validity of the sandwich element*

The two previous examples establish that the present finite element yields good results for a class of sandwich structures. This class is characterized by thin and soft cores, *i.e.* h_l/h_c about 10 and E_l/E_c greater than 10^3. These orders of magnitude correspond to metal/polymer sandwiches, that are especially designed to reduce noise for widely distributed products.

In other areas as aerospace industry, sandwich structures are build up in view of large bending stiffness. In these sandwiches, the core thickness is at least of the same order as the face one. One can wonder whether the present element still works in such cases. In this respect, we did new comparisons between our sandwich element and the 2D element, that is considered as the reference, for two thickness ratios h_c/h_l= 1 and 7 three values of the core modulus E_c = 27.216 10^4, 27.216 10^6 and 27.216 10^8. The Young modulus of the core is assumed to be complex and constant with η_c=0.2. The other data are the same as for beam 1. From the results presented in Table 5, the sandwich element remains more and less valid for h_c=h_l, the error being about 2.6% for frequency. As expected, it yields more significant errors for the thick core case. Nevertheless, the loss factor is correctly evaluated in all tested cases. In such case, a finer description of the core stress is necessary see for instance [GAN 96].

	E_l/E_c =77.16 10^4		E_l/E_c =77.16 10^4		E_l/E_c =77.16	
	f	η/η_c	f	η/η_c	f	η/η_c
h_c/h_l=1	17.73 * 18.20 +	$2.40\ 10^{-1}$ $2.40\ 10^{-1}$	45.95 47.15	$2.72\ 10^{-1}$ $2.72\ 10^{-1}$	55.06 56.50	$5.53\ 10^{-3}$ $5.78\ 10^{-3}$
h_c/h_l=7	17.04 19.52	$4.12\ 10^{-1}$ $4.12\ 10^{-1}$	81.85 93.58	$7.21\ 10^{-1}$ $7.20\ 10^{-1}$	175.47 200.45	$4.89\ 10^{-2}$ $4.91\ 10^{-2}$

Table 5. *First frequency and loss factor ratio of beam 1 predicted by the ANM*

***** 2D-discretisation results, + Present sandwich element results.

3.2. *Sandwich plate*

In this section, the present finite element analysis is applied to the vibrations of rectangular plates . Two tests are presented, that differ from one another by the plate dimensions, the material properties and the boundary conditions.

3.2.1. *Plate 1*

This example has been studied experimentally by the steel company USINOR. The material properties are the same as for the beam 1 and the plate dimensions are given in Table 6. In the experiment, the plate was suspended and excited by a harmonic force at its center. The polymer was considered at 50°C. The experimental values of the natural frequency and of the loss factor are presented in Table 7 and compared to those obtained by the proposed finite element analysis coupling to the ANM. Free edge boundary conditions were assumed. One can note a good agreement between the numerical and experimental results.

Plate 1	
Length	L= 300mm
Width	l = 30 mm
Thickness of the face layer	h_1 = 0.4 mm
Thickness of the core	h_c= 0.05 mm

Table 6. *Dimensions for the plate 1*

Present sandwich shell element		Experimental results	
Frequency(Hz)	Loss factor	Frequency(Hz)	Loss factor
48	$2.42\ 10^{-2}$	48	$2.25\ 10^{-2}$
213	$8.80\ 10^{-2}$	211	$8.45\ 10^{-2}$
450	$1.25\ 10^{-2}$	459	$1.37\ 10^{-2}$
760	$1.27\ 10^{-2}$	797	$1.91\ 10^{-2}$

Table 7. *Natural frequencies and loss factors of the plate 1 at 50°C*

3.2.2. *Plate 2*

The following example is a sandwich plate which is simply supported along the edges. The same plate has been studied in [CUP 95] and [JOH 81]. The Young modulus of the viscoelastic layer is assumed to be complex and constant as for beam 2. In reference [JOH 81], the loss factor and natural frequency of the plate were predicted by NASTRAN program using the MSEM. The elastic layers were modeled with quadrilateral or triangular plate elements having two rotations and three translations per node. The viscoelastic layer was modeled with solid elements. Another analytical-numerical predictions of the loss factor and natural frequency are given in reference [CUP 95]. Of course, the latter is only applied when the plate is simply supported. Here, we limit ourselves to an approximate computation of eigenvalues by MSEM. In Table 9, the results with the proposed finite element are presented and compared with those in references [CUP 95], [JOH 81], that have also used MSEM. The three solutions differ only by the discretization method.

Elastic layers	
Young's modulus :	$E_1 = 6.89\ 10^{10}$ N/m^2
Poisson's ratio :	$\upsilon_c = 0.3$
Mass density :	$\rho_1 = 2740$ kg/m^3
Thickness :	$h_1 = 0.762$ mm
Viscoelastic layer	
Elastic modulus :	$E_c = 2670.08\ 10^3$ N/m^2
Poisson's ratio :	$\upsilon_c = 0.49$
Mass density :	$\rho_c = 999$ kg/m^3
Thickness :	$h_c = 0.254$ mm
Material damping	$\eta_c = 0.5$
Whole plate	
Length	L = 348 mm
Width	l = 304.8 mm

Table 8. *Material properties and dimensions for the plate 2*

Present sandwich shell element		NASTRAN results [JON 81]		Analytical results [CUP 95]	
frequency	Loss factor	frequency	Loss factor	frequency	Loss factor
60.3	0.205	57.4	0.176	60.3	0.190
116.4	0.210	113.2	0.188	115.4	0.203
131.9	0.204	129.3	0.188	130.6	0.199
181.3	0.183	179.3	0.153	178.7	0.181
198.9	0.176	196.0	0.153	195.7	0.174

Table 9. *Natural frequencies and loss factors of the plate 2. Solutions by MSEM*

The agreement between the results obtained using the proposed finite method and the analytical results is good, because the two techniques are based on the same assumptions (Section 2). Some differences are observed with the NASTRAN computation that requires more degrees of freedom through the thickness. This can be due to an inconsistent account of the continuity of the displacement field. The relevance of linear 3D elements for bending analysis is also questionable.

3.3. *Sandwich cylindrical shell*

In this section, the presented finite element method is applied to discretize a sandwich cylindrical shell with diametrically opposed rectangular cutouts, which are placed in the middle between the two ends (Fig. 2). The geometrical data are presented in Table 10. Only one eighth of the shell has been discretized. The edge of the cylinder is clamped, the cutouts are stress free. Symmetry boundary conditions are prescribed elsewhere. The mesh involves 6920 degrees of freedom. The Young modulus of the viscoelastic layer is assumed to be complex and constant as in plate 1. The material data are the same as in table 1 and two cases have been considered for the core damping : a small damping (η_c=0.2) and a large one (η_c=1.0). In Tables 11, 12, we present the values of the loss factor and natural frequency predicted by the ANM (exact) and by the MSEM (approximated).

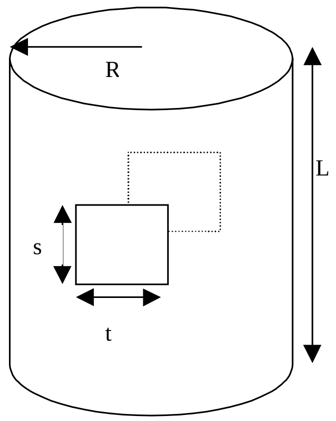

Figure 2. *Sandwich cylindrical shell*

Whole cylindrical shell	
Length	L = 200 mm
Radius	R = 100 mm
Thickness of elastic layers	h_l = 0.6 mm
Thickness of the core	h_c = 0.045 mm
Rectangular cutout	
Length	s = 80 mm
Width	t = 79.5 mm

Table 10. *Dimensions of the sandwich cylindrical shell*

ANM results		MSEM results	
Frequency(Hz) ($X10^4$)	Loss factor	Frequency(Hz) ($X10^4$)	Loss factor
7.7	$1.29\ 10^{-2}$	7.7	$1.45\ 10^{-2}$
43.4	$1.97\ 10^{-2}$	43.1	$2.21\ 10^{-2}$
126.3	$2.66\ 10^{-2}$	125.1	$2.96\ 10^{-2}$
260.1	$3.82\ 10^{-2}$	257.2	$4.14\ 10^{-1}$
279.1	$6.32\ 10^{-1}$	273.3	$6.92\ 10^{-1}$

Table 11. *Natural frequencies and loss factors of the cylindrical shell when* η_c*=0.2*

ANM results		MSEM results	
Frequency(Hz) ($X10^4$)	Loss factor	Frequency(Hz) ($X10^4$)	Loss factor
7.8	$4.95\ 10^{-2}$	7.7	$7.28\ 10^{-2}$
44.0	$7.58\ 10^{-2}$	43.1	$1.10\ 10^{-2}$
128.8	$1.03\ 10^{-2}$	125.1	$1.48\ 10^{-2}$
267.0	$1.54\ 10^{-2}$	257.2	$2.07\ 10^{-1}$
294.4	$2.37\ 10^{-1}$	273.3	$3.46\ 10^{-1}$

Table 12. *Natural frequencies and loss factors of the cylindrical shell when* η_c*=1.0*

It appears that the Modal Strain Energy Method involves significant errors in the evaluation of the loss factors, especially for large damping: the error is about 10% for a small η_c and 30% for large one. Here we limited ourselves to the cheap version of MSEM by computing the strain energy corresponding to the real mode. More relevant values of the loss factor could be obtained by introducing a complex mode [SON 81], but this leads to a higher computational cost because the computation of the complex mode involves the inversion of a complex matrix.

4. Conclusions

In this paper, a class of finite elements has been developed for damped sandwich structures. The basic idea is to extend in a finite element framework classical assumptions about the displacement field in a sandwich [RAO 78]. The obtained finite elements have only eight degrees of freedom per node that are the longitudinal displacements of the elastic layers, the deflection and three rotations. In this paper, we have only tested a triangular plate element with three nodes, the formulation being based on the Discrete Kirchhoff Theory. Several examples have been presented to validate this technique for different geometries (beam, plate, and cylindrical shell). A good agreement has been obtained between the proposed method and known experimental, analytical and numerical results.

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