A finite element algorithm for cavitation in hydrodynamic lubrication

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ABSTRACT. Issued from a mass conservation cavitation model for a slightly compressible fluid, a specific finite element discretization and a related fixed-point algorithm are introduced. Convergence of this algorithm is proved. Moreover, the behavior of the solution of the discrete problem towards the solution of the continuous problem is studied. Numerical results are given for one and two-dimensional problems.

RÉSUMÉ. A partir d'un modèle de cavitation conservatif pour un fluide faiblement compressible nous introduisons une discrétisation éléments finis spécifique et proposons un algorithme de résolution de type point fixe. La convergence de cet algorithme est démontrée. Nous étudions aussi la convergence de la solution du problème discrétisé vers celle du problème continu. Des résultats numériques sont présentés dans le cas de problèmes unidimensionnels et bidimensionnels .

KEYWORDS: cavitating fluid film, hydrodynamic lubrication, Reynolds equation, numerical algorithm, finite element analysis, convergence proofs.

MOTS-CLÉS : lubrification hydrodynamique, cavitation, Equation de Reynolds, algorithme, éléments finis, convergence.

1. Introduction

Cavitation is a well-known phenomenon in fluid mechanic when fluid is no longer homogeneous and takes diphasic aspect with the appearance of air bubbles. This kind of phenomenon can be found in most of the lubricated devices and engineers try to describe it in terms of the physical variable commonly used in the context of the lubrication: the pressure. This is not an easy thing because whereas cavitation is mostly recognized as a three-dimensional phenomenon even for thin film flow, the pressure in lubrication problems is assumed to be constant through the gap height. Common practice was to modify full film results by setting the negative pressures (or the pressures below the cavitation pressure level) to the cavitation pressure value, obtaining by the way the so-called half-Sommerfeld model [DOW 75]. Another method was to use the same procedure inside an over-relaxation Gauss Seidel iterative loop. It can be proved [CRY 71] that it is equivalent to consider the Reynolds boundary condition $\partial p/\partial n = 0$ on the free boundary between the full film and the cavitation area. Both methods are easy to implement and reasonably accurate in term of load capacity. However the mass flow conservation is often violated.

The starting point of another widely used model comes back to the work of [FLO 57] in which a new variable is introduced to describe the cavitation as « a proportion », namely θ, of the surface locally covered by the fluid, taking into account the presence of air bubbles. This model which has the advantage to keep the two dimensional feature in the description of the flow can be written in the form of the following free boundary problem.

Find p and θ and a splitting of the domain into Ω^+ and Ω^0 such that:

$$p > 0, \quad \theta = 1, \quad div\left(\frac{h^3}{12\mu}\vec{\nabla}p\right) = div\left(\frac{h}{2}\vec{U}\right) \quad on \quad \Omega^+ \qquad [1]$$

$$p = 0, \quad 0 < \theta < 1, \quad div\left(\frac{h}{2}\theta\vec{U}\right) = 0 \quad on \quad \Omega^0 \qquad [2]$$

$$\frac{h^3}{6\mu}\frac{\partial p}{\partial n} = (1-\theta h)\vec{U}.\vec{n} \quad on\ the\ \ free\ boundary\ \overline{\Omega^+} \cap \overline{\Omega^0} \qquad [3]$$

h is the gap, μ is the viscosity, $\vec{U}$ is related to the velocities of the surrounding surfaces and *n* is the unitary normal vector to the free boundary.

From a mathematical aspect, this equation is highly nonlinear, mostly with respect to the relation between *p* and θ. Moreover it is an elliptic equation in the non-cavitated area where only *p* acts and an hyperbolic one in the cavitated area where only θ appears. In some sense, the free boundary condition [3] can be considered as a Rankine-Hugoniot condition [DOW 75]. So it is not surprising that solving numerically this equation has not been an easy job and various strategies have been proposed. In [LUN 69] an iterative strategy "try and check" for the free boundary is proposed without too much details. Elrod introduced in [ELR 81] a slight compressibility of the fluid, so that equations are easier to numerically manage. Moreover, this model can cope with starvation effects. Although mass flow conservation is gained, oscillations near the cavitation boundary often occur. To prevent such oscillations, a modification of the Elrod algorithm by an upwind procedure to cope with the convection term is proposed in [VIJ 89]. In [BAY 86] a new algorithm is introduced in which at each node, only one of the two unknowns p or θ acts. More recently, in [BON 95], a modified version of the Murty's algorithm is used. Based upon the complementary formulation of the Reynolds variational inequality, Murty's scheme [MUR 74] is applied without modification around the rupture free boundary where condition $\partial p/\partial n = 0$ is valid and has to be modified in the vicinity of the reformation free boundaries. Another approach uses time marching approach so that the solutions of equations [1][2][3] are obtained as stabilized solutions after time integration of the unsteady Reynolds equations. Various time discretization methods can be used like characteristics method [BAY 90, BAY 98] or space-time conservation elements [CIO 00]. In that last paper, evidence is shown of discrepancies in the results obtained by various methods, even for a one dimensional slider bearing, especially if starved inlet is considered.

All these algorithms are mostly finite difference one and few convergence theorems neither for the iterative process itself nor for the validity of the discrete approximation with respect to the continuous problem seem to exist. It is the goal of this paper to support by a rigorous way a cavitation model closed to the one proposed by Elrod in [ELR 81], and a related finite element algorithm whose convergence will be proved.

In the second section, the variational formulation of the initial (p,θ) free boundary problem is recalled and it is shown how to derive by different ways some "compressible" approximations. A fixed-point algorithm is proposed in the third section, deduced from the one proposed in [ALT 80] for the computation of a free boundary problem in a porous medium. The basic idea being to introduce at the discrete level a unique unknown which describes at each node either the pressure or the saturation. In the present situation of lubricated slightly compressible flow, it will be shown how to adapt this procedure. Sufficient conditions for its convergence are given when the compressibility parameter is small. In the following section, convergence of the discrete solution to the one of the continuous problem is studied and a specific finite element is proposed. At last, some numerical results are given for one and two-dimensional problems.

2. Cavitation models in lubrication

2.1. *Classical models and related approximations*

It is convenient to write the problem in a divergence form which does not take into account the boundary conditions, so [1][2][3] becomes:

Find $p \geq 0, \quad 0 \leq \theta \leq 1, \quad \theta = 1 \quad if \quad p > 0$ such that:

$$div\left(\frac{H^3}{12\mu}\vec{\nabla}p - \theta\frac{H}{2}\vec{U}\right) = 0 \qquad [4]$$

where the derivation is taken in a distributional sense.

For the mathematical aspect, the main difficulty is linked to the $(p \rightarrow \theta)$ relation which can be described by the Heaviside graph Y (figure 1) and gives evidence to the fact that for p equals zero, θ is not uniquely determined.

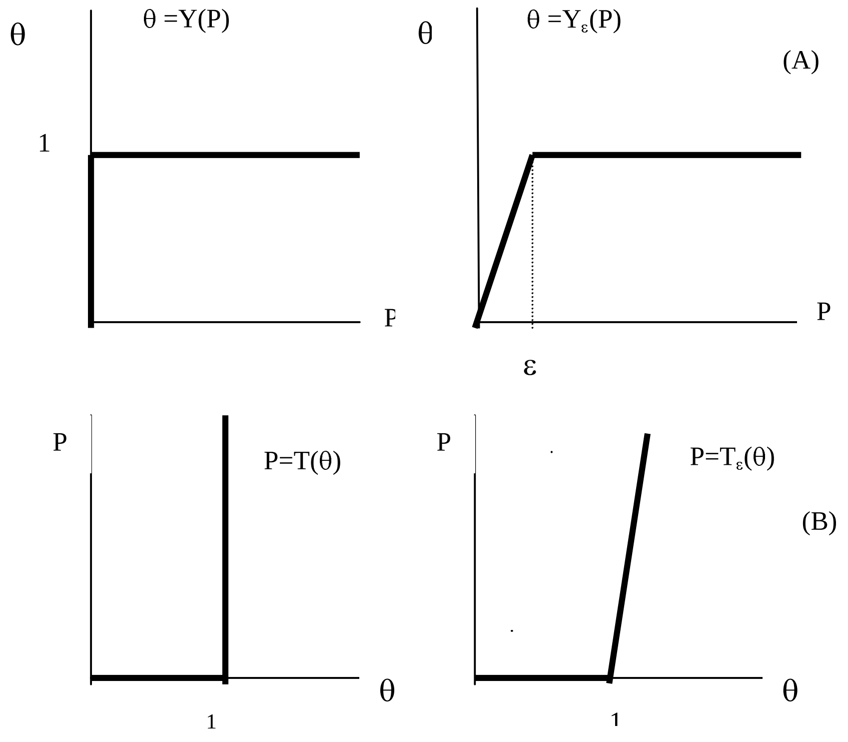

Figure 1. *Graph of the Saturation-Pressure (A) and of the Pressure-Saturation (B) relation and their related approximations*

To overcome the difficulty, one way [BAY 84] is to approximate the Heaviside graph by a function Y_ε (figure 1) such that, for a small parameter $\varepsilon > 0$:

$$Y_\varepsilon(t) = \begin{cases} \dfrac{t}{\varepsilon} & if \quad 0 < t < \varepsilon \\ 1 & if \quad t > \varepsilon \end{cases}$$

Then for each ε , one has to solve the problem with only one unknown.

Find $p,\ p \geq 0$ such that:

$$div\left(\frac{H^3}{12\mu}\vec{\nabla}p\right) = div\,(Y_\varepsilon(p)\frac{H}{2}\vec{U}) \qquad [5]$$

This problem is still a nonlinear one, but the formulation gives way to a fixed point procedure in a somewhat natural manner by fixing p in the right hand side of [5] and then by solving a sequence of classical linear elliptic problems. Unfortunately, this approach does not lead to numerical convergence as cyclic phenomena are observed.

Another way to obtain a problem with only one unknown is to approximate in figure 1 the relation $p = T(\theta)$ instead of the relation $\theta = Y(p)$ by introducing a function T_ε (figure 1) defined by:

$$T_\varepsilon(t) = \begin{cases} 0 & if \quad t \leq 1 \\ \dfrac{t-1}{\varepsilon} & if \quad t > 1 \end{cases}$$

This approximation leads to the following relation:

$p = \dfrac{1}{\varepsilon}(\theta - 1)^+$, where $(\ )^+$ is the positive part of ( ).

The only unknown kept is θ and the related problem to be solved becomes:

Find $\theta.\quad \theta \geq 0$ such that:

$$div\quad (\frac{H^3}{12\mu}\nabla(\theta-1)^+) = div\,(\varepsilon\theta\frac{H}{2}\vec{U}) \qquad [6]$$

Although being a problem with only one unknown, this approximation seems to lead to a slightly more complicated problem than [5]. The primary interest of [6] is that it can be obtained in a more physical way by taking into account the compressibility of the fluid, and it is the basis of the widely used algorithm proposed by [ELR 81].

2.2. *Cavitation and compressibility*

So far, the lubricant has been supposed to be incompressible. However, the Reynolds equation for a compressible flow is well known and can be written [FRE 90]:

$$div\ (\rho H^3 \vec{\nabla} p / 12\mu) = div\ (\rho H / 2\vec{U}) \qquad [7]$$

where ρ is the density of the fluid. For a slightly compressible fluid like water or oil, the law linking p and ρ is usually described by:

$$p \cong \frac{1}{\varepsilon}\log(\rho), \quad \rho > 1, \quad p > 0 \qquad [8]$$

where ε is of the order of magnitude of 10^{-6}. Taking into account the possible existence of a cavitation area where p is zero and $\rho \le 1$ leads to generalize [8] in :

$$p \cong \frac{1}{\varepsilon}\log((\rho - 1)^+ + 1), \quad \rho \ge 0 \qquad [9]$$

Rewriting [7] in term of ρ, we gain with derivation in the distributional sense :

$$div\left(\frac{H^3}{12\mu}\vec{\nabla}(\rho - 1)^+ - \varepsilon\rho\frac{H}{2}\vec{U}\right) = 0 \qquad [10]$$

which is nothing else than equation [6] with ρ instead of θ.

It is to be note that once $\theta \equiv \rho$ is computed, then the pressure deduced by these two approximations will not be exactly the same :

$$p = (either) \quad \frac{1}{\varepsilon}(\theta - 1)^+, \quad (or) \quad \frac{1}{\varepsilon}\log((\rho - 1)^+ + 1)$$

However, for small compressibility effects $\log(1 + (\rho - 1)^+)$ is equivalent to $(\rho - 1)^+$ so that these two values of p are very close.

2.3. *Variational formulation for the compressible model*

Introducing $u = \dfrac{\rho - 1}{\varepsilon}$ as the primary unknown, equation [10] becomes:

$$div\left(\frac{H^3}{12\mu}\vec{\nabla}u^+\right) = div\,((1+\varepsilon u)\,\frac{H}{2}\vec{U}) \qquad [11]$$

From this section on, to be able to define the useful functional spaces, we consider a specific device such as a journal bearing with a circumferential supply line. However, the same kind of study is valid for other boundary conditions, corresponding to other working spaces (see sec 2.4)

Let us denote by $(0,2\pi R)\times(0,L)$ the surface of the (half-)bearing, p the pressure, s the relative velocity in the circumferential direction of the shaft with respect to the bearing and H the gap. We introduce rescaled variables (x, y) so that:

$$(x, y) \in \Omega = (0,2\pi)\times(0,1)$$

Boundary conditions are assumed for the rescaled pressure $\overline{p} = \dfrac{p}{6\mu sR}$

$\overline{p} = \overline{p}_a > 0$ along the supply line y = 1

$\overline{p} = 0$ (the atmospheric pressure) along the line y = 0

$\overline{p}$ is 2π -periodic in the x-direction.

To introduce the variational formulation of [11], we consider different working spaces for u and for u^+ because H^1 regularity is needed for u^+ but L^2 regularity is sufficient for u.

Recalling the change of unknowns:

$$u = \frac{\rho - 1}{\varepsilon}, \quad \overline{p} = \tfrac{1}{\varepsilon}\log((\rho-1)^+ + 1)$$

we set:

$$V_0 = \left\{ \varphi \in H^1(\Omega),\ 2\pi - periodic\ in\ x,\ \varphi = 0\ for\ y = 0\ and\ y = 1 \right\}$$

$$Va = \left\{ \varphi \in H^1(\Omega),\ 2\pi - periodic\ \ in\ x,\ \varphi = 0\ for\ y = 0,\ \varphi = u_a\ for\ y = 1 \right\}$$

where:

$$u_a = \left(e^{\varepsilon p_a} - 1\right)/\,\varepsilon$$

Then, the variational formulation of [11] is:

Find $u \in L^2(\Omega)$ such that $u^+ \in V_a$,

$$\int_\Omega H^3 \left(\frac{\partial u^+}{\partial x} \frac{\partial \varphi}{\partial x} + (\frac{R}{L})^2 \frac{\partial u^+}{\partial Y} \frac{\partial \varphi}{\partial y} \right) dxdy = \int_\Omega (1+\varepsilon(u^+ - u^-))H \frac{\partial \varphi}{\partial x} dxdy, \quad \forall \varphi \in V_0 \text{ [12]}$$

recalling that $u^+ = \sup(u, 0)$ $u^- = \sup(-u, 0)$ so that $u = u^+ - u^-$.

2.4. *Models for other boundary conditions*

For a journal bearing with a longitudinal supply line located on $\{x = 0\}$ at a given pressure $p_a > 0$ and zero pressure boundary conditions on $\{y = 0\}, \{y = 1\}$ *and* $\{x = 2\pi\}$, formulation [12] is still valid by changing the definitions of V_0 and V_a.

$$V_0 = \{\varphi \in H^1(\Omega), \varphi(0, y) = \varphi(2\pi, y) = \varphi(x,0) = \varphi(x,1) = 0\}$$
$$V_a = \{\varphi \in H^1(\Omega), \varphi(0, y) = u_a, \varphi(x,0) = \varphi(x,1) = \varphi(2\pi, y) = 0\}$$

If the supply pressure is equal to the cavitation pressure (=0), then the cavitation may occur by a starvation effect at the vicinity of the supply line $\{x = 0\}$. It is then necessary to introduce an additional data g(y) on $\{x = 0\}$ related to the inlet flow. The variational formulation becomes:

Find $u \in L^2(\Omega)$, such that $u^+ \in V_1$ and:

$$\int_\Omega H^3 \left(\frac{\partial u^+}{\partial x} \frac{\partial \varphi}{\partial x} + (\frac{R}{L})^2 \frac{\partial u^+}{\partial y} \frac{\partial \varphi}{\partial y} \right) dxdy = \int_\Omega (1+\varepsilon(u^+ - u^-))H \frac{\partial \varphi}{\partial x} dxdy - \int_{\{x=0\}} g(y) H(0) \varphi(0,y) dy$$
$$\forall \varphi \in V_1 = \{\varphi \in H^1(\Omega), \varphi(x, 0) = \varphi(x,1) = \varphi(2\pi, 0) = 0 \}$$

It is to be noticed that the zero boundary conditions on $\{x = 0\}$ is not included in the formulation and is not mandatory. It will be only obtained [BAY 86] when the input flow whose value is controlled by the data g(y) is not great enough to fill the gap around the boundary $\{x = 0\}$. For a given gap H, the maximum and minimum value for g(y) can be found to prevent simultaneously overpressure on Γ_0 and allows the non-cavitated area Ω^+ to be non empty.

3. Finite element approximation

The goal is to define a discrete version of the variational formulation [12]. To that purpose let us consider $\tau_h = \{T_i\}$ a family of triangulations relying on a small parameter $h > 0$, each T_i satisfying the suitable assumptions for a basis of finite elements (P_1-Lagrange for instance). Let $\{m_i\}, i \in N_h$ be the set of the nodes in $\overline{\Omega}$.

Due to the boundary conditions, we introduce some subsets of the index set:

$$N_{1h} = \{i \in N_h,\ m_i \in \overline{\Omega},\ m_i \notin \overline{\Gamma}_0 \cup \overline{\Gamma}_a\}$$

$$N_{oh} = \{i \in N_h, m_i \in \overline{\Gamma}_0\}, \quad N_{ah} = \{i \in N_h, m_i \in \overline{\Gamma}_a\}$$

where $\Gamma_0 = \{(x, y), 0 < x < 2\pi,\ y = 0\}$ and $\Gamma_a = \{(x, y), 0 < x < 2\pi,\ y = 1\}$.

To approximate u^+ which lies in $H^1(\Omega)$, we consider a basis of finite element $\{\omega_i\}$, with $\omega_i \geq 0$, such that the associated finite dimensional subspace V_h is included in:

$$V_{per} = \{\phi \in H^1(\Omega),\ 2\pi - x\ periodic\}$$

To approximate u which belongs to $L^2(\Omega)$, we consider a set of characteristic functions $\{\chi_i\}$ such that $\sum_{i \in Nh} \chi_i(x) = 1 \quad \forall x \in \Omega$. The corresponding finite dimensional spaces are:

$$V_h = \left\{ v = \sum_i v_i \omega_i,\ i \in N_h, v_i \in R \right\}$$

$$V_{oh} = \{ v \in V_h, \quad v_i = 0\ \forall\ i \in N_{0h} \cup N_{ah} \}$$

$$L_h = \left\{ \gamma = \sum_i \gamma_i \chi_i, i \in N_h, \gamma_i \in R \right\}$$

We introduce the bilinear form :

$$a_r(v, w) = \int_\Omega H^3 \left(\frac{\partial v}{\partial x}\frac{\partial w}{\partial x} + \left(\frac{R}{L}\right)^2 \frac{\partial v}{\partial y}\frac{\partial w}{\partial y} \right) dxdy \qquad [13]$$

This leads to the discrete finite dimensional approximation :

find $(v_h, \gamma_h) \in V_h \times L_h$ such that:

$$a_r(v_h, \phi_h) = \int_\Omega \left(1 + \varepsilon(v_h - \gamma_h\right) H \frac{\partial \phi_h}{\partial x} dxdy \qquad [14]$$

for every ϕ_h in V_{0h}, with the constraints :

$$v_h \geq 0 \; on \; \Omega, \; v_h = 0 \; on \; \Gamma_0, \; v_h = u_a \; on \; \Gamma_a \qquad [15]$$

$$0 \leq \gamma_h \leq \frac{1}{\varepsilon} \text{ on } \Omega, \; \gamma_h = 0 \; on \; \Gamma_a \qquad [16]$$

$$v_h(m_i)\gamma_h(m_i) = 0 \quad \forall i \in N_h \qquad [17]$$

It is to be noticed however that the number of degrees of freedom for γ_h is greater than the one for v_h, which is exactly the number of equations implicitly contained in $[14]$. More precisely, $v_h = 0 \; on \; \Gamma_0$ while γ_h is unknown.

To obtain additional equations, keeping in mind the mass flow conservation equation in the cavitation area, we set:

$$\int_\Omega \left(1 - \varepsilon\gamma_h\right) H(x) \frac{\partial \omega_i}{\partial x} dxdy = 0 \quad \forall i \in N_{oh} \qquad [18]$$

It will be proved further that [11]-[18] is a well-posed problem whose solution converges towards that of the continuous problem.

The matrices and vectors $A = (a_{ij}), B = (b_{ij}), E = (e_{ij}), F = (f_i)$ are defined as follows:

$$a_{ij} = \int_\Omega H^3(x) \left(\frac{\partial \omega_i}{\partial x} \frac{\partial \omega_j}{\partial x} + \left(\frac{R}{L}\right)^2 \frac{\partial \omega_i}{\partial y} \frac{\partial \omega_j}{\partial y} \right) dxdy, \quad i \in N_{1h}, j \in N_h$$

$$b_{ij} = \int_\Omega H(x) \omega_j \frac{\partial \omega_i}{\partial x} dxdy \quad i \in N_{1h}, j \in N_h$$

$$e_{ij} = \int_\Omega H(x) \chi_j \frac{\partial \omega_i}{\partial x} dxdy \quad i \in N_{oh} \cup N_{1h}, j \in N_h$$

$$f_i = \int_\Omega H(x) \frac{\partial \omega_i}{\partial x} dxdy \quad i \in N_{oh} \cup N_{1h}$$

$$a_{ij} = b_{ij} = 0 \quad if \quad i \in N_{0h} \quad j \in N_h$$

In the following, we will introduce vectors $V_h, \Gamma_h \ldots$associated to functions like $v_h, \gamma_h \ldots$ For the sake of simplicity, the components will be denoted without the subscript h, for example $V_h = (v_i)$ *and* $\Gamma_h = (\gamma_i), i \in N_h$.

Using the previous notations, we get:

$$(A - \varepsilon B) V_h + \varepsilon \Gamma_h = F$$

Taking [14] [15] [16] into account, we define the vector $U_h = (u_i)_{i=1}^{N_h}$ such that:

$$u_i^+ = \max(u_i, 0) = v_i, \; u_i^- = \max(-u_i, 0) = \gamma_i \qquad [19]$$

Then using [19], problem [14]-[18] is equivalent to problem (Q):

$$problem\ (Q) \quad \left| \begin{array}{l} Find\ U_h \in R^{N_h}\ such\ that \\ (A - \varepsilon B) U_h^+ + \varepsilon E U_h^- = F \\ u_i = u_a > 0 \quad \forall i \in N_{ah} \quad u_i^+ = 0 \quad \forall i \in N_{oh} \\ 1 + \varepsilon\, u_i \geq 0 \quad \forall i \in N_h \end{array} \right.$$

In order to prove the existence of a solution of the discrete problem (Q), a fixed point form associated to U_h is well adapted. Let us define an application C_h from R^{N_h} into $R^{N_{oh} \cup N_{1h}}$ by its components:

$$C_{hi}(U_h) = f_i - \sum_{j \neq i,\, j \in Nh} (\varepsilon e_{ij} u_j^- + a'_{ij}\, u_j^+) \quad i \in N_{0h} \cup N_{1h}$$

where $a'_{ij} = a_{ij} - \varepsilon b_{ij}$ and an application A_h from R^{N_h} into R^{N_h} :

$$A_{hi}(U_h) = \begin{cases} C_{hi}(U_h)/a'_{ii} & if\ C_{hi}(U_h) \geq 0\ and\ i \in N_{1h} & [20] \\ -C_{hi}(U_h)/\varepsilon e_{ii} & if\ C_{hi}(U_h) < 0\ and\ i \in N_{1h} & [21] \\ u_a & if \quad i \in N_{ah} & [22] \\ -C_{hi}(U_h)/\varepsilon e_{ii} & if\ C_{hi}(U_h) < 0 \quad and \quad i \in N_{oh} & [23] \\ 0 & if\ C_{hi}(U_h) \geq 0 \quad and \quad i \in N_{oh} & [24] \end{cases}$$

PROPOSITION 3.1.– *Under the assumptions:*

$$a'_{ii} > 0, a'_{ij} \leq 0, \quad \forall i \neq j, \quad i \in N_{ih}, \quad j \in N_h \qquad [25]$$

$$e_{ii} < 0, e_{ij} \geq 0, \quad \forall i \neq j, \quad i \in N_{ih} \cup N_{oh}, \; j \in N_h \qquad [26]$$

problem (Q) is equivalent to following problem (R) *problem (R)* : Find $U_h \in R^{N_h}$ such that $U_h = A_h(U_h),\ I + \varepsilon U_h \geq 0$

PROOF.–

Rewriting each component of a solution to *problem (Q)* and taking into account the sign conditions, we have:

$$a'_{ii}\ u_i^+ + \varepsilon e_{ii}\ u_i^- = C_{hi}(U_h)$$

Now, for $i \in N_{1h}$ and $u_i \geq 0$, $C_{hi}(U_h) \geq 0$ and we get [20] while for $i \in N_{1h}$ and $u_{hi} < 0$, $C_{hi}(U_h) < 0$ and we get [21]. If $i \in N_{oh}$, then $u_i^+ = 0$ and this implies $u_{h,i} \leq 0$ so that:

$$u_i = -\frac{C_{hi}(U_h)}{\varepsilon e_{ii}} \leq 0$$

If $C_{hi}(U_h) \geq 0$ we necessarily have [24], else [23]. So the equality $u_i = A_{hi}(U_h)$ is gained and the reverse is proved by the same arguments.

Now we will use monotonicity arguments to prove the existence result.

PROPOSITION 3.2.– *Under assumptions [25][26],* $W_h = (w_i)$ *with* $w_i = -1/\varepsilon$ *is a sub solution to problem (R).*

PROOF.– We have to prove that $A_{hi}(W_h) \geq -1/\varepsilon$ for every index *i*. This is obvious for positive components of $A_h(W_h)$ so that it suffices to consider the components such that $C_{hi}(W_h) < 0, \forall i \in N_{0h} \cup N_{1h}$.

$$C_{hi}(W_h) = f_i - \sum_{\substack{j \neq i \\ j \in N_h}} e_{ij} = \int_\Omega H(x)(1 - \sum_{j \neq i} \chi_j(x)) \frac{\partial \omega_i}{\partial x}\, dxdy$$

From assumption $\sum_{j \in N_h} \chi_j = 1$, we deduce that $C_{hi}(W_h) = e_{ii}$.

For $i \in N_{1h} \cup N_{oh}$, $e_{ii} < 0$, then $A_{hi}(W) = -C_{hi}(w)/\varepsilon e_{ii} = -1/\varepsilon$

To define an upper solution is less obvious. The cavitation phenomenon is a direct consequence of the lack of the maximum principle for the Reynolds equation as the left hand side does not have a constant sign. So introducing a discrete Reynolds-like equation with left hand side $\beta.\inf\left(\frac{dH}{dx}, 0\right)(\beta > 0)$ is a way to obtain a super solution for small ε. We set $\varepsilon_1 = \frac{H\min^3}{4\pi^2\alpha}$ where $\alpha = \max_{(x,y)\in\Omega} |H'(x)|$ and:

$$\varepsilon_2 = \frac{\beta - 1}{u_a} \text{ with } \beta > 1 \qquad [27]$$

PROPOSITION 3.3.– *For* $\varepsilon < \varepsilon_1$ *defined in [27], the problem:*

Find $\overline{w}_h$ *in* V_{oh} *such that :*

$$a_r(\overline{w}_h, \phi_h) - \varepsilon \int_\Omega H(x)\overline{w}_h \frac{\partial \phi_h}{\partial x} dxdy = -\beta \int_\Omega \inf\ (H'(x), 0)\phi_h dxdy, \quad \forall \phi_h \in V_{oh}$$

has a unique solution. Moreover $\overline{w}_h$ *is positive.*

PROOF.– Let us first prove that the bilinear form :

$$a(w,\phi) = a_r(w,\phi) - \varepsilon \int_\Omega H(x) w \frac{\partial \phi}{\partial x} dxdy$$

is V_{oh} elliptic.

$$a(\phi,\phi) = a_r(\phi,\phi) + \varepsilon \int_\Omega H'(x) \frac{\phi^2}{2} dxdy$$

Considering the subset A of Ω: $A = \{H'(x) < 0\}$ and using the Poincaré's inequality :

$$-\int_A H'(x)\phi^2 dxdy \le \int_\Omega \alpha\phi^2 dxdy \le 4\pi^2\alpha \int_\Omega (\frac{\partial \phi}{\partial x})^2 dxdy + \int_\Omega \left(\frac{R}{L}\right)^2 (\frac{\partial \phi}{\partial y})^2 dxdy$$

and we have:

$$a(\phi,\phi) \ge \int_\Omega \left(H_{\min}^3 - 4\pi^2\alpha\varepsilon\right)\left((\frac{\partial \phi}{\partial x})^2 + \left(\frac{R}{L}\right)^2 (\frac{\partial \phi}{\partial y})^2\right) dxdy \ge (\varepsilon_1 - \varepsilon) 4\pi^2 \alpha \|\phi\|_{V_{oh}}^2$$

The ellipticity induces existence and uniqueness for $\overline{w}_h$.

To prove the positivity of $\overline{w}_h$, it is not possible to choose $(\overline{w}_h)^- = \sup(-\overline{w}_h, 0)$ as a test function as in the continuous case, because the function $(\overline{w}_h)^-$ does not always belongs to V_h.

Let us introduce:

$$\bar{\phi}_h = \sum_{i \in N_{ih}} \bar{w}_i^- \omega_i$$

By construction, $\bar{\phi}_h$ belongs to V_{0h}, so that :

$$a(\bar{w}_h, \bar{\phi}_h) = -\beta \int_\Omega \inf\,(H'(x), 0)\bar{\phi}_h\, dxdy \qquad [28]$$

Moreover, by the definition of $\bar{\phi}_h$:

$$a(\bar{w}_h, \bar{\phi}_h) = -a(\bar{\phi}_h, \bar{\phi}_h) + \sum_{i,j \in N_{ih}} (a_{ij} - \varepsilon b_{ij}) \bar{w}_i^+ \times \bar{w}_j^- \qquad [29]$$

For $i = j$, $\bar{w}_i^+ \times \bar{w}_j^-$ is zero. For $i \neq j, a'_{ij} = a_{ij} - \varepsilon b_{ij}$ is negative so that [29] induces $a(\bar{w}_h, \bar{\phi}_h) \leq 0$, while [28] implies $a(\bar{w}_h, \bar{\phi}_h) \geq 0$ because β and $\bar{\phi}_h$ are positive. Then $a(\bar{w}_h, \bar{\phi}_h)$ is zero which implies in turn that

$$a\,(\bar{\phi}_h, \bar{\phi}_h) = 0$$

From the definition of $\bar{\phi}_h$, we gain $\bar{w}_i^- = 0$ for every i and the positivity of $\bar{w}_h$ is proved.

Now, $\bar{w}_h$ allows to get a super solution. Let us denote by $\bar{W}_h$ the vector on R^{N_h} with components $\bar{w}_i$ and Ψ_h the vector whose components are the values of the linear function $\Psi(x, y) = u_a y$ on each node, we have

PROPOSITION 3.4.– *Under assumptions [25] [26] and* $\varepsilon < \min(\varepsilon_1, \varepsilon_2)$ *(see [27]),* $\bar{\bar{W}}_h = \bar{W}_h + \Psi_h$ *is a super solution to the fixed point problem (R).*

PROOF.– As $\bar{\bar{W}}_h$ is positive and $\bar{\bar{w}}_i = 0$ for $i \in N_{0h}$ and $\bar{\bar{w}}_i = u_a$ for $i \in N_{ah}$, we only have to prove that $A_{hi}(\bar{\bar{W}}_h) \leq \bar{\bar{w}}_i$ for $i \in N_{1h}$ and $C_{hi}(\bar{\bar{W}}_h) \geq 0$ (see [20]. Due to the fact that $\bar{w}_i \geq 0$, this is equivalent to:

$$J_i = f_i - \sum_{j \in N_h} a'_{ij}\,(\bar{w}_j - \Psi_j) \leq 0$$

We compute for any ϕ in $V_h \subset V$:

$$a_R(u_a y, \phi) = u_a \int_\Omega H^3(x) \frac{\partial \phi}{\partial y}\, dxdy = 0$$

This can be written :

$$\sum_{j\in N_h} a_{ij}\psi_j = 0, \quad \forall i \in N_{1h}$$

so that :

$$J_i = f_i - \sum_{j\in N_h} a'_{ij}\,\overline{w}_j + \varepsilon\sum_j b_{ij}\psi_j \;.$$

From the definition of $\overline{W_h}$, we have:

$$\sum_{j\in N_h} a'_{ij}\,\overline{w_j} = \beta\int_\Omega \inf(H'(x),0)\omega_i\,dxdy$$

then:

$$J_i = \int_\Omega H(x)\frac{\partial\omega_i}{\partial x}dxdy - \beta\int_\Omega \inf\;(H'(x),0)\omega_i\,dxdy + \varepsilon\sum_j\int_\Omega H(x)\omega_j\frac{\partial\omega_i}{\partial x}dxdy$$

$$= -\int_{\{H'>0\}} H'(x)\omega_i(1+\varepsilon u_a y)dxdy - \int_{\{H'<0\}}\{H'(x)\omega_i(1+\varepsilon u_a y-\beta\}dxdy$$

Taking $\varepsilon < \dfrac{\beta-1}{u_a}$ is a sufficient condition to ensure $J_i \le 0$.

THEOREM 3.5.– *Under assumptions [25] [26] and* $\varepsilon < \min(\varepsilon_1,\varepsilon_2)$ (see [27]), *problem (Q) has at least one solution* U_h *such that :*

$$-1/\varepsilon \le u_i \le \overline{w}_i$$

PROOF.– By definition [20]-[24] A_h is a continuous mapping, and we will show that A_h is monotone, that is : If $U_h = (u_i)$ *and* $V_h = (v_i)$ are such that $u_i \le v_i$ for every i, then $A_{hi}(U_h) \le A_{hi}(V_h)$ for every i.

– for $i \in N_{ah}$, A_{hi} is constant;

– for the other cases, we have to prove first the monotonicity of C_h, then the one of A_h can be immediately deduced as it is the product of A_h by a positive constant or 0. For instance for $i \in N_{oh}$,

$$C_{hi}(U_h) - C_{hi}(V_h) = -\sum_{\substack{j\ne i\\ j\in N_h}} \varepsilon\, e_{ij}(u_j^- - v_j^-)$$

$u_j \le v_j, \forall_j$ *implies* $u_j^- \ge v_j^-, \forall_j$ then from [26] we get $C_{hi}(U_h) \le C_{hi}(V_h)$.

As a conclusion, we have shown that A_h is a monotone continuous mapping in R^{N_h} with a sub- and a super solution, so that A_h has at least one fixed point which is a solution to problem (R) and equivalently to problem (Q).

COMPUTATIONAL ALGORITHM.– From theorem 3.5 it is easy to define an algorithm to obtain a numerical solution of *problem (Q):* Starting from the sub solution $U^0 = (-1/\varepsilon)$ or from the super solution $U^0 = (\overline{\overline{w_i}})$, we build two sequences $U^{n+1} = A_h(U^n)$ which are known to converge towards the fixed point solution to *problem (Q)*. As these sequences stay respectively below and above the solution, the present algorithm allows to obtain lower and upper bounds for the required solution.

We will show in the next section that the finite element approximation is a convergent approximation from the continuous problem as h tends to zero.

4. Convergence results

It will be shown that any fixed point solution of problem (Q) tends to a solution of the continuous problem [12] when h tends to zero. In that sense the convergence theorem can be considered as an existence theorem for problem [12]. For the convergence proof, various properties of the discretization are needed. In a second part of the section, it will be shown that peculiar finite elements discretization fulfils the required conditions.

4.1. *Assumptions related to the discretization*

We need assumptions related to the free boundary problem at the limit.

DEFINITION 4.1.– *A pair of vectors* (V_h, Γ_h) *with* $V_h = (v_i), \Gamma_h = (\gamma_i)$ *satisfies condition [30] : "if there exists a positive constant C, such that* $\gamma_i \geq 0$ *for* $i \in N_{1h}$ *implies* $v_i \leq C\,h$ *"*

We will give in §4.3 an example of discretization such that technical conditions [27] and [30] are satisfied.

4.2. *Convergence theorem*

Let U_h be a solution to the discrete problem (Q), setting

$$v_h = \sum_{i \in N_h} u_i^+ \omega_i \quad \text{and} \quad \gamma_h = \sum_{i \in N_h} u_i^- \chi_i \,, \text{ and } \varepsilon_3 = \frac{H_{\min}^3}{2\pi H_{\max}} \qquad [31]$$

we have the theorem:

THEOREM 4.2.– *Assuming [27] [30] and* $\varepsilon \le \varepsilon_3$ *, then there exists* v^* *in* $H^1(\Omega)$ *and* γ^* *in* $L^\infty(\Omega)$ *such that for a subsequence,* (v_h) *converges to* v^**, weakly in* $H^1(\Omega)$ *and* (γ_h) *converges to* γ^**, weakly in* $L^2(\Omega)$*. Moreover* $u^* = v^* - \gamma^*$ *is solution of [12].*

PROOF.– By proposition 3.1, v_h and γ_h defined by [31] satisfy [14]-[18]. We set $\tilde{v}_h = v_h - u_a y$ as a test function ϕ_h in [14].

For $i \in N_{1h}, \omega_i(x,y) = 0$ *on* $\Gamma_0 \cup \Gamma_a$ and is periodic for x=0 and x=1, so that:

$$a_r(u_a y, \omega_i) = u_a \left(\frac{R}{L}\right)^2 \int_\Omega H^3(x) \frac{\partial \omega_i}{\partial y} dxdy = u_a \left(\frac{R}{L}\right)^2 \int_{\partial\Omega} H^3(x)\omega_i \cos(n,y) dv = 0$$

Consequently $a_r(u_a y, \tilde{v}_h) = 0$. Then:

$$a_r(\tilde{v}_h, \tilde{v}_h) = \int_\Omega (1 + \varepsilon\tilde{v}_h - \varepsilon\gamma_h + \varepsilon u_a y) H(x) \frac{\partial \tilde{v}_h}{\partial x} dxdy$$

$$H^3_{\min} \|\tilde{v}_h\|^2 \le \int_\Omega (1 - \varepsilon\gamma_h + \varepsilon u_a y) H(x) \frac{\partial \tilde{v}_h}{\partial v} dxdy + \varepsilon \int_\Omega H(x) \tilde{v}_h \frac{\partial \tilde{v}_h}{\partial x} dxdy$$

Using Cauchy Schwarz's and Poincaré's inequality, we obtain:

$$H^3_{\min} \|\tilde{v}_h\|^2 \le (1 + \varepsilon u_a) H_{\max} \sqrt{2\pi} \|\tilde{v}_h\| + \varepsilon H_{\max} 2\pi \|\tilde{v}_h\|^2$$

$$(H^3_{\min} - \varepsilon H_{\max} 2\pi) \|\tilde{v}_h\| \le (1 + \varepsilon u_a) H_{\max} \sqrt{2\pi}$$

Assumption $\varepsilon \le \varepsilon_3$ shows that $\|\tilde{v}_h\|$ and consequently $\|v_h\|$ are bounded in $H^1(\Omega)$ with respect to ε and h. Using now the sub solution $W_h = \left(w_i = -\frac{1}{\varepsilon}\right)$, it is obvious that $\|\gamma_h\|_{L^\infty(\Omega)} \le \frac{1}{\varepsilon}$.

This last result shows that for fixed ε , $\|\gamma_h\|_{L^\infty(\Omega)}$ is bounded with respect to h, but the bound is ε -dependent. Thus the first part of the theorem is proved and we know that $v^* \ge 0, \gamma^* \ge 0$ and $1 + \varepsilon v^* - \varepsilon\gamma^* \ge 0$ almost everywhere in Ω .

Using a classical interpolation argument in finite element approximation, for any □ in V there exists ϕ_h in V_h such that ϕ_h converges strongly in $H^1(\Omega)$ towards ϕ. So we can pass to the limit in [14]:

$$a_r(v^*,\phi) = \int_\Omega (1+\varepsilon v^* - \varepsilon\gamma^*) H(x) \frac{\partial \phi}{\partial x} dxdy, \quad \forall \phi \in V$$

Assumption [30] is useful to prove that (v^*, γ^*) is actually a solution to the continuous problem. This technical argument can be found in [CHA 87] and is similar to that in [ALT 80] (th. 3.4) inducing the result:

$$\{\gamma^* > 0\} \subset \{v^* = 0\} \qquad a.e. \ in\ \Omega.$$

4.3. *Special discretizations*

Consider a regular hexagonal triangulation in Ω (see fig.2), corresponding to meshes

$$h_x = \frac{\pi}{N_x}, \quad h_y = \frac{1}{2N_y}, \quad h = \max\,(h_x, h_y) \qquad [32]$$

V_h is the classical P_1-finite element subspace of V_{per} corresponding to piecewise-linear function ω_i in which the periodicity is introduced in a natural way by equalling the degrees of freedom for $x = 0$ and $x = 2\pi$.

L_h is the space of linear combinations of the characteristic functions χ_i of the shadowed set (fig.2) corresponding to the node m_i for $i \in N_{1h}$, and its intersection with Ω for $i \in N_{oh} \cup N_{ah}$.

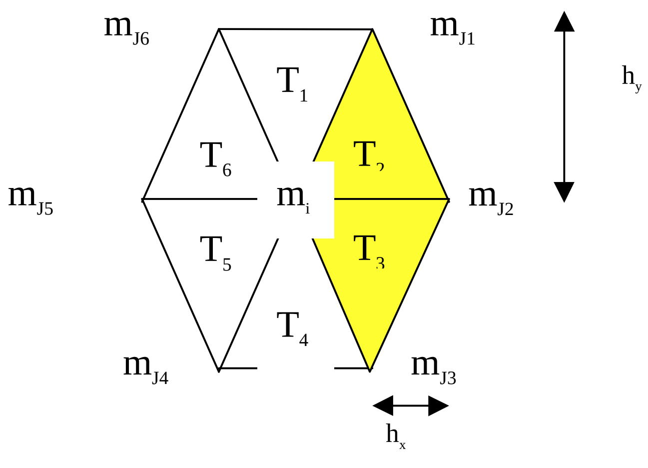

Figure 2. *Hexagonal finite element and notations*

REMARK 4.3.– It will be difficult to consider such a special hexagonal triangulation for a more general nonrectangular Ω or in the case of the use of an automatic triangulation. It is probably possible to show without further restrictions that a general regular triangulation fulfils the required assumptions, following same kind of arguments as in [PIE 82].

In the following we will show that the assumptions given in the existence and convergence theorems are satisfied for the particular hexagonal finite element introduced in the present subsection. The arguments used are very similar to that in [ALT 80] so attention will be focused on the particular difficulties arising in relation to the variable ε linked to the compressibility model.

PROPOSITION 4.4.– *If the meshes ratio* $\tau = \frac{R}{L}\frac{h_x}{h_y}$ *is such that* $\tau \le 1$, *for sufficiently small* ε *(* $\varepsilon \le \varepsilon_4$ *see [33]), assumption [25] is fulfille*d.

PROOF.– Using notations of figure 2:

$$a_{ij_2} = (\tau^2 - 1)/4h_x^2 \int_{T_2UT_3} H^3(x)dxdy \quad \le 0$$

$$b_{ij_2} = -\frac{1}{2h_x}\int_{T_2UT_3} H(x)\omega_{j_2}dxdy < 0 \text{ so that } a'_{ij_2} \le 0 \text{ as soon as } \varepsilon \le \frac{2\tau}{h_y}\frac{H_{\min}{}^3}{H_{\max}}.$$

This condition is mesh-dependent, but a rough bound is valid $\left(h_y \le \frac{1}{2}\right)$ so that for $\varepsilon \le \frac{4\tau H_{\min}^3}{H_{\max}}$ then $a'_{ij_2} \le 0$. Similar technical and easy computations show that for $\varepsilon \le \varepsilon_4$ assumptions [25] are satisfied where :

$$\varepsilon_4 = \min\left(\frac{1-\tau^2}{4\pi}, 4\tau, \frac{2H_{\max}}{3\max|H'|}(3\tau + \frac{1}{\tau})\right)\frac{H_{\min}^3}{H_{\max}} \qquad [33]$$

PROPOSITION 4.5.– *If* $\tau \le 1$ *and* $\varepsilon \le \frac{\varepsilon_4}{3}$ *[33], condition [30] is satisfied for the hexagonal finite element.*

PROOF.– The details are to be found in [CHA 87]. $\tau \le 1$ induces condition [30] for the incompressible model $(\varepsilon = 0)$. The condition $\varepsilon \le \frac{\varepsilon_4}{3}$ shows that it is still true for the compressible model if the "compressibility ε" is small enough.

Gathering the results of propositions 4.4 and 4.5, we can conclude that for the particular hexagonal finite element introduced in that section, for $\tau \le 1$ and for

$\varepsilon \leq \min(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ the cavitation model has at least one solution, the discrete model has also at least one solution that converges for small h to the continuous one.

5. Numerical results

To make more clearly evidence of the influence of the various parameters introduced in the numerical procedure namely the compressibility parameter ε, the number of degrees of freedom N, and the required precision PREC. We first consider a one-dimensional parabolic slider bearing with starved inlet, like the one introduced in [CIO 00]. Physical date are given in table 1.

Parameter	Value	Units
Length	$7.62\text{x}10^{-2}$	m
Minimum Height	$2.54\text{x}10^{-2}$	m
Maximum Height	$5.08\text{x}10^{-2}$	m
Velocity	4.57	m/s
Viscosity	0.039	Pas

Table 1. *Physical Conditions for Slider Bearings*

Instead of periodic boundary conditions, we have to consider Dirichlet boundary conditions and the problem is described by the one-dimensional equation deduces from [12] by cancelling the derivatives with respect of y. Positive part of u is approximated by classical one dimensional P_1 approximation and negative part by $\sum \lambda_i \chi_i$, χ_i being the characteristic function of $[x_{i-1}, x_i]$. The input flow g is 0.55.

The iterations have been carried on until the relative difference between the computed output flow at x = 1 and the given input flow at x = 0 is less than PREC. All the computations were conducted with the value PREC = 10^{-6}. As a comparison point, we add in the table the results obtained by the purely incompressible approach using "$\varepsilon = 0$" described in [BAY 86]. Table 2 gives for various values of ε and degrees of freedom N the relative number RITER of iterations and the value of the load $W = \int_\Omega p(x)dx$.

ε= 0	W = 0.6974	W = 0.7368	W = 0.7503	W = 0.7571
	RITER = 1	RITER = 5.6	RITER = 15	RITER = 30
$\varepsilon=10^{-1}$	W = 0.1788	W = 0.1841	W = 0.1859	W = 0.1868
	RITER = 0.1	RITER = 0.5	RITER = 1.2	RITER = 2.2
$\varepsilon=10^{-2}$	W = 0.5529	W = 0.5809	W = 0.5905	W = 0.5953
	RITER = 0.6	RITER = 3	RITER = 7.6	RITER = 14.2
$\varepsilon=10^{-3}$	W = 0.6769	W = 0.7179	W = 0.7309	W = 0.7374
	RITER = 0.9	RITER = 5.2	RITER = 13.6	RITER = 26.5
$\varepsilon=10^{-4}$	W = 0.6956	W = 0.735	W = 0.7484	W = 0.7551
	RITER = 1	RITER = 5.6	RITER = 14.8	RITER = 29.2
$\varepsilon=10^{-5}$	W = 0.6972	W = 0.7366	W = 0.7501	W = 0.7569
	RITER = 1	RITER = 5.6	RITER = 14.8	RITER = 29.2
$\varepsilon=10^{-7}$	W =0.6974	W = 0.7368	W = 0.7503	W = 0.7571
	RITER = 1	RITER = 5.6	RITER = 14.8	RITER = 29.2

Table 2. *Load W and relative number of iterations RITER (reference RITER=1 for N=200, e=0) compressible parameter e and degrees of freedom (PREC = 10^{-6}*

Table 3 shows the same outputs, for constant values of N = 800 and $\varepsilon = 10^{-5}$ for various values of PREC.

PREC = 10^{-2}	W = 0.6053	RITER = 1
PREC = 10^{-3}	W = 0.7407	RITER = 2.5
PREC = 10^{-4}	W = 0.7553	RITER = 3
PREC = 10^{-5}	W = 0.7568	RITER = 3.1
PREC = 10^{-6}	W = 0.7569	RITER = 3.1
PREC = 10^{-7}	W = 0.7569	RITER = 3.1

Table 3. *Load W and relative number of iterations RITER (reference RITER=1 for PREC=10^{-3}) for various values of PREC*

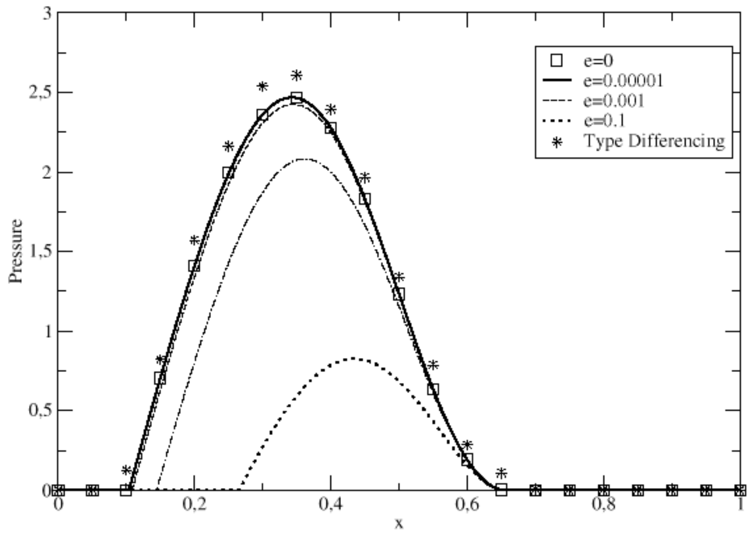

Figure 3. *Pressure versus ε*

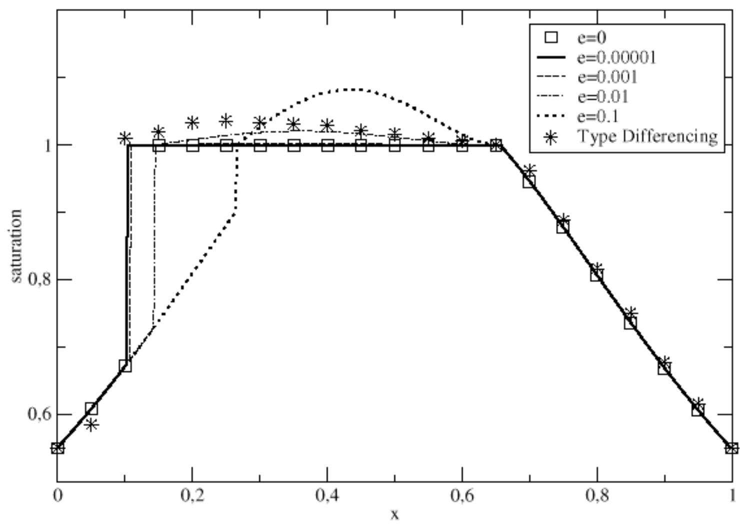

Figure 4. *θ –profile versus ε*

Figures 3 and 4 depict the pressure and the θ profiles for various values of ε, including the incompressible case and results issued from [CIO 00] by the type differencing method.

These results suggest that as soon as ε is less than 10^{-3}, the load and the computation cost do not depend on ε. For ε greater than 10^{-3}, the number of iteration decreases as ε increases; this shows numerically the regularising effect of introducing the compressibility. The necessity of choosing a small value of PREC and a sufficiently great number of degrees of freedom is also clear.

At last, we consider a two-dimensional bearing with periodic boundary conditions. The data are: supply pressure = 19 x 10^5 Pa, eccentricity = 0.8, R/L = 0.5, ε =10^{-5}, degrees of freedom = 900.

Figure 5 depicts the cavitation area and is compared with the one obtained in [LUN 69] and that obtained by the Christopherson algorithm [CRY 71].

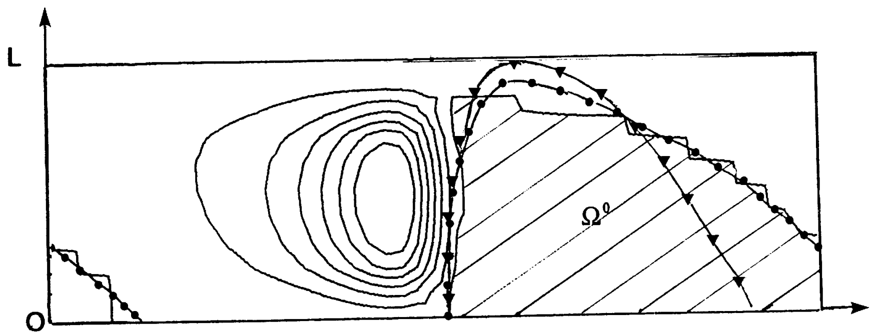

Figure 5. *Pressure distribution and cavitation area computed by*
___ *the present alg.* •• *the Lundholm alg.* •• *the Christopherson alg.*

The importance of the parameter $\tau = R\,h_x/Lh_y$ is illustrated in figure 6 (cf. Th 4.2) in which we give the relative error between the input flow Q_g and the output flow Q_l for various degrees of freedom (600 to 900 nodes), various τ and various numbers of iterations (NITER = 300 to 600):

$$Q_g = \int_{\Gamma_0} H^3 \frac{\partial p}{\partial x} dx \quad Q_l = \int_{\Gamma_a} H^3 \frac{\partial p}{\partial Y} dx$$

Computing time of the present code appears to be very sensitive to the ratio τ , the optimum value of which is about 0.5.

To obtain a discrepancy between output and input flows at most of 10% requires about 300 to 400 iterations while the convergence of the load is obtained before half the overall computational time.

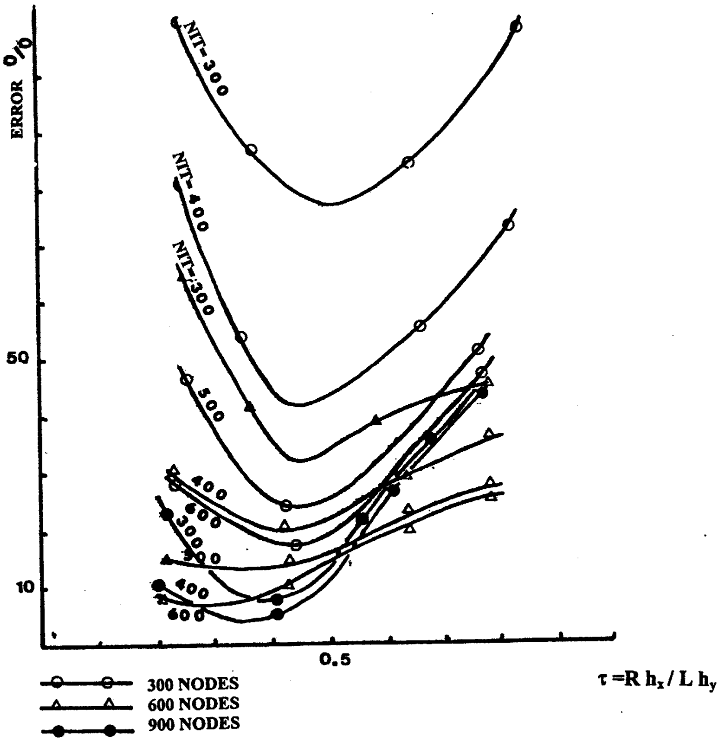

Figure 6. *Error between input and output flow as a function of the number of nodes, the value* τ *and NITER*

6. Concluding remarks

The convergence of the fixed-point algorithm corresponding to *(problem Q)* has been proved in theorem 4.2. It can be shown that the implementation of a Gauss Seidel process is also a convergent one. The convergence being theoretically at least as fast as for the present Jacobi algorithm. The numerical results confirm the fact with an improvement of 40% in the number of required iterations for a given value of PREC.

It could seem surprising that for one-dimensional devices, so many degrees of freedom and a so small value of PREC are needed to obtain a good convergence. This can be explained by the stiffness of the problem, which lies only in the x-direction. So that the 1-dimensional case is more difficult that the 2-dimensional one in which the introduction of the second dimension renders the solution globally smoother.

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