On the numerical modelling of ductile damage with an implicit gradient-enhanced formulation

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ABSTRACT. *This paper presents a phenomenological framework to incorporate ductile damage in an elasto-plastic model. Based on a gradual reduction of the yield stress upon damage evolution, the theory is first presented in a small deformation context, after which a generalization to a geometrically nonlinear hyperelasto-plastic formulation is made. In both cases, an isotropic damage variable is used that is computed from a nonlocal field variable. This approach guarantees the well-posedness of the underlying mathematical problem, and thus prevents any form of pathological mesh dependence. Special attention is focused on the type of nonlocality that is used to drive the ductile damage. Lagrangian and Eulerian nonlocal kernels are compared and their influence on the material length scale is evaluated. Examples are given for a small and a large deformation problem, which highlights the applicability in view of its practical implementation to simulate forming processes.*

RÉSUMÉ. *Cet article présente une approche phénoménologique pour l'incorporation de l'endommagement ductile dans un modèle élasto-plastique. Fondée sur la réduction progressive de la contrainte d'écoulement par l'endommagement, la théorie est d'abord présentée dans un contexte de déplacements infinitésimals, après lequel une généralisation envers une formulation géométriquement non-linéaire en hyperélasto-plasticité suit. Les deux cas font usage d'une variable d'endommagement isotrope, calculée d'un champs non-local. Cette approche garantie que le problème mathématique associé est bien posé, et élimine toute forme de sensitivité au maillage. Une attention particulière est donnée au type de non-localité utilisé pour le calcul de l'endommagement. Des kernels Lagrangiens et Eulériens sont comparés et leur influence sur la longueur intrinsèque est évaluée. Des exemples sont donnés pour un problème à petites et un problème à grandes déformations, ce qui illustre l'applicabilité en vue de son implémentation pratique pour la simulation de processus de formage.*

KEY WORDS: *ductile damage, fracture, gradient damage, hyperelasto-plasticity, higher-order regularization, nonlocality*

MOTS-CLÉS : *endommagement ductile, rupture, emdommagement aux gradients, hyperélasto-plasticité, régularisation, nonlocalité*

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1. Introduction

The numerical modelling of ductile damage and fracture in engineering materials and particularly in metals has gained a considerable interest in recent years. Many of the available solution strategies focus on the initiation process, which is of great importance in manufacturing processes where any damage is to be avoided. However, there is also an important class of practical problems, in which damage, failure and crack propagation are essential steps in the production process, e.g., cutting, blanking, drilling, etc.. Analysis of these processes is mainly based on phenomenological knowledge. Lengthy trial and error procedures along with a variety of empirical guidelines are used to develop and optimize the process. Nowadays, it is expected that numerical techniques may be a versatile alternative, provided that they are capable to give a reliable and adequate description of the ductile fracture process, i.e., initiation and propagation of a crack in a ductile material.

It is known from the analysis of damage and fracture in the infinitesimal deformation theory, that the adequate modelling of the evolution of damage requires a higher-order continuum in which either a gradient or a nonlocal approach is used. In the context of ductile damage, explicit gradient-enhanced small deformation theories have been developed in the past ten years, see for instance [DB 92, PAM 94, SVE 97, RAM 98]. So far, only few extensions to a large deformation framework are available, e.g., [MIK 99] where explicit gradients have been incorporated in the constitutive model. Integral nonlocal approaches applied to softening plasticity have been studied in [NIL 98] on the basis of thermodynamical considerations. Steinmann [STE 99] investigated a geometrically nonlinear gradient damage formulation, applicable to rubberlike materials. In contrast to the explicit gradient enhancements used in the cited gradient plasticity models, this paper is based on the use of an implicit (and hence nonlocal) gradient formulation. The nonlocal character of this gradient formulation has been proven recently by Peerlings et al. [PEE 99], whereas nonlocal damage models have been used with considerable success by several authors [BAŽ 88, TVE 95]. The small deformation solution of the implicit gradient version has now been elaborated and tested by Engelen et al. [ENG 01]. Only basic features of this solution strategy will be reviewed here.

Several finite plasticity formulations nowadays exist [NAG 90, MIE 98b, MIE 98a, ALF 98]. Many models are based on a hypoelastic stress response, which is commonly obtained by generalization of the corresponding infinitesimal framework. It is known that elastic deformations need to be small in order to apply such models, since no stored energy function exists that ensures true elastic behaviour (i.e., reversible deformations without energy dissipation). This restriction does not hold for a hyperelastic stress response, where a stored energy function does exist, that depends on the invariants of the right Cauchy-Green deformation tensor for isotropic materials. On the basis of this approach, hyperelasto-plasticity frameworks have been proposed and implemented by Simo et al. [SIM 85, SIM 88a] and Simo [SIM 88b]. The volumetric and deviatoric response is fully decoupled on the level of the stored energy potential. This hyperelasto-J_2-plasticity model is taken as the point of departure for the incorporation of a ductile damage evolution. A major difficulty in numerical descriptions of ductile damage, is the adequate incorporation of physical mechanisms. In metals, ductile damage starts if the number of dislocation barriers prevents further plastification and leads to void initiation, growth and coalescence, the underlying damage mechanisms of the frequently applied Gurson model [GUR 77] . A lot of empirical and micromechanical research has been performed in this area, which has to embedded in the damage initiation and evolution in a later stage. The paper subsequently presents the small and large deformation framework, the incorporation of a nonlocally driven ductile damage variable in the field function, some computational aspects for the large deformation model, as well as several numerical examples for different cases. A discussion on the use of nonlocal models in the presence of large deformations is also made, where a material or a spatial framework do not lead to the same interpretation of the underlying 'material' length scale, that is mostly used as a constant in the small deformation context. Note that the ductile damage parameter appears as an additional internal state variable, a concept that is widely used in internal state variable plasticity and damage [LEM 90, KRA 96].

2. Underlying elasto-plasticity formulations

2.1. *Small deformation elasto-plasticity model*

The elasto-plastic framework used within the infinitesimal deformation assumption is a standard isotropic von Mises elasto-plastic model. The constitutive relation for the stress rate tensor versus the elastic strain rate tensor is typically given by

$$\dot{\boldsymbol{\sigma}} = {}^4\boldsymbol{C} : \dot{\boldsymbol{\varepsilon}}_e \qquad [1]$$

The plastic state is characterized with a yield function f

$$f(\boldsymbol{\sigma}, \varepsilon_p) = \sigma_{eq}(\boldsymbol{\sigma}) - \sigma_y(\varepsilon_p) \qquad [2]$$

where σ_{eq} is the von Mises equivalent stress

$$\sigma_{eq}(\boldsymbol{\sigma}) = \sqrt{-3J_2(\boldsymbol{\sigma}^d)} = \sqrt{\tfrac{3}{2}\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d} \qquad [3]$$

and σ_y the yield stress. A linear or a nonlinear hardening rule may be considered, e.g.,

$$\sigma_y(\varepsilon_p) = \sigma_{y0} + h\,\varepsilon_p \qquad [4]$$

$$\sigma_y(\varepsilon_p) = \sigma_{y0} + h\,\varepsilon_p + (h_\infty - h_0)(1 - e^{-\delta\varepsilon_p}) \qquad [5]$$

In here, ε_p is the effective plastic strain, h is the linear hardening modulus and h_∞, h_0, $\delta > 0$ are nonlinear hardening parameters. The effective plastic strain is defined by

$$\varepsilon_p = \int_0^t \dot{\varepsilon}_p(t')dt' \qquad \text{where} \qquad \dot{\varepsilon}_p = \sqrt{\tfrac{2}{3}\dot{\boldsymbol{\varepsilon}}_p : \dot{\boldsymbol{\varepsilon}}_p} \qquad [6]$$

The plastic strain rate tensor $\dot{\boldsymbol{\varepsilon}}_p$ is extracted from an associate flow rule, given by

$$\dot{\boldsymbol{\varepsilon}}_p = \dot{\gamma} \boldsymbol{N} \qquad \text{with} \qquad \boldsymbol{N} = \frac{\partial f}{\partial \boldsymbol{\sigma}} \tag{7}$$

It can be easily shown that the equations [2], [3], [6] and [7] lead to the following identity

$$\dot{\gamma} = \dot{\varepsilon}_p \tag{8}$$

The set of equations is completed with the standard Kuhn-Tucker loading/unloading relations

$$\dot{\gamma} \geq 0, \quad f(\boldsymbol{\sigma}, \varepsilon_p) \leq 0, \quad \dot{\gamma} f(\boldsymbol{\sigma}, \varepsilon_p) = 0 \tag{9}$$

and the consistency condition

$$\dot{\gamma} \dot{f} = 0 \tag{10}$$

The solution of this elasto-plastic problem follows standard rules, which can be found in many textbooks.

2.2. *Large deformation hyperelasto-plasticity model*

The large deformation formulation which is used in this paper, is based on a rate-independent hyperelasto-plastic model presented by [SIM 88a, SIM 88b, SIM 98], in which a correction has to be made in order to comply with the assumed isochoric plastic flow. The model is presented in a format that tends towards the infinitesimal solution of the previous section for small deformations. This large deformation model presents many features which are well-known in the classical infinitesimal theory of plasticity, although it has not been obtained through ad hoc extensions of the small strain theory. It is based on the adequate implementation of finite deformation kinematics in the elastic part and through the application of general principles of associative plasticity in the plastic part. A hyperelastic stress-strain relation is used for the elastic predictor. In this section, the essential algorithmic steps are highlighted, as well as the modifications made with respect to Simo's original model. The numerical implementation then follows identical lines as given in [SIM 88b].

Based on micromechanical considerations of crystallographic slip, a multiplicative decomposition of the deformation gradient tensor $\boldsymbol{F}$ is performed according to

$$\boldsymbol{F} = \boldsymbol{F}_e \cdot \boldsymbol{F}_p \tag{11}$$

Such a multiplicative split implies the existence of an intermediate state which is obtained if the current state Ω is relaxed to a (local) stress-free configuration Ω_p, where only plastic deformations exist. Note that $\boldsymbol{F}_p$ and $\boldsymbol{F}_e$ are only defined up to an arbitrary rigid body motion of the intermediate state. The different configurations and the well-known associated kinematic tensors (total, elastic, plastic right or left Cauchy-Green deformation tensors, stretch tensors and strain tensors) are shown in Figure 1. The following pull-back push-forward relations are then valid

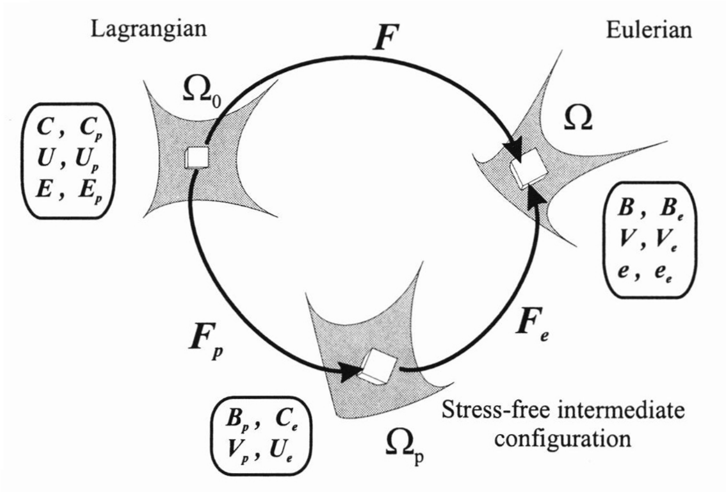


Figure 1. *The multiplicative split and associated deformation tensors*

$$\boldsymbol{B}_e = \boldsymbol{F}\cdot\boldsymbol{C}_p^{-1}\cdot\boldsymbol{F}^T \quad [12]$$

$$\overset{\nabla}{\boldsymbol{B}}_e = \boldsymbol{F}\cdot[\dot{\boldsymbol{C}_p^{-1}}]\cdot\boldsymbol{F}^T \quad [13]$$

$$\left[\overset{\nabla}{\boldsymbol{B}}_e\right]^d = \boldsymbol{F}\cdot[\dot{\boldsymbol{C}_p^{-1}}]^D\cdot\boldsymbol{F}^T \quad [14]$$

The elastic left Cauchy-Green deformation tensor $\boldsymbol{B}_e$ is the push-forward (with $\boldsymbol{F}$) of the inverse plastic right Cauchy-Green deformation tensor $\boldsymbol{C}_p^{-1}$. The (objective and covariant) Lie-derivative of $\boldsymbol{B}_e$ is the push-forward of the material time derivative of $\boldsymbol{C}_p^{-1}$, which also holds for their corresponding spatial and material deviatoric parts, respectively given by

$$\left[\overset{\nabla}{\boldsymbol{B}}_e\right]^d = \overset{\nabla}{\boldsymbol{B}}_e - \tfrac{1}{3}\mathrm{tr}(\overset{\nabla}{\boldsymbol{B}}_e)\boldsymbol{I} \quad [15]$$

$$[\dot{\boldsymbol{C}_p^{-1}}]^D = [\dot{\boldsymbol{C}_p^{-1}}] - \tfrac{1}{3}\left([\dot{\boldsymbol{C}_p^{-1}}]:\boldsymbol{C}\right)\boldsymbol{C}^{-1} \quad [16]$$

Assuming that the plastic flow is isochoric means that the volume change ratio J depends on elastic deformations only

$$\det(\boldsymbol{F}_p) = 1 \quad \longrightarrow \quad J = \det(\boldsymbol{F}) = \det(\boldsymbol{F}_e) = J_e \qquad [17]$$

The kinematic constraint which follows from this assumption can be rephrased as

$$\overset{\nabla}{\boldsymbol{B}}_e : \boldsymbol{B}_e^{-1} = 0 \qquad [18]$$

which is different from the facilitating assumption made by [SIM 88a], where

$$\mathrm{tr}(\overset{\nabla}{\boldsymbol{B}}_e) = \overset{\nabla}{\boldsymbol{B}}_e : \boldsymbol{I} = 0 \qquad [19]$$

was forwarded. Unfortunately, satisfaction of [19] does not lead to an isochoric plastic flow. Equation [18] should have been used instead.

The Kirchhoff stress tensor $\boldsymbol{\tau}$ is computed from the elastic deformations, using an isotropic hyperelastic stress-strain response.

$$\boldsymbol{\tau} = \frac{K}{2}\left(J^2 - 1\right)\boldsymbol{I} + G\bar{\boldsymbol{B}}_e^d \qquad [20]$$

where K and G equal the bulk and shear modulus respectively. The stored energy function W that corresponds to this hyperelastic relation is given by

$$W = \tfrac{1}{2}K\left[\tfrac{1}{2}(J^2 - 1) - \ln J\right] + \tfrac{1}{2}G\,[\mathrm{tr}(\boldsymbol{B}_e) - 3] \qquad [21]$$

The yield function is the classical von Mises-Huber function, formulated in terms of the Kirchhoff stress tensor

$$f(\boldsymbol{\tau}, \varepsilon_p) = \tau_{eq} - \tau_y(\tau_{y0}, \varepsilon_p) \leqslant 0 \qquad [22]$$

which closely resembles equation [2]. The von Mises equivalent stress τ_{eq} is defined like in the infinitesimal case by

$$\tau_{eq}(\boldsymbol{\tau}) = \sqrt{-3\,J_2(\boldsymbol{\tau}^d)} = \sqrt{\tfrac{3}{2}\boldsymbol{\tau}^d : \boldsymbol{\tau}^d} \qquad [23]$$

The earlier proposed linear and a nonlinear hardening rules now read

$$\tau_y(\varepsilon_p) = \tau_{y0} + h\,\varepsilon_p \qquad [24]$$

$$\tau_y(\varepsilon_p) = \tau_{y0} + h\,\varepsilon_p + (h_\infty - h_0)(1 - e^{-\delta\varepsilon_p}) \qquad [25]$$

The effective plastic strain ε_p is now defined with respect to the elastic left Cauchy-Green deformation tensor

$$\varepsilon_p = \int_0^t \dot{\varepsilon}_p(t')\,dt' \qquad \text{with} \qquad \dot{\varepsilon}_p = \sqrt{\frac{3}{2a^2}\left[\overset{\nabla}{\boldsymbol{B}}_e\right]^d : \left[\overset{\nabla}{\boldsymbol{B}}_e\right]^d} \qquad [26]$$

where a is a coefficient that will be determined later in order to equate the large and small deformation models in the case of infinitesimal displacements. Simo [SIM 88a] derived an associative flow rule for this hyperelasto-plastic formulation, through the application of the principle of maximum plastic dissipation. The following deviatoric flow rule was obtained

$$\left[\overset{\nabla}{\boldsymbol{B}}_e\right]^d = -a\,\dot{\gamma}\,\boldsymbol{N}^d \qquad \text{where} \qquad \boldsymbol{N}^d = \frac{\boldsymbol{\tau}^d}{\tau_{eq}} \tag{27}$$

The relation between the plastic multiplier γ and the effective plastic strain ε_p may be extracted by combining equations [22], [23], [26] and [27]

$$\dot{\varepsilon_p} = \frac{a}{3}\,\dot{\gamma} \tag{28}$$

The formulation of the model is again completed with the standard Kuhn-Tucker loading-unloading conditions and the consistency condition

$$\dot{\gamma} \geqslant 0, \quad f(\boldsymbol{\tau}, \varepsilon_p) \leqslant 0, \quad \dot{\gamma}\, f(\boldsymbol{\tau}, \varepsilon_p) = 0 \quad \dot{\gamma}\, \dot{f}(\boldsymbol{\tau}, \varepsilon_p) = 0 \tag{29}$$

The plastic incompressibility condition for infinitesimal displacements becomes

$$\overset{\nabla}{\boldsymbol{B}}_e : \boldsymbol{B}_e^{-1} \implies [\dot{\boldsymbol{C}}_p^{-1}] : \boldsymbol{C}_p = 0 \approx \mathrm{tr}([\dot{\boldsymbol{C}}_p^{-1}]) \tag{30}$$

by means of which the following simplifications can be made if the limit towards the infinitesimal framework is taken

$$\left[\overset{\nabla}{\boldsymbol{B}}_e\right]^d \approx [\dot{\boldsymbol{C}}_p^{-1}]^D \approx [\dot{\boldsymbol{C}}_p^{-1}]^d \approx [\dot{\boldsymbol{C}}_p^{-1}] \approx -\dot{\boldsymbol{C}}_p \approx -2\,\boldsymbol{D}_p \approx -2\,\dot{\boldsymbol{\varepsilon}}_p \tag{31}$$

Consequently, the flow rule [27] in this infinitesimal limit case reads

$$\dot{\boldsymbol{\varepsilon}}_p \approx -\frac{a}{2}\,\dot{\gamma}\,\frac{\boldsymbol{\sigma}^d}{\sigma_{eq}} \tag{32}$$

Hence, if the coefficient a equals 3, all equations coincide with the infinitesimal elasto-plasticity framework.

The algorithm is completed with an elastic predictor - plastic corrector scheme, see [SIM 88b, SIM 98] for a similar example. The algorithm can be implemented in a finite element framework without difficulties, including the desired consistent tangent operator.

3. Incorporating ductile damage

The addition of ductile damage during plastic flow is based on a progressive reduction of the yield stress once the failure process initiates. Within an isotropic framework, this can be achieved with one single damage variable, in a similar way to the stiffness reduction in damage mechanics. Similar arguments as the one used by Kachanov can be used to motivate such an approach, since the initiation of microcracks (intergranular) and voids can be observed in this stage of the deformation. Based on the knowledge which has been acquired in the field of the computational modelling of material instabilities, it is now known that a continuum solution can only be obtained if the principle of local action is abandoned. A higher-order or a nonlocal constitutive theory [DB 92, PAM 94, SVE 97, RAM 98, GEE 00] has to be used, in order to obtain a set of well-posed partial differential equations. In this paper, an implicit gradient enrichment is used, which combines the computational efficiency of gradient type theories with the integral nonlocal concept. It has been shown that such an implicit approach presents a true nonlocal character [PEE 99], in the sense that an equivalent integral format exists, in which the nonlocal variable is a long-range weighted average of field of local variables. The solution strategy for the infinitesimal case will be highlighted briefly, after which the extension towards the hyperelasto-plastic formulation is scrutinized. Computational details and algorithmic aspects for the infinitesimal elasto-plasticity framework can be found in [ENG 01]. Note that the inherent relation between this ductile damage variable and its associated thermodynamic kinematic quantity is not further considered here, where the nonlocal character of the damage requires special attentions, see [NIL 98, GB 99, GAN 99].

3.1. *Small deformations*

Assuming a fully isotropic material behaviour, a ductile damage parameter $0 \leqslant \omega_p \leqslant 1$ is introduced, which leads to a gradual reduction of the yield stress in the softening stage. Void nucleation, void growth and coalescence, growing out to cracks are the underlying physical mechanisms. The yield function given in [2] is now transformed to the following damage-sensitive yield function

$$f(\sigma, \varepsilon_p) = \sigma_{eq} - (1 - \omega_p)[\sigma_y(\sigma_{y0}, \varepsilon_p)] \leqslant 0 \qquad [33]$$

Evidently, damage will affect the stress tensor, but unlike damage mechanics the ductile damage does not enter the hyperelastic constitutive relation and deformations are thus not reversible. The ductile damage ω_p is computed from a history variable κ, which is the ultimate value of the nonlocal variable $\bar{\psi}$ in the deformation history of the considered material point. The 'implicit gradient' approach refers to the equation that is used to extract the nonlocal field variables $\bar{\psi}$ from the field of their local counterparts ψ. This is done through the solution of a partial differential equation of the Helmholtz type

$$\bar{\psi} - \ell^2 \nabla^2 \bar{\psi} = \psi \qquad [34]$$

along with the Neumann boundary condition

$$\vec{\nabla}\bar{\psi}\cdot\vec{n} = 0 \qquad \text{on the boundary } \Gamma \text{ with outward normal } \vec{n} \qquad [35]$$

The parameter ℓ is a length parameter (often called the intrinsic length) that is related to the size of the influence zone of the nonlocal averaging function. Determining $\bar{\psi}$ from such a gradient formulation is equivalent to an integral nonlocal format, see [BAŽ 88], where the nonlocal variable is computed as a weighted average of the local field. Note that the Laplacian used in the nonlocality equation [34] can be defined with respect to the undeformed or deformed state. In the case of infinitesimal displacements however, this difference is not relevant. The relation between the ductile damage and the nonlocal field variable is governed by a damage evolution law $\omega_p(\kappa)$ which quantifies the damage growth in terms of the field of kinematic variables. Phenomenological examples of such laws are

$$\omega_p = min\{\frac{\kappa - \kappa_i}{\kappa_c - \kappa_i}, 1\} \qquad [36]$$

$$\omega_p = 1 - e^{-\beta\kappa} \qquad [37]$$

where the damage grows respectively linearly or exponentially (initially fast increase) or towards its ultimate value 1 at failure. The parameters β, κ_i and κ_c are material parameters. These equations only influence the evolution of damage once it has initiated. The initiation of damage is controlled by the proper scalar function for the local field variable ψ. Its choice should be founded on micromechanical considerations and known mechanisms in ductile damage initiation and evolution, which is still subject of future research. In this contribution, the damage controlling field variable ψ is taken equal to the effective plastic strain measure ε_p, i.e.,

$$\psi = \varepsilon_p \qquad [38]$$

In the case of infinitesimal displacements, the constitutive equations for elasto-plasticity enriched with this ductile damage approach have been implemented in a computational strategy and solved within a finite element framework [ENG 01]. It is shown that such a formulation is well suited to solve softening and failure in ductile materials up to complete failure.

3.2. *Large deformations*

3.2.1. *Material and spatial nonlocality*

Extending the damage evolution proposed in the infinitesimal case to large deformations raises a fundamental problem with respect to the nature of the nonlocality. This is easily understood from the Helmholtz equation [34], which may take two possible formats in the geometrically nonlinear case, i.e.,

$$\bar{\psi} - \ell_0^2 \nabla_0^2 \bar{\psi} = \psi \qquad \text{with} \quad \vec{\nabla}_0 \bar{\psi} \cdot \vec{n}_0 = 0 \qquad [39]$$

$$\bar{\psi} - \ell^2 \nabla^2 \bar{\psi} = \psi \qquad \text{with} \quad \vec{\nabla} \bar{\psi} \cdot \vec{n} = 0 \qquad [40]$$

The first equation corresponds to the Lagrangian (or material) averaging of the local field, while the second equation reflects the Eulerian (or spatial) averaging case. Steinmann [STE 99] already addressed this difference in his analysis for large deformation quasi-brittle damage model. He showed that only the Lagrangian averaging solution seemed to inherit the properties of the infinitesimal model.

Equation [40] can be pulled back towards the Lagrangian configuration, which yields

$$\bar{\psi} - \ell^2 (\boldsymbol{F}^{-T} : \vec{\nabla}_0 \otimes \boldsymbol{F}^{-T}) \cdot \vec{\nabla}_0 \bar{\psi} - \ell^2 \boldsymbol{C}^{-1} : \vec{\nabla}_0 \otimes \vec{\nabla}_0 \bar{\psi} = \psi \qquad [41]$$

Besides a symmetric term that depends inversely on the deformation, a second non-symmetric term appears. Clearly, the relative influence of both terms gradually changes with the deformation, which may be particularly important in localization zones. This is well illustrated with the one-dimensional counterpart of [41].

$$\bar{\psi} + \frac{\ell^2}{\lambda^3} \frac{\partial \lambda}{\partial X} \frac{\partial \bar{\psi}}{\partial X} - \frac{\ell^2}{\lambda^2} \frac{\partial^2 \bar{\psi}}{\partial X^2} = \psi \qquad [42]$$

where X is the material coordinate, x the spatial coordinate and λ the stretch ratio. In localization zones $\frac{\partial \lambda}{\partial X}$ becomes large, while the influence of λ varies from traction to compression. The coefficients are small in tension and large in compression. In the case of locally uniform stretch, the non-symmetric term locally equals zero and the nonlocal averaging equation can be written as

$$\bar{\psi} - [\ell_0(\lambda)]^2 \frac{\partial^2 \bar{\psi}}{\partial X^2} = \psi \qquad [43]$$

The material length parameter $\ell_0 = \ell/\lambda$ now depends on the deformation, which is contrast to the one-dimensional Lagrangian equation derived from [39]

$$\bar{\psi} - \ell_0^2 \frac{\partial^2 \bar{\psi}}{\partial X^2} = \psi \qquad [44]$$

where the material length parameter ℓ_0 is constant. Nonlocal spatial averaging (one averages over distances that are fixed in space and independent of the deformation of the material) is thus performed over a volume that is constant in space but variable in the material, while nonlocal material averaging (one averages over the initial undeformed distances between the material particles or voids) is applied over a constant material volume. In the case of tension ($\lambda > 1$, typically the case with tensile cracks) the material volume over which the spatially nonlocal kernel acts vanishes, while it becomes extremely large in the case of compression ($0 < \lambda < 1$).

A spatial nonlocal formulation [40] with a constant spatial averaging volume, leads to a nonlocal variable that tends to the local variable with increasing tensile deformations. It is also clear that spatial and material nonlocal formulations will behave quite differently in tension or compression. Note that equation [34] has already been used with a non-constant length scale within an infinitesimal damage-mechanics framework, see [GEE 98, GEE 00]. It was shown that a length scale that decreases with deformation, leads to a solution that tends towards the local ill-posed solution upon complete damage. In fact, local deformations in the neighbourhood of cracks will always tend to large values, while they tend to zero in the surrounding unloading material. Although material nonlocality seems more relevant than spatial nonlocality, there is no obvious reason why the nonlocal kernel should be independent of the deformation. The non-trivial answer to this question should ensue from micromechanics or physics and certainly constitutes a challenge for future research.

3.2.2. *Computational predictor-corrector algorithm*

The solution of the damage-enhanced material behaviour is performed with an elastic predictor - plastic corrector scheme during a time increment from t to $t + \Delta t$. The predictor corresponds to a fully elastic increment Δt, i.e., the plastic flow increment is zero. The intermediate stress free configuration is thus preserved in that state.

$$\left[{}^{\star}\boldsymbol{C}_p^{t+\Delta t}\right]^{-1} = \left[\boldsymbol{C}_p^{t}\right]^{-1} \quad \text{and} \quad {}^{\star}\varepsilon_p^{t+\Delta t} = \varepsilon_p^{t} \qquad [45]$$

The small star symbol is used to indicate that the considered quantity corresponds with the predictor state. The incremental deformation tensor $\boldsymbol{F}_\Delta = [\boldsymbol{F}^{t+\Delta t}]\cdot[\boldsymbol{F}^{t}]^{-1}$, quantifies the deformation at time $t + \Delta t$ with respect to the configuration at time t. The predictor of the isochoric left Cauchy-Green deformation tensor is then obtained through

$${}^{\star}\bar{\boldsymbol{B}}_e^{t+\Delta t} = \bar{\boldsymbol{F}}_\Delta\cdot\bar{\boldsymbol{B}}_e^{t}\cdot[\bar{\boldsymbol{F}}_\Delta]^T \qquad [46]$$

and hence the trial Kirchhoff stress tensor as

$${}^{\star}\boldsymbol{\tau}^{t+\Delta t} = \frac{K}{2}\left([J^{t+\Delta t}]^2 - 1\right) + G\left[{}^{\star}\bar{\boldsymbol{B}}_e^{d}\right]^{t+\Delta t} \qquad [47]$$

Note that the ductile damage does not evolve in the predictor phase, since the hyperelastic relation is independent from ω_p.

The next step consists in the integration of the flow rule, which is performed with a pull-back and push-forward procedure in order to ensure incremental objectivity. The objective Eulerian flow rule is pulled-back to the invariant Lagrangian configuration, after which a time discretization is carried out using an implicit Euler backward scheme. The discretized Lagrangian flow rule is then pushed forward to the spatial description again. Following this procedure with respect to [27], followed by a multiplication with $[J^{t+\Delta t}]^{-2/3}$ and substitution of the elastic predictor [46] permits to rephrase the discretized flow rule in the current state as

$$\left[\bar{\boldsymbol{B}}_e^d\right]^{t+\Delta t} = \left[{}^{\star}\bar{\boldsymbol{B}}_e^d\right]^{t+\Delta t} - a\,\Delta\gamma\,[J^{t+\Delta t}]^{-2/3}\left[\boldsymbol{N}^d\right]^{t+\Delta t} \qquad [48]$$

which may also be written in the following format

$$\left(1 + a\,G\,[J^{t+\Delta t}]^{-2/3}\,\frac{\Delta\gamma}{\tau_{eq}^{t+\Delta t}}\right)[\bar{\boldsymbol{B}}_e^d]^{t+\Delta t} = \left[{}^{\star}\bar{\boldsymbol{B}}_e^d\right]^{t+\Delta t} \qquad [49]$$

Repeating the pull-back/integration/push-forward scheme to rate equation [28] gives for $a = 3$

$$\varepsilon_p^{t+\Delta t} = \varepsilon_p^t + \Delta\gamma \qquad [50]$$

The increment of the plastic flow $\Delta\gamma$ may now be determined from equation [49] by taking the square root of the double inner product of this equation with itself, and making use of the definition of the equivalent Kirchhoff stress τ_{eq} and the deviatoric part of $\boldsymbol{\tau}$ in equation [20], which yields

$$\left(1 + a\,G\,[J^{t+\Delta t}]^{-2/3}\,\frac{\Delta\gamma}{\tau_{eq}^{t+\Delta t}}\right)\tau_{eq}^{t+\Delta t} = {}^{\star}\tau_{eq}^{t+\Delta t} \qquad [51]$$

Using a linear hardening law[24] enriched with ductile damage, i.e., the form $\tau_y = (\tau_{y0} + h\varepsilon_p)(1 - \omega_p)$, with a constant hardening modulus h, and making use of the trial value ${}^{\star}f = {}^{\star}\tau_{eq} - (\tau_{y0} + h\varepsilon_p^t)(1 - \omega_p^t)$ of the yield function permits to rewrite equation [51] and extract the plastic multiplier by means of [50]

$$\Delta\gamma = \frac{{}^{\star}f_{\omega_p}^{t+\Delta t}}{h(1 - \omega_p^{t+\Delta t}) + G\,[J^{t+\Delta t}]^{-2/3}} \qquad [52]$$

For the nonlinear hardening rule [25], equation [51] has to be solved iteratively for $\Delta\gamma$ with a local Newton scheme. Once $\Delta\gamma$ is determined, the equivalent von Mises stress can be extracted directly from [51]. Alternatively, it can be noticed from the equations [49] and [51] that $\boldsymbol{N}^d = {}^{\star}\boldsymbol{N}^d$, which permits to compute $[\bar{\boldsymbol{B}}_e^d]^{t+\Delta t}$ and thus $\boldsymbol{\tau}^{t+\Delta t}$ directly from equation [48].

The spherical part of the flow rule is fully determined through the isochoric plastic flow assumption. Plastic incompressibility is enforced by computing the spherical part such that $\det(\boldsymbol{B}_e) = J^2$ holds. Using the principal invariants $J_1^d = 0$, J_2^d, J_3^d of $\boldsymbol{B}_e^d$, which can be computed after the return mapping of the deviatoric part, it is easy to show that the trace of $\boldsymbol{B}_e$ is the solution of the following cubic polynomial equation

$$[\mathrm{tr}(\boldsymbol{B}_e)]^3 + 9J_2^d[\mathrm{tr}(\boldsymbol{B}_e)] + 27(J_3^d - J^2) = 0 \qquad [53]$$

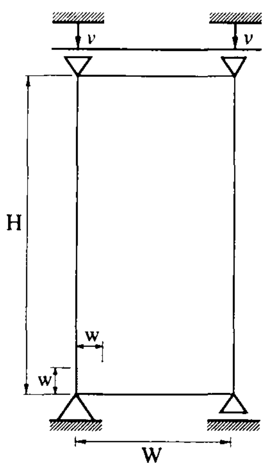
The solution $\mathrm{tr}(\boldsymbol{B}_e)$ of this equation, and the result of the return mapping scheme in terms of $\boldsymbol{B}_e^d$, permits to determine the complete elastic left Cauchy-Green deformation tensor. Note that the elastic deformation tensor $\boldsymbol{F}_e$ is not computed explicitly in this algorithm, which means that the unknown rotation tensor, up to which the intermediate configuration is defined, does not have to be quantified.

The solution strategy presented above can be implemented in a finite element framework without big difficulties, since it is a natural combination of the small deformation framework [ENG 01] and Simo's work [SIM 88a, SIM 88b]. In any case, i.e., material or spatial nonlocality, a consistent tangent operator can be determined.

4. Examples and comparisons

4.1. *Small deformation analysis*

Several examples for small deformation gradient-enhanced ductile damage were already given in [ENG 01]. Only one example is therefore presented here. A two-dimensional plate is loaded in compression as indicated in figure 2, where the material parameters are also given. The problem has been investigated with several discretizations, where the expected mesh-objectivity of the result has been confirmed. The development of the shear bands and the localization process in the softening branch is illustrated in figure 3, where the effective plastic strain is depicted. It can be noticed that the softening branch is modelled up to complete failure, where the deformation



Height	H	0.12 m
Width	W	0.06 m
Thickness	T	10^{-3} m
Imperfection	w × w	0.01 m
Young's modulus	E	20000 MPa
Poisson's ratio	ν	0.49
Hardening modulus	h	2000 MPa
Initial yield stress	σ_{y0}	20 MPa
in imperfection		18 MPa
Length scale	ℓ	0.01 m

Figure 2. *Compression of a plate with an initial imperfection*

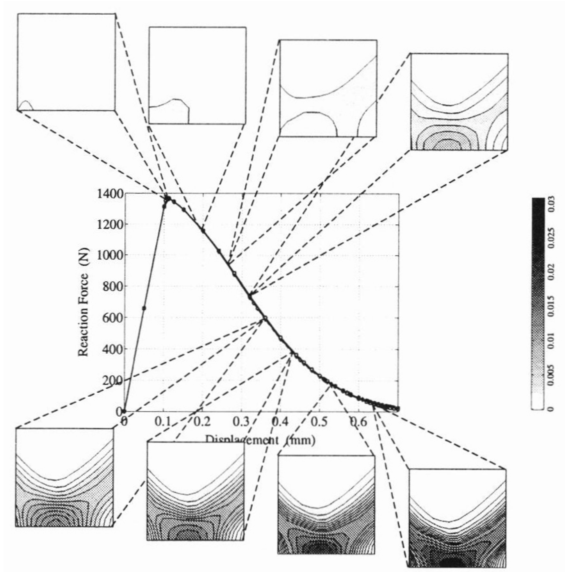


Figure 3. *Shear band development and localization in a softening plate*

localizes in a zone that expands to certain width determined by ℓ, after which the further localization takes place in a smaller band that narrows progressively.

4.2. *Large deformation analysis*

A first trivial case, that validates the assumptions made in the infinitesimal framework and its extensions to the geometrically nonlinear framework, is found by comparing the plate compression example for both frameworks. The associated force displacement curves are depicted in figure 4. It can be noticed that results almost overlap, which is essentially due to the fact that deformations remain small in this example. If applications towards metals are envisaged, deformations inevitably become much larger. To illustrate this, an axisymmetric tensile bar has been modelled, for which the characteristics are listed in table 1, (geometry was taken from Simo [SIM 88b]). No imperfection was used, since physical softening will be triggered automatically by the cross-section reduction in the necking area. The tensile bar fails due to geometrical and physical softening in the necking zone. The analysis was again made with both the infinitesimal and the large deformation framework. Results are shown in figure 5. Large difference now appear, already soon after the initial yield point. Note that this strongly influences the localization of deformation and the softening behaviour of the material in the failure stage.

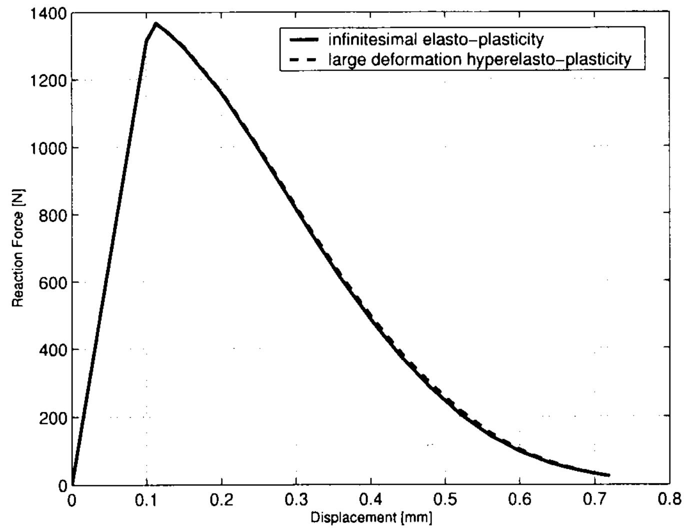


Figure 4. *Plate compression solved with the infinitesimal and geometrically nonlinear approach*

Length	L	53.3 mm
Radius	R	6.4 mm
Bulk modulus	K	164 GPa
Shear modulus	G	80 GPa
Initial yield stress	τ_{y0}	450 MPa
Residual flow stress	$\tau_{y\infty}$	715 MPa
Hardening modulus	h	129 MPa
Initial κ_i ($\omega_p = 0$)	κ_i	5 %
Critical κ_c ($\omega_p = 1$)	κ_c	150 %
Softening slope parameter	β	1
Length scale	ℓ	10 mm

Table 1. *Characteristics of the axisymmetric tensile bar*

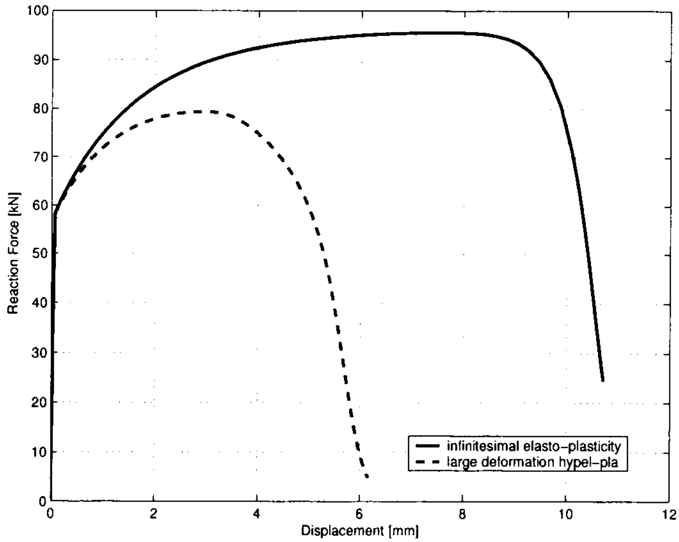


Figure 5. *Axisymmetric tensile bar under failure*

The same example was used to make a comparison between material and spatial averaging. The force-displacement curve is given in figure 6. In contrast to the observations made by Steinmann [STE 99], differences are rather small in elasto-plasticity. This is mainly due to the fact that deformations are irreversible here. Furthermore, the pull-back analysis of the spatial averaging solution, points out that differences can be expected if deformations are very large, i.e., at the end of the crack initiation process which is at the very end of the failure stage of a tensile bar. No singularities are present in this example and no crack propagation occurs, which means that these results may not be generalized ad hoc. Future work will undoubtedly clarify this point.

5. Conclusions

A small and a large deformation elasto-plasticity framework, enhanced with an isotropic ductile damage variable, has been presented. The solution strategy has been emphasized, where it has been shown that this formulation is particularly well suited to model damage initiation and evolution in real engineering problems. In spite of the phenomenological character of the framework, it overcomes the well-known problems in continuum modelling of damage which were not solved to this level yet for large deformation elasto-plastic behaviour. Future research must address issues like damage initiation and evolution in terms of the complex deformation history (including the

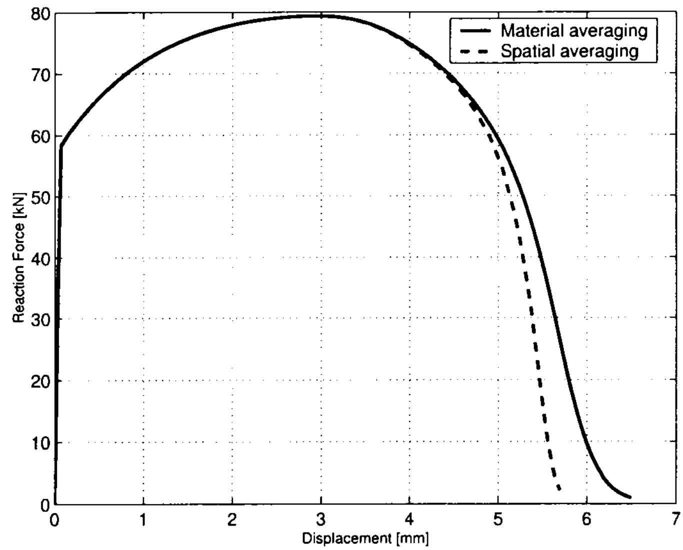


Figure 6. *Axisymmetric tensile bar with material/spatial nonlocality*

influence of hydrostatic pressure, micromechanical theories, e.g., Gurson, etc.), as well the use of advanced remeshing techniques that allow the transition of smooth damage zones into discrete cracks.

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