Mathematical and numerical aspects of an elasticity-based local approach to fracture

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ABSTRACT. *In local approaches to fracture, crack initiation and growth are regarded as the ultimate consequences of a gradual, local loss of material integrity. Standard degradation models, however, are unable to deal with localisation instabilities and the singular strains at the crack tip. As a consequence, finite element analyses become highly sensitive to the spatial discretisation. Adding nonlocal terms to the models enhances their ability to describe fracture processes. An essential role is played by the treatment of the boundary of the equilibrium problem, particularly the internal boundary which represents the crack contour. If the appropriate boundary conditions are applied and the crack region is removed from the discrete equilibrium problem, finite element analyses become mesh-objective.*

RÉSUMÉ. *Les approches locales de rupture adressent les problèmes de l'initiation et de l'accroissement de fissures comme les conséquences ultimes d'une perte locale et graduelle de l'intégrité d'un matériau. Des modèles standards de dégradation, par contre, sont inaptes à traiter les instabilités de localisation et les déformations singulières au coin d'une fissure. Par conséquent, les analyses éléments finis deviennent fortement dépendantes à la discrétisation spatiale. L'ajout de termes non locaux aux modèles rectifie leur capacité de décrire les processus de fissuration. Un rôle essentiel est attribué au traitement du contour du problème d'équilibre, et plus particulièrement au contour interne qui représente les bords de la fissure. Si les conditions aux limites appropriées sont appliquées et si la région fissurée est éliminée du problème discret d'équilibre, les analyses aux éléments finis deviennent objectives par rapport aux maillages utilisées.*

KEY WORDS: *fracture, damage, fatigue, localisation, finite element method, mesh sensitivity*

MOTS-CLÉS : *fissuration, endommagement, fatigue, localisation, éléments finis, dépendance au maillage*

1. Introduction

Component failure due to the formation and growth of cracks is traditionally modelled by fracture mechanics. Fracture mechanics theory uses global criteria to determine under which conditions a pre-existing crack will grow and thus lead to complete fracture of the component. Additional criteria predict the rate and direction of crack growth. In situations where the surrounding material is relatively unaffected by the presence of a crack, this type of modelling is highly successful. In many engineering materials, however, the concentration of deformation and stress near the crack tip produces irreversible changes in the microstructure of the material and in the microstructural processes which govern its behaviour. Examples are plastic flow and/or void formation in ductile materials, microcracking in concrete and fibre pull-out or delamination in fibre-reinforced polymers. In return, these changes may have a considerable influence on – or indeed govern – the crack growth process.

In situations where interactions between crack and microstructural damage play an important role, fracture mechanics may not be the most suitable modelling tool. A more natural treatment of these problems is provided by the so-called local approach to fracture, in which the change of material behaviour is modelled explicitly [LEM 86, CHA 88]. The development and growth of a crack is regarded as the ultimate consequence of the local degradation process. A crack is represented by a region in which the material integrity has been completely lost and which therefore cannot sustain any stress. The internal boundary which describes the crack contour expands when material in front of the crack tip fails. As a result, no separate fracture criteria are needed: the rate and direction of crack growth follow from the constitutive behaviour. As an additional advantage, crack initiation and crack growth can be described within the same framework, so that it is not necessary to define an – often arbitrary – initial crack.

In local approaches to fracture, the degradation of material properties is often modelled using continuum damage mechanics [KAC 58, RAB 69, CHA 88, LEM 96]. In their standard form, this and other types of degradation modelling (e.g., softening plasticity) are usually not suited to describe crack initiation and crack growth, because they cannot properly describe the accompanying localised deformations. As a consequence, the deformation and damage growth are often observed to localise in a surface (i.e., in a vanishing volume) right at – or even before – the onset of fracture. In numerical analyses, this localisation results in an extreme sensitivity to the spatial discretisation of the problem. Upon refinement of the discretisation, the solution converges to one in which the fracture process is instantaneous and does not dissipate any energy. It is emphasised that this pathological behaviour is not due to the numerical treatment, but to a shortcoming of the underlying continuum modelling. It can be removed by introducing nonlocality in the constitutive modelling, either by integral terms or by higher-order gradients [BAŽ 84, PC 87, FRÉ 96, PEE 96]. The enriched continuum formulations which are thus obtained preclude the localisation of deformation in a vanishing volume and the resulting instability. As a result, crack growth rate predictions remain finite and a positive amount of energy is dissipated.

Finite element analyses using nonlocal continuum models are mesh-insensitive once the discretisation is sufficiently fine to accurately capture the solution. However, these methods require some modifications of the standard algorithms. The nonlocality introduces either an additional integration step or an extra set of equations. Special attention must be paid to the treatment of the additional terms near boundaries. This is particularly true in fracture problems, where boundary conditions must also be applied at the internal boundary between the crack and the remaining material. Furthermore, frequent remeshing and an adaptive step size selection may be needed to accurately describe the crack path and special solution control techniques may be required to deal with instabilities and bifurcations [PC 91, GEE 99, PEE 99, PEE 00].

This paper summarises the algorithmic ingredients which are essential for reliable and accurate fracture analyses using the continuum approach. An elasticity-based damage framework is used here (Section 2), but most issues are equally relevant in other degradation models (e.g., softening plasticity). Emphasis is on preventing pathological localisation and mesh sensitivity. Since these phenomena find their origin in the mathematical (continuum) modelling, it is useful to first study the difficulties arising at this level; this is done in Section 3. Section 4 shows how these difficulties can be avoided by using an enriched formulation. Some aspects of the finite element implementation are discussed in Section 5 and results are given in Section 6.

2. Constitutive modelling and the local approach to fracture

Continuum damage mechanics uses a set of continuous damage variables to represent microstructural defects (microcracks, microvoids) in a material. If it is assumed that the development of damage does not introduce anisotropy, a single, scalar damage variable can be used to describe the local damage state. This damage variable D is defined such that $0 \leq D \leq 1$, where $D = 0$ represents the initial, undamaged material and $D = 1$ represents a state of complete loss of material strength. If the influence of damage is added to standard linear elasticity, the classical stress-strain relation of elasticity-based damage mechanics is obtained [LEM 90]:

$$\sigma_{ij} = (1 - D)\, C_{ijkl}\, \varepsilon_{kl}, \qquad [1]$$

where σ_{ij} denotes the Cauchy stress components, C_{ijkl} the standard elasticity tensor and ε_{kl} the linear strains. An important application of relation [1] is to quasi-brittle fracture (e.g., concrete, fibre-reinforced polymers). Here, however, we will use it to model high-cycle fatigue, in which plastic deformations also remain negligible [PAA 93, PEE 99, PEE 00].

In the fatigue model, the growth of damage is related to the local deformation of the material. For this purpose a damage loading function is introduced in terms of the strain components:

$$f(\tilde{\varepsilon}, \kappa) = \tilde{\varepsilon} - \kappa, \qquad [2]$$

with $\tilde{\varepsilon}$ a positive, scalar equivalent measure of the actual strain state and κ a threshold variable, which is taken constant here: $\kappa = \kappa_0$. For the equivalent strain the von Mises strain, scaled such that it equals the axial strain in uniaxial tension, will be used:

$$\tilde{\varepsilon} = \frac{1}{1+\nu}\sqrt{-3J_2}, \qquad [3]$$

where

$$I_1 = \varepsilon_{kk}, \qquad J_2 = \tfrac{1}{6}I_1^2 - \tfrac{1}{2}\varepsilon_{ij}\varepsilon_{ij}. \qquad [4]$$

The damage variable remains constant when $f \leq 0$; the behaviour is then linear elastic. Notice that positive values of κ_0 therefore imply the existence of a fatigue threshold. Strain states for which $f > 0$ lead to damage growth only for continued deformation and as long as the critical value $D = 1$ has not been reached, i.e., if $\dot{f} \geq 0$ and $D < 1$. When these three conditions are satisfied, the damage rate is governed by an evolution law which reads in its most general form

$$\dot{D} = g(D, \tilde{\varepsilon})\,\dot{\tilde{\varepsilon}}, \qquad [5]$$

where the dependence of the damage growth rate on the equivalent strain rate has been taken linear in order to avoid rate effects.

It is immediately clear from equation [1] that no stresses can be transferred for $D = 1$. In the local approach to fracture this critical state is used to represent a crack by a region of completely damaged material (Ω_c in Figure 1). In the remaining part of the domain, and particularly next to the crack, some (noncritical) damage may have been developed (Ω_d in the figure), while other areas may still be unaffected by damage (Ω_0). In the latter region the material has retained its virgin stiffness. Under the influence of further straining the damage variable will increase in those parts of the body where the conditions for damage growth are met. This will often be the case particularly near the strain concentration in front of the crack tip. When the critical value $D = 1$ is reached in this region, the completely damaged zone Ω_c will start to expand, thus simulating crack growth. The direction and rate of crack growth are governed by the damage growth locally near the crack tip, hence the term local approach to fracture.

For reasons of computational efficiency, the stress-strain behaviour is sometimes partly uncoupled from the growth of damage. The material in the damaged zone Ω_d then retains its virgin stiffness until the damage variable becomes critical. Upon reaching $D = 1$, the elastic stiffness is then suddenly decreased from its virgin value to zero. This method is sometimes referred to as *uncoupled* or *semi-coupled* approach, while the full model, in which the stresses are governed by relation [1], may be called *coupled* or *fully coupled* [PAA 93, LEM 96]. It is obvious that the uncoupled approach can only be followed if the influence of damage prior to failure is relatively small, which may be true in the high-cycle fatigue case considered here. However, this also cancels part of the advantage of the local approach, namely that it can account for the influence of material damage on crack growth.

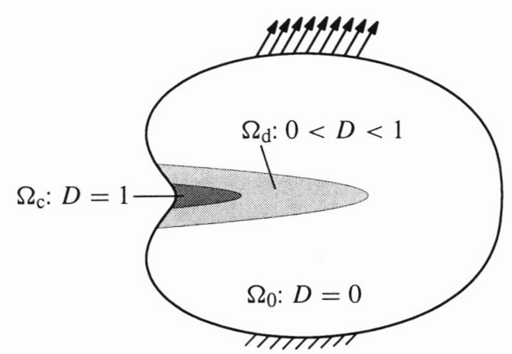

Figure 1. *Damage distribution in a continuum*

Both in the coupled and uncoupled approach it is important to realise that the local, complete loss of strength in Ω_c implies that stresses are zero for any arbitrary deformation (see equation [1]). As a consequence, the equilibrium equations are meaningless in this region. This can be seen for the elasticity-based damage model by substituting relations [1] in the standard equilibrium equations

$$\frac{\partial \sigma_{ij}}{\partial x_i} = 0. \qquad [6]$$

Making use of the right minor symmetry of the elasticity tensor (i.e., $C_{ijkl} = C_{ijlk}$) the equilibrium equations can then be written as the system of second-order partial differential equations

$$(1 - D)\, C_{ijkl} \frac{\partial^2 u_k}{\partial x_i \partial x_l} - \frac{\partial D}{\partial x_i} C_{ijkl} \frac{\partial u_k}{\partial x_l} = 0. \qquad [7]$$

For a given damage field $D(\mathbf{x}) < 1$, the displacement components u_k can be determined from this differential system and the corresponding kinematic and dynamic boundary conditions. In a crack however, where $D \equiv 1$, both terms in [7] vanish. Consequently, the partial differential equations degenerate and the boundary value problem becomes ill-posed. This indefiniteness must be avoided by limiting the equilibrium problem to the subdomain $\tilde{\Omega} = \Omega_0 \cup \Omega_d$ where $D < 1$. At the boundary between crack and remaining material the natural boundary condition $n_i \sigma_{ij} = 0$ must be applied, with the vector $\mathbf{n}$ normal to the boundary. A free boundary problem is thus obtained, in which the position of the internal boundary (the crack front and crack faces) follows from the growth of damage, see also [BUI 80].

3. Localisation and mesh sensitivity

Analyses using the local approach as described in the previous section will usually quickly result in a situation where all further growth of damage is concentrated in a surface. Since this means that the volume of material that participates in the damage development vanishes, no work is needed for the crack to propagate, even if the specific work needed by the damage process is positive. Furthermore, the crack traverses the remaining cross-section instantaneously, instead of by a small increment in every loading cycle. From a mathematical point of view, two phenomena play a role in this pathological localisation of damage growth: loss of ellipticity of the rate equilibrium equations and singularity of the damage rate. The latter cause is often not recognised in the literature, but is actually the most important in crack growth analyses, since it is responsible for the instantaneous and perfectly brittle crack growth. Loss of ellipticity, on the other hand, may result in premature initiation of cracks, before the damage variable has reached the critical level $D = 1$ in a stable way.

The rate equilibrium equations lose ellipticity when the acoustic tensor $n_i \bar{C}_{ijkl} n_l$, where $\bar{C}_{ijkl}$ denotes the tangent stiffness, becomes singular for some unit vector $\mathbf{n}$, i.e., when

$$\det\left(n_i \bar{C}_{ijkl} n_l\right) = 0. \qquad [8]$$

For the fully coupled fatigue damage model the tangential stiffness $\bar{C}_{ijkl}$ is given by

$$\bar{C}_{ijkl} = (1 - D)C_{ijkl} - g(D, \tilde{\varepsilon})C_{ijmn}\varepsilon_{mn} \frac{\partial \tilde{\varepsilon}}{\partial \varepsilon_{kl}}. \qquad [9]$$

As the damage variable grows, the second term in this expression may become of the same order as the first term, so that condition [8] may indeed be met at some point. The vector $\mathbf{n}$ is then the normal to a characteristic surface segment of the set of rate equilibrium equations. Since solutions of linear partial differential equations with smooth coefficients can have discontinuities or discontinuous derivatives only across characteristic surfaces, this opens the possibility of jumps in the velocity solution. A discontinuity of the velocity across a characteristic surface results in a strain rate singularity on this surface, which in turn renders the damage rate singular (see equation [5]). This means that for continued loading the damage variable immediately becomes critical on the characteristic segment and thus that instantaneous failure of this surface is predicted. In order to follow the stress drop resulting from this instantaneous loss of stiffness, material adjacent to the characteristic segment must unload elastically, so that the growth of damage indeed localises in the surface segment.

Loss of ellipticity plays an important role in localisation of deformation and damage in static fracture, where damage growth is fast right at the onset [PEE 96]. In fatigue, however, the initial growth of damage is usually slow and the ellipticity of the rate equilibrium problem is therefore preserved until near the end of the fatigue life.

Indeed, it can be easily seen that for the uncoupled approach loss of ellipticity cannot occur before the damage variable becomes critical. Since the tangential stiffness tensor $\bar{C}_{ijkl}$ equals the elasticity tensor C_{ijkl} in this case, the characteristic determinant equals

$$\det\left(n_i C_{ijkl} n_l\right) = \mu^2(\lambda + 2\mu), \qquad [10]$$

where λ, μ denote Lamé's constants. Expression [10] is always positive and the problem therefore remains elliptic until $D = 1$. As a result, the damage growth remains stable and affects a finite volume throughout the crack initiation phase in fatigue.

Once a crack has been initiated, however, the continued growth of damage nevertheless localises in a surface and the predicted crack growth becomes nonphysical again. This localisation during crack growth is not due to loss of ellipticity, but is related to the strain singularity which is inevitably present at the tip of the crack. Once the damage variable becomes critical in a certain point and a crack is thus initiated, the displacement and velocity must become discontinuous across this crack. This implies that the strain (rate) field at the crack tip becomes singular. Since the damage growth rate is directly related to the equivalent strain, the strain singularity at the tip results in an infinite damage rate. For continued deformation all stiffness is therefore lost instantaneously at the most critical point in front of the crack tip and the crack thus starts to propagate. Since the material adjacent to the crack must unload elastically in order to follow the resulting stress drop, the width of the crack remains zero. This implies that the strain and damage growth rate at the crack tip remain singular as the crack grows and consequently that the crack grows at an infinite rate. No work is needed in this fracture process, since it involves damage growth in a vanishing volume.

It is emphasised once more that this mechanism of damage localisation and instantaneous crack growth is activated even if the rate equilibrium equations remain elliptic until $D = 1$. Loss of ellipticity may cause premature initiation of a crack and thus result in perfectly brittle crack growth, but the pathological propagation behaviour is nevertheless due to the singularity of the damage rate at the crack tip. This is true even in models of static fracture, but the problem is aggravated in this case by the fact that a crack is initiated shortly after the onset of damage as a result of loss of ellipticity.

Finite element solutions try to follow the nonphysical behaviour of the continuum model as described above, but are limited in doing so by their finite spatial resolution. In standard finite element methods the displacement field must be continuous. The displacement jumps and singular strains of the actual solution can therefore only be approximated by high, but finite displacement gradients in the finite element solution. As a consequence, a finite volume is involved in the damage process, and a positive amount of energy is dissipated. Also, because the damage growth rate at the tip of the damage band remains finite, the crack propagates at a finite velocity. When the spatial discretisation grid is refined, however, the finite element approximation becomes more accurate in the sense that the displacement gradients which describe the discontinuities become stronger. Consequently, the predicted fracture energy becomes smaller and the crack propagates faster. In the limit of vanishingly small elements, the actual solution is retrieved, i.e., a vanishing fracture energy and an infinite crack growth rate. This convergence of the finite element approximation to the actual, nonphysical solution of the problem is the origin of the apparent mesh sensitivity of damage models and other continuous descriptions of fracture.

An example of the apparent mesh sensitivity is given in Figure 2. The diagram shows the steady-state fatigue crack growth rate predicted by a finite element analysis versus the size of the elements which were used in the analysis. The problem geometry, loading conditions and modelling for which these results have been obtained will be detailed in Section 6. Fully coupled as well as uncoupled analyses have been done. The dependence of the crack growth per cycle $\mathrm{d}a/\mathrm{d}N$ on the element size h is quite strong in both approaches: a decrease of the element size by roughly one decade leads to an increase of the crack growth rate by almost three decades. In the limit $h \to 0$ the crack growth rate clearly goes to infinity, as predicted by the discussion above.

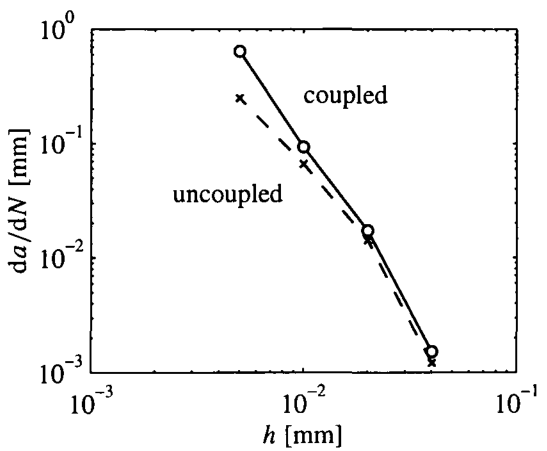

Figure 2. *Predicted fatigue crack growth rate versus element size*

4. Nonlocal modelling

An effective method to avoid pathological localisation of damage is to add nonlocal terms to the constitutive model. This approach has been successfully applied to damage models of a number of failure mechanisms [BAŽ 84, PC 87, SAA 89, TVE 95]. The spatial interactions resulting from the nonlocality prevent the damage growth from localising in a surface. Instead, the damage growth occupies a finite band, the width of which is related to the internal length scale provided by the nonlocality. In its traditional, integral form, nonlocality can be introduced in the fatigue model of Section 2 by rewriting the loading function [2] and the damage evolution law

[5] in terms of a new field variable, the nonlocal equivalent strain $\bar{\varepsilon}$:

$$f(\bar{\varepsilon}, \kappa) = \bar{\varepsilon} - \kappa \quad [11]$$

$$\dot{D} = g(D, \bar{\varepsilon})\,\dot{\bar{\varepsilon}}. \quad [12]$$

The nonlocal equivalent strain is defined by (cf. [PC 87])

$$\bar{\varepsilon}(\mathbf{x}) = \int_{\tilde{\Omega}} \psi(\mathbf{y}; \mathbf{x})\tilde{\varepsilon}(\mathbf{y})\, \mathrm{d}\Omega, \quad [13]$$

with $\psi(\mathbf{y}; \mathbf{x})$ a weight function which usually decays rapidly with the distance $|\mathbf{y} - \mathbf{x}|$. The nonlocality is apparent from equations [12] and [13]: the damage growth and thus the stresses in $\mathbf{x}$ are influenced by the value of the strain in other points $\mathbf{y} \neq \mathbf{x}$.

Instead of the integral definition [13] of the nonlocal strain, we will use the boundary value problem given by

$$\bar{\varepsilon} - c\nabla^2\bar{\varepsilon} = \tilde{\varepsilon}, \quad [14]$$

$$\frac{\partial \bar{\varepsilon}}{\partial n} = 0, \quad [15]$$

to define the nonlocal equivalent strain $\bar{\varepsilon}$ [PEE 96]. The solution of this problem can formally be written as:

$$\bar{\varepsilon}(\mathbf{x}) = \int_{\tilde{\Omega}} G(\mathbf{y}; \mathbf{x})\tilde{\varepsilon}(\mathbf{y})\, \mathrm{d}\Omega, \quad [16]$$

where $G(\mathbf{y}; \mathbf{x})$ denotes the Green's function associated with it [PEE 99, PEE 01]. Expression [16] takes exactly the same form as equation [13] for the nonlocal model. This means that the enhanced damage model based on the differential equation [14] is a member of the class of nonlocal models defined by [13]. The parameter c in equation [14], which is of the dimension length squared, sets the internal length scale of the model and thus determines the degree to which damage growth localises.

It should be noted that in the presence of cracks, equation [14] is defined only on the domain $\tilde{\Omega}$ where the damage variable has not yet become critical. This is not only natural, since the equilibrium problem is defined only on $\tilde{\Omega}$, but also necessary because the right-hand side $\tilde{\varepsilon}$ is not uniquely defined in the cracked region as a result of the indefiniteness of the displacement field (Section 2). The boundary condition [15] associated with equation [14], as well as the standard boundary conditions for the equilibrium problem, must therefore be defined on the boundary $\tilde{\Gamma}$ of $\tilde{\Omega}$. This means that they are imposed not only at the boundary of the problem domain, but also at the internal boundary which represents the crack contour and that they move with the crack contour as the crack grows. In terms of the original nonlocal formulation, the integration in equation [13] must be limited to $\tilde{\Omega}$ and must therefore be re-evaluated as the crack grows. The necessity of this identical treatment of internal and external boundaries does not seem to have been recognised in the literature. If the – nonphysical – strains in the crack are included in the integral in [13] the computed nonlocal equivalent strain and damage rate at the crack tip are too high, resulting in an overprediction of the rate of crack growth. Furthermore, since the nonlocal strain at the crack faces continues to increase as the crack opens, the damage variable continues to grow at the crack faces. As a consequence, the width of the crack region continues to increase along the entire crack surface, until it finally occupies the entire domain. Both effects can also be observed in numerical analyses if the cracked zone is not properly separated from the remaining material.

It has been argued in Section 3 that the standard, local damage model predicts the immediate initiation of a crack when the displacement field becomes discontinuous upon loss of ellipticity of the rate equilibrium problem. It can be easily shown for the nonlocal model that this loss of ellipticity no longer occurs. The characteristic determinant associated to the set of rate equilibrium equations and equation [14] is given by [PEE 99]

$$\det\left(n_i \bar{C}^{\star}_{ijkl} n_l\right) = c(1-D)^3 \mu^2 (\lambda + 2\mu), \qquad [17]$$

where $\bar{C}^{\star}_{ijkl}$ $(i, l = 1, 2, 3,\ j, k = 1, 2, 3, 4)$ contains the coefficients of the second-order derivatives in the combined set of equations. Expression [17] is positive for all $\mathbf{n}$ as long as $D < 1$. This means that the partial differential system is elliptic throughout the initiation phase. However, when $D = 1$ somewhere in the domain, and a crack is initiated, a strain singularity may be unavoidable at the crack tip. It is important that the damage growth rate remains finite, because the crack growth would otherwise be instantaneous (see Section 3). Since the damage growth rate depends on the nonlocal equivalent strain $\bar{\varepsilon}$ in the nonlocal formulation, this implies that $\bar{\varepsilon}$ must remain finite at the crack tip in order to have a finite crack growth rate.

An analytical expression for the nonlocal strain can be obtained for a crack in an infinite, linear elastic medium. It is emphasised that this situation is not entirely representative of a crack in a damaged medium, because the influence of damage on the deformation near the crack tip is not accounted for. However, the analysis is illustrative of the way in which the nonlocality removes the damage rate singularity and thus localisation of damage in a surface. A plane crack in an infinite medium is considered, which is loaded in mode I. Furthermore, a plane stress state is assumed throughout the medium. The asymptotic local equivalent strain field can then be determined from linear fracture mechanics and shows the usual $r^{-1/2}$-singularity:

$$\tilde{\varepsilon}(r, \vartheta) = \frac{K_{\mathrm{I}}}{2E\sqrt{2\pi r}} \sqrt{(1 - \cos\vartheta)(5 - 3\cos\vartheta)}. \qquad [18]$$

The nonlocal equivalent strain is now obtained by solving the boundary value problem [14]–[15] for this source term. This results for the nonlocal strain at the crack tip in [PEE 99]:

$$\bar{\varepsilon}(0,0) = \frac{K_{\mathrm{I}}\Gamma^2\left(\frac{3}{4}\right)}{\pi^{3/2}E\sqrt[4]{c}}\left(1 + \frac{1}{2\sqrt{3}}\operatorname{arctanh}\left(\tfrac{1}{2}\sqrt{3}\right)\right), \qquad [19]$$

with $\Gamma(\alpha)$ the gamma function. This expression is indeed finite for $c > 0$, so that the damage growth rate at the crack tip remains finite in the enhanced model. This in turn means that in this simplified situation a finite crack growth rate is obtained instead of the instantaneous fracture predicted by the standard, local damage model. As was mentioned earlier, this does not necessarily imply that singularities are avoided also in the full, coupled model, where the singularity of $\tilde{\varepsilon}$ may be stronger. However, numerical simulations (see Section 6) seem to indicate that this is indeed the case.

5. Aspects of the finite element implementation

In mathematical terms the essential difference between the nonlocal damage formulation introduced in Section 4 and the classical, local damage models consists of the additional linear partial differential equation [14]. This equation must be solved simultaneously with the standard equilibrium equations. For finite element implementations this means that $\bar{\varepsilon}$ must be discretised in addition to the displacement components. The discrete form of [14] follows from the standard transition to a weak form and Galerkin discretisation of the nonlocal equivalent strain [PEE 96]:

$$\int_{\bar{\Omega}} \left(c\,\bar{\mathbf{B}}^{\mathrm{T}}\bar{\mathbf{B}} + \bar{\mathbf{N}}^{\mathrm{T}}\bar{\mathbf{N}}\right) \mathrm{d}\Omega\, \mathbf{e} = \int_{\bar{\Omega}} \bar{\mathbf{N}}^{\mathrm{T}}\tilde{\varepsilon}\, \mathrm{d}\Omega, \qquad [20]$$

where the matrices $\bar{\mathbf{N}}$ and $\bar{\mathbf{B}}$ contain the interpolation functions of the nonlocal equivalent strain and their derivatives and the column matrix $\mathbf{e}$ contains the nodal values of $\bar{\varepsilon}$. The discrete form of the equilibrium equations follows in the standard way:

$$\int_{\bar{\Omega}} \mathbf{B}^{\mathrm{T}}\boldsymbol{\sigma}\, \mathrm{d}\Omega = \int_{\bar{\Gamma}} \mathbf{N}^{\mathrm{T}}\mathbf{t}\, \mathrm{d}\Gamma, \qquad [21]$$

with the matrices $\mathbf{N}$ and $\mathbf{B}$ containing the displacement interpolation functions and their derivatives, respectively, and the column matrices $\boldsymbol{\sigma}$ and $\mathbf{t}$ the Cauchy stresses and boundary tractions. The finite element interpolations of the displacements and the nonlocal strain need to satisfy only the standard, $\mathcal{C}_0$-continuity requirements. The order of each of the interpolations can be selected independently, although some combinations may result in stress oscillations [PEE 99].

Since damage growth is defined by relation [12] in a rate format, it must be integrated over each time increment of the numerical analysis in order to obtain the damage at the end of the increment. Standard integration rules may be used for this purpose, but in high-cycle fatigue analyses it may be advantageous to use a more sophisticated integration, which takes into account the cyclic character of the loading [PEE 99, PEE 00]. After discretisation in time by either method, [20] and [21] become a set of nonlinear algebraic equations, which can be solved for instance using a Newton-Raphson scheme, see references [PEE 99, PEE 00] for details. It is at this point that the gain in efficiency of an uncoupled approach becomes apparent: if the effect of damage growth on the stiffness is neglected while the damage variable is noncritical, the problem will often remain linear as long as no additional elements fail (equation [20] may be nonlinear for nonproportional loading and for some equivalent strain definitions). As a result, the tangent stiffness matrix remains constant and one iteration suffices in each increment to reach equilibrium.

It has already been argued that the cracked region, Ω_c, should not be part of the equilibrium problem domain because the equilibrium equations are not meaningful in it. Accordingly, the equilibrium equations and the additional equation [14] are defined only on the domain $\tilde{\Omega} = \Omega \setminus \Omega_c$ and boundary conditions are provided at the boundary $\tilde{\Gamma}$ of $\tilde{\Omega}$. For the finite element formulation this means that the discretisation of the equilibrium problem must also be limited to the noncritical domain $\tilde{\Omega}$. The difficulty is, however, that this effective domain will gradually shrink as the predicted crack growth progresses. Consequently, the problem domain must be redefined in the numerical analysis for each increment of crack growth and a new finite element discretisation must be defined. This remeshing is often avoided by using the original domain Ω even if this domain contains a crack. The material in the crack is then given a small residual stiffness in order to avoid singularity of the discrete equilibrium equations. It is then argued that the stresses which are still transferred by the crack influence equilibrium only marginally if the residual stiffness is sufficiently small. This may indeed be true in local damage models, in which the nonphysical strains in the crack do not influence the surrounding material. But if this approach is followed for nonlocal damage models, the nonlocal equivalent strain maps the large strains which may be computed in the cracked region onto the surrounding material in which the damage variable is not (yet) critical. This does not only result in faster growth of damage in front of the crack and consequently in higher predicted crack growth rates, but also in damage growth at the faces of the crack, thus causing the thickness of the crack region to increase unboundedly. The numerical implementation should therefore reflect the mathematical separation of the crack and the remaining domain by adapting the finite element mesh to the growth of the crack. This can be achieved without full remeshing of the problem if completely damaged elements are removed from an otherwise fixed finite element mesh. However, the crack contour always follows the (initial) grid lines in this approach, which means that a fine discretisation is needed in a relatively large region.

6. Application

The nonlocal damage formulation has been used to model crack initiation and growth due to fatigue. Reference is made to [PEE 99, PEE 00] for details of the damage modelling. The problem geometry of Figure 3 has been considered. The thickness of the specimen is 0.5 mm. The lower edge of the specimen is fixed in all directions, while fully reversed vertical displacement cycles with an amplitude of 0.0048 mm are forced upon its top edge. Because of symmetry, only half of the specimen has been modelled in the finite element analyses. The reference mesh contains a regular grid of elements with an edge length $h = 0.04$ mm in an area of approximately $0.65 \times 0.12\,\text{mm}^2$ at the notch tip (indicated in Figure 3). The discretisation has been successively refined in this area to $h = 0.02, 0.01$ and 0.005 mm. Quadrilateral plane-stress elements with bilinear displacement and nonlocal strain interpolations and a constant damage variable have been used. Both fully coupled and uncoupled analyses have been done. In these analyses, elements were removed when the damage variable exceeded 0.999999, after which led to this critical damage value was recomputed starting from the converged state in the previous increment [PEE 99].

Figure 4 shows the crack initiation and growth process as simulated using the finest of the four meshes and the coupled approach. The area which is shown in this figure is the refined area indicated in Figure 3. The stress concentration at the notch tip initially leads to a concentration of damage at the tip. After 4210 cycles a crack is initiated, i.e., the damage variable becomes critical in an element which is then removed from the mesh. For continued cycling the crack grows along the symmetry axis. The crack width decreases as the damage zone which was formed before crack initiation is traversed. Beyond this damage zone the crack width becomes stationary at 0.04 mm, which is of the same order as the internal length $\sqrt{c} = 0.1$ mm.

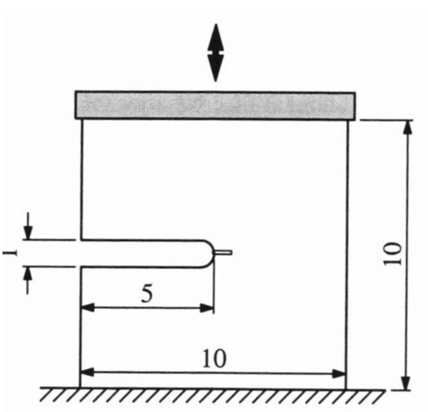

Figure 3. *Problem geometry and loading conditions of the fatigue problem (dimensions in mm)*

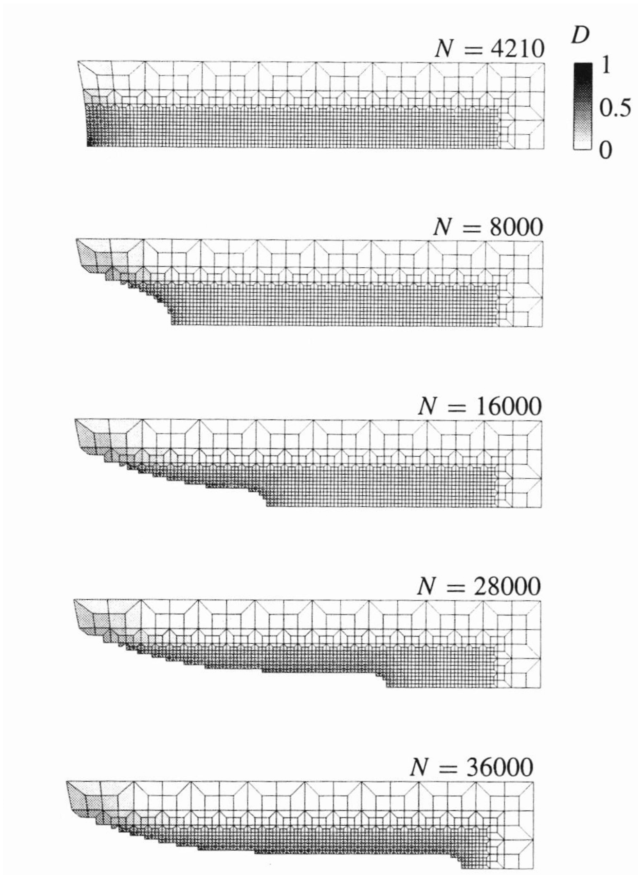

Figure 4. *Damage and crack growth at the notch tip in the* $h = 0.005$ *mm mesh (coupled approach)*

The influence of the finite element discretisation on the crack shape is shown in Figure 5, in which the final crack pattern has been plotted for the four discretisations. The coarsest mesh (Figure 5(a)) gives a rather crude approximation of the crack shape and necessarily overestimates the width of the steady-state part of the crack because this width is smaller than the element size. But the $h = 0.02$ and 0.01 mm meshes give a good approximation of the crack shape in the finest discretisation. The steady-state width of the crack does not vary between the three finest discretisations. The final damage and crack patterns obtained with the uncoupled approach are almost identical to the ones shown in Figure 5. However, there is a slight difference in the number of loading cycles needed to reach these states. This is illustrated in Figure 6, which shows the length of the crack, a, versus the number of loading cycles, N, for the four meshes in the coupled as well as the uncoupled approach. For an increasingly refined discretisation the growth curves converge to a response with a finite number of

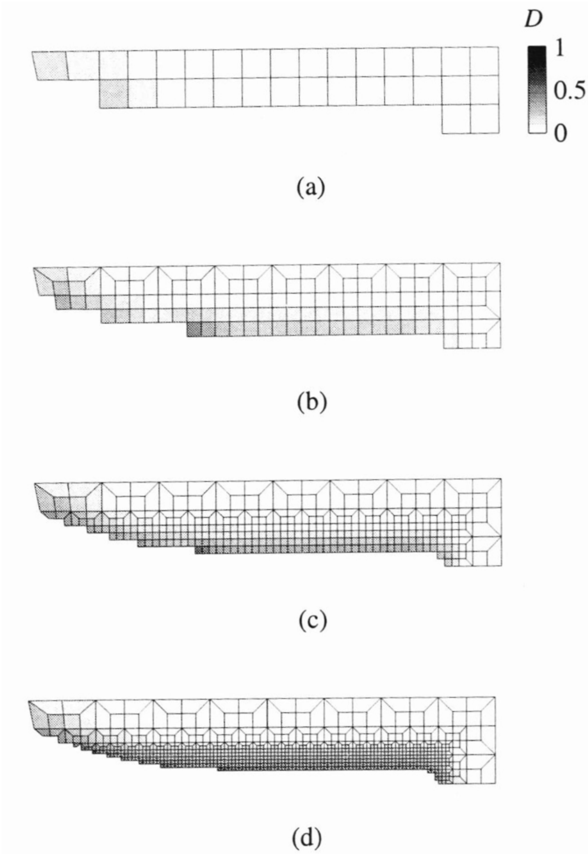

Figure 5. *Final crack pattern in the (a) $h = 0.04\,mm$, (b) $h = 0.02\,mm$, (c) $h = 0.01\,mm$ and (d) $h = 0.005\,mm$ meshes (coupled approach)*

cycles to crack initiation and a finite growth rate, instead of the instantaneous growth predicted by the local model. In the uncoupled model, the crack is initiated slightly later and grows slightly slower than in the fully coupled model. This is due to the fact that the damage in front of the crack tip has no influence on the deformation, which is therefore smaller. This results in a smaller damage rate and thus in slower crack growth. The steady-state crack growth rate obtained in both approaches has been plotted versus the element size h in Figure 7. In contrast with the local damage model (Figure 2) the growth rate in the nonlocal models becomes practically constant as the element size is reduced.

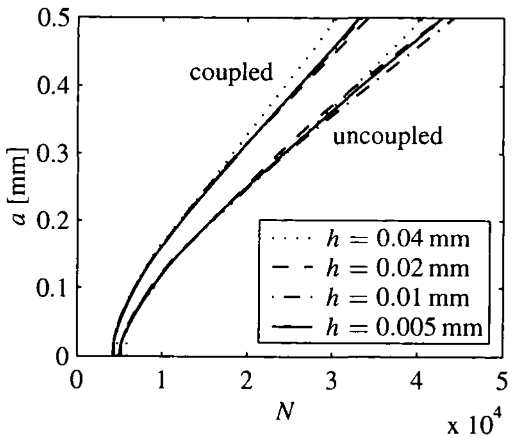

Figure 6. *Influence of the element size on the predicted crack growth in the nonlocal damage model*

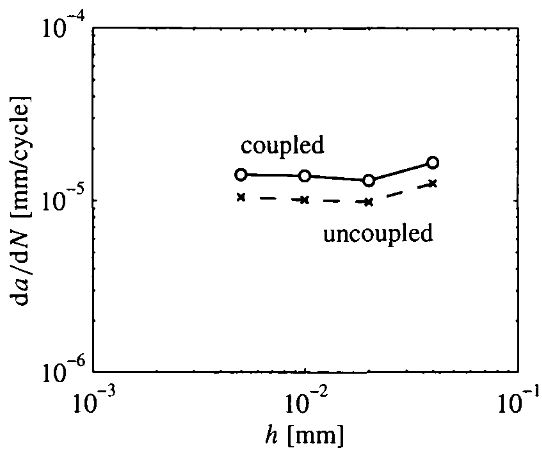

Figure 7. *Influence of the element size on the steady-state crack growth rate*

7. Discussion and concluding remarks

A key issue in the development of fracture models based on a continuum damage approach is their ability to correctly describe the localised deformations which are typical of fracture problems. If this issue is not properly addressed, the damage process which represents the initiation and growth of cracks tends to localise in a vanishing volume. A perfectly brittle response is then obtained, even if the constitutive relations have been designed to show a gradual loss of strength. This pathological localisation of damage is not so much caused by loss of ellipticity of the rate equilibrium equations, but rather due to singularities at the crack tip. It therefore occurs not only in fully coupled analyses, but also in uncoupled analyses, in which the damage variable does not immediately affect the constitutive behaviour. As a result of the singularities, the material in front of the crack fails immediately and in a vanishing volume, even if the rate equilibrium equations do not first lose ellipticity. The crack traverses the remaining cross section at an infinite growth rate and the thickness of the corresponding damage band is zero.

The nonphysical behaviour of the standard models can be effectively removed by the introduction of nonlocality in the constitutive relations. This can be achieved by including an additional partial differential equation in the equilibrium problem. As a result, the localisation of damage is limited to the scale of the intrinsic length which is introduced by the nonlocality. Crack growth is no longer instantaneous and a positive volume takes part in the damage process which describes the crack growth. This also means that a positive amount of work is needed for the crack growth and that the fracture process is thus no longer perfectly brittle.

Additional boundary conditions must be provided in the nonlocal model, not only at the boundary of the problem domain, but also at the internal boundary which describes the crack contour. The latter ensures that the crack is well separated from the remaining part of the continuum and that nonphysical deformations which may be computed in the cracked region do not affect the growth of damage at the crack faces. The numerical implementation of the nonlocal model must reflect this separation. This means that the spatial discretisation of the equilibrium problem must be adapted for each increment of crack growth. If this separation is not made rigorously, the damage growth rate may be overestimated and nonphysical damage growth may be predicted at the faces of the crack. In this contribution, a rigorous but crude approach has been followed: completely damaged elements are removed from the finite element mesh. Meaningful, mesh-objective numerical solutions have been obtained with this technique for the nonlocal formulation of the coupled as well as the uncoupled problem. Although reliable and useful for development purposes, the approach is not very suitable for practical problems. The location of crack initiation and the direction of crack growth are usually not known in advance. In this case, adaptive spatial discretisation techniques are needed to follow the free boundary which represents the crack contour and to accurately describe the high deformation gradients at its tip.

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