Numerical aspects of nonlocal damage analyses

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ABSTRACT. *Constitutive models based on nonlocal variables provide an effective and mechanically sound solution to the ill-posedness of the boundary value problem in the presence of damage induced softening. However, the averaging of constitutive variables entails other computational problems like the lack of symmetry of the tangent operator in a finite element approximation. In the present paper, an isotropic local damage model with symmetric tangent matrix is presented. Two alternative nonlocal versions of the same model are comparatively discussed. It is shown how the symmetry of the tangent matrix in the finite element approximation can be preserved formulating the nonlocal model within the context of the thermodynamic nonlocal theory recently proposed by Borino et al. The computational implications of the adopted regularization technique are discussed by means of a simple one-dimensional example.*

RÉSUMÉ. *Les lois de comportement non locales permettent de reformuler de façon efficace et mécaniquement satisfaisante le problème aux limites initialement mal posé en présence d'adoucissement induit par l'endommagement. Cependant, l'utilisation de variables moyennes pose des problèmes numériques, comme le manque de symétrie de l'opérateur tangent dans une discrétisation par éléments finis. Dans ce papier, on présente un modèle d'endommagement isotrope local avec matrice tangente symétrique. Deux versions différentes non locales de ce modèle sont comparées. On montre comment la symétrie de la matrice tangente algorithmique peut être préservée en formulant le modèle non local avec la théorie non locale thermodynamique proposée récemment par Borino* et al. *Les conséquences numériques de la méthode de régularisation adoptée sont illustrées par un exemple simple monodimensionel.*

KEYWORDS: *finite elements, damage, nonlocal regularization.*

MOTS-CLÉS : *éléments finis, endommagement, régularisation non locale.*

1. Introduction

In many instances of practical interest, the initiation of fracture is preceded by a significant strain localization phase in which the material is macroscopically integer and inelastic phenomena tend to be confined in a narrow region. In this phase, the use of continuum models with softening, like *e.g.* damage models, is justified. However, the strain softening behavior due to the development of material damage is well known to produce unrealistic mesh sensitivity in standard finite element application. Zero energy dissipation is expected in the limit since strains tend to localize on a zero volume region as the mesh is refined. In statics, the failure of classical discretization methods can be explained, from the mathematical point of view, with the boundary value problem loosing ellipticity as a consequence of the softening material behavior. The ill-posedness of the boundary value problems reflects the fact that standard continuum mechanics theories are not appropriate when the microscopic material heterogeneity is characterized by an internal length which is not negligible if compared to the typical macroscopic length of the structure, so that the range of the microscopic interaction forces has to be considered long with respect to the macroscopic scale (see *e.g.* [GAN 00] for a recent discussion).

Among the several regularization techniques proposed in the literature, one of the most computationally convenient seems the one based on the formulation of a nonlocal continuum (see [ERI 81] for nonlocal plasticity). The idea is that the long range nature of the microscopic interaction forces is taken into account at the macroscale by expressing the material constitutive law in terms of one or more nonlocal variables defined as suitable weighted averages of their local values over the interaction domain. In the formulation of a nonlocal model, several choices have to be made such as the definition of the nonlocal variable (variables), the definition of the weight function and the definition of the interaction domain.

The adopted choices have important numerical consequences in finite element implementations (see [JIR 98] for a discussion of other aspects): the corrector phase of the iterative procedure, typically carried out at each Gauss point separately, may cease to be local [STR 96]; the consistent tangent matrix becomes non-symmetric [BAZ 88], [PIJ 95], [JIR 99]. The lack of symmetry has important consequences both from the theoretical and computational point of view. In particular a non symmetric model is not suitable for variational approaches and non symmetric solvers have to be used in numerical applications, with consequent increase of computing costs.

In the present paper the discussion is confined to isotropic damage models. Within this context, it is shown that it is possible to formulate a very general isotropic local model endowed with a symmetric consistent tangent matrix. The model considered is based on the definition of two damage variables affecting the shear and bulk moduli separately. The consistent tensor of tangent elastic moduli is derived and it is shown that it is symmetric provided that associative evolution equations are assumed for both damage and kinematic internal variables.

A nonlocal formulation of the model is then proposed, based on the thermodynamic formulation of Borino *et al.* [BOR 99], [BEN 00]. While in [BEN 00] a kinematic internal variable was assumed as the primal nonlocal variable, in the model here proposed the primal nonlocal variables are the damage variables. Following [BOR 99], the nonlocality is transferred onto the conjugate variables which in the present case are the energy release rates, by means of an energy equivalence which allows to eliminate the so called nonlocality residual [ERI 81]. It is shown that, unlike in [BEN 00], the proposed model maintains the attractive feature that all constitutive computations can be performed locally, at the Gauss point level [PIJ 87], [COM 00], and that it gives rise to a symmetric finite element tangent stiffness matrix.

A one-dimensional problem is studied for a simplified version of the model, with a single damage variable. The results obtained with the proposed dual nonlocal formulation and with the standard nonlocal formulation of [COM 00] are compared.

2. A "symmetric" isotropic local damage model

Let $\mathbf{e} = \boldsymbol{\varepsilon} - 1/3\mathbf{I}\varepsilon_v$ be the deviatoric part of the strain tensor $\boldsymbol{\varepsilon}$, ε_v being its volumetric part and $\mathbf{I}$ the second order identity tensor. The free energy density potential under isothermal conditions for the proposed damage model is defined as

$$\Psi = \frac{1}{2}(1-d_G)2G_0\mathbf{e}:\mathbf{e} + \frac{1}{2}(1-d_K)K_0\varepsilon_v^2 + h(\xi) \qquad [1]$$

where G_0 and K_0 are the initial elastic and shear moduli, respectively, d_G and d_K are shear and volumetric damage variables and ξ is a scalar variable of kinematic nature. The state equations defining the conjugate static variables are given by

$$\begin{gathered} \mathbf{s} = \frac{\partial \Psi}{\partial \mathbf{e}} = (1-d_G)2G_0\mathbf{e}; \quad p = \frac{\partial \Psi}{\partial \varepsilon_v} = (1-d_K)K_0\varepsilon_v; \quad \chi = \frac{\partial \Psi}{\partial \xi} = \frac{\partial h}{\partial \xi} \\ Y_G = -\frac{\partial \Psi}{\partial d_G} = \frac{1}{2}(2G_0\mathbf{e}:\mathbf{e}); \quad Y_K = -\frac{\partial \Psi}{\partial d_K} = \frac{1}{2}K_0\varepsilon_v^2 \end{gathered} \qquad [2]$$

where $\mathbf{s} = \boldsymbol{\sigma} - \mathbf{I}p$ is the stress deviator and $p = 1/3\sigma_{kk}$ is the mean stress; χ is a static internal variable and Y_G, Y_K represent the elastic energy release rates.

The activation of damage is governed by the following activation functions and loading-unloading conditions

$$f\left(Y_G, Y_K, \chi\right) \leq 0, \quad \dot{\gamma} \geq 0; \quad \dot{\gamma} f = 0 \qquad [3]$$

$\dot{\gamma}$ being a scalar dissipation multiplier. The associative evolution equations are given by

$$\dot{d}_G = \frac{\partial f}{\partial Y_G}\dot{\gamma}; \quad \dot{d}_K = \frac{\partial f}{\partial Y_K}\dot{\gamma}; \quad \dot{\xi} = -\frac{\partial f}{\partial \chi}\dot{\gamma} \qquad [4]$$

Finally, the rate of dissipation density is given by

$$D = Y_G \dot{d}_G + Y_K \dot{d}_K - \chi \dot{\xi} \qquad [5]$$

REMARK 1.– The presence of separate damage variables d_G, d_K adds flexibility to the model. The activation function may be defined in a form more suited for materials with non-symmetric tension-compression behavior like concrete and the separate evolution equations for deviatoric and volumetric damages allow for a varying Poisson's coefficient while preserving the isotropic nature of the model. □

REMARK 2.– The scalar internal variable ξ accounts for material rearrangements at the microscale due to damage development. Damage is the only dissipation mechanism considered in this model. □

In finite element applications, the constitutive law is integrated within a time-step, in the corrector phase of the iterative procedure, according to an Euler backward-difference scheme. This implies computing all derivatives in [4 at the end of the step. At the end of the corrector phase, a relation between stress and strain increments is implicitly obtained: $\Delta\boldsymbol{\sigma} = \Delta\boldsymbol{\sigma}(\Delta\boldsymbol{\varepsilon})$. In the subsequent predictor phase, the consistent tangent elastic tensor is computed by differentiating this relation under the assumption of continuous loading in the increment, i.e.

$$\delta f = \frac{\partial f}{\partial Y_G}\delta Y_G + \frac{\partial f}{\partial Y_K}\delta Y_K + \frac{\partial f}{\partial \chi}\delta\chi = 0 \qquad [6]$$

with all quantities evaluated at the end of the step. Taking into account eqs. [4_3 and 2_3], one has

$$\delta\gamma = A\left(\frac{\partial f}{\partial Y_G}\delta Y_G + \frac{\partial f}{\partial Y_K}\delta Y_K\right) \quad \text{with} \quad \begin{aligned} A &= \left(\frac{\partial f}{\partial \chi}\frac{\partial^2 h}{\partial \xi^2}\frac{\partial f}{\partial \chi}\right)^{-1} \\ \delta Y_G &= 2G_0 \mathbf{e} : \delta\mathbf{e} \\ \delta Y_K &= K_0 \varepsilon_v \delta\varepsilon_v \end{aligned} \qquad [7]$$

After some algebra, the explicit expression of the consistent tangent elastic tensor is obtained as:

$$\delta\boldsymbol{\sigma} = \mathsf{D}^{ed} : \delta\boldsymbol{\varepsilon}$$

$$\mathsf{D}^{ed} \equiv 2G_0\left(1-d_G\right)\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I} + \left[K_0\left(1-d_K\right) - \frac{2}{3}G_0\left(1-d_G\right) - K_0^2 A\left(\frac{\partial f}{\partial Y_K}\right)^2 \varepsilon_v^2\right]\mathbf{I}\otimes\mathbf{I} +$$
$$-4G_0^2 A\left(\frac{\partial f}{\partial Y_G}\right)^2 \mathbf{e}\otimes\mathbf{e} - 2G_0K_0A\frac{\partial f}{\partial Y_K}\frac{\partial f}{\partial Y_G}\varepsilon_v\left(\mathbf{e}\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{e}\right) \qquad [8]$$

where $\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I}$ denotes the fourth order symmetric identity tensor of components $\left(\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I}\right)_{ijhk} = 1/2\left(\delta_{ih}\delta_{jk} + \delta_{ik}\delta_{jh}\right)$.

3. Nonlocal version of the "symmetric" isotropic damage model

The nonlocal version of the model is obtained substituting one of the constitutive variables by its weighted average over the whole domain Ω of the structure. The averaged quantity reflects the effect of the interaction at the microscale between the considered material point and the neighboring points. The decaying effect of the interaction with the distance is taken into account by the weighting function. In the literature, there exist several proposals concerning the choice of the nonlocal variable (see [JIR 98] and [GAN 99] for a recent discussion on the subject). From the computational standpoint, the most convenient choices are those which allow to carry out the constitutive calculations locally at each Gauss point, without introducing any coupling at constitutive level as, *e.g.*, in [PIJ 87] and [COM 00], where the strain invariants have been selected as nonlocal variables. In the present context, this would imply defining two nonlocal variables as follows

$$\bar{Y}_G\left(\mathbf{x}\right) = \int_\Omega W\left(\mathbf{x},\mathbf{s}\right)Y_G\left(\mathbf{s}\right)ds, \quad \bar{Y}_K\left(\mathbf{x}\right) = \int_\Omega W\left(\mathbf{x},\mathbf{s}\right)Y_K\left(\mathbf{s}\right)ds \qquad [9]$$

with:

$$W\left(\mathbf{x},\mathbf{s}\right) \equiv \frac{W_0\left(\|\mathbf{x}-\mathbf{s}\|\right)}{\bar{W}\left(\mathbf{x}\right)}, \quad W_0\left(\|\mathbf{x}-\mathbf{s}\|\right) \equiv \exp\left(-\frac{\|\mathbf{x}-\mathbf{s}\|^2}{2l_c^2}\right), \quad \bar{W}\left(\mathbf{x}\right) \equiv \int_\Omega W_0\left(\|\mathbf{x}-\mathbf{s}\|\right)ds \qquad [10]$$

l_c being a material internal length related to the width of the localization zone. The particular definition of the weighting function W accounts for the effect of the boundary on the nonlocal interaction at the microscale and allows to reproduce in a simple way a uniform field. In other words, if Y_G is constant over the body, it seems logic and desirable that also $\bar{Y}_G$ be uniform. However, the adopted definition of W in [10]$_1$ is such that $W(\mathbf{x},\mathbf{s}) \neq W(\mathbf{s},\mathbf{x})$. This lack of symmetry of the weighting function entails that also the consistent tangent operator is not symmetric for the nonlocal model [BAZ 88], [PIJ 95], [JIR 99] even if the consistent tangent operator of the underlying local model is symmetric.

A more rigorous treatment of the boundary effect could be inspired to homogeneization techniques for periodic structures in the proximity of geometric boundaries (see *e.g.* [LEG 97]).

The non-symmetric nonlocal version of model [1]-[5] is governed by eqs. [1], [2] and by the following activation conditions and evolution equations

$$\begin{gathered} f\left(\bar{Y}_G, \bar{Y}_K, \chi\right) \leq 0, \quad \dot{\gamma} \geq 0; \quad \dot{\gamma} f = 0 \\ \dot{d}_G = \frac{\partial f}{\partial \bar{Y}_G}\dot{\gamma}; \quad \dot{d}_K = \frac{\partial f}{\partial \bar{Y}_K}\dot{\gamma}; \quad \dot{\xi} = -\frac{\partial f}{\partial \chi}\dot{\gamma} \end{gathered} \qquad [11]$$

A symmetric nonlocal formulation of the same local damage model can be achieved following the thermodynamic nonlocal approach of Borino *et al.* [BOR 99], [BEN 00]. An application of that theory to the present model which preserves the computational advantages of the above non-symmetric nonlocal model is obtained by assuming that the damage variables are the variables reflecting at the macroscale the microscopic interaction due to the heterogeneity of the material and, therefore, have to be considered nonlocal [BAZ 88]. Hence, one can set:

$$\begin{gathered} \Psi = \frac{1}{2}\left(1-\bar{d}_G\right)2G_0\mathbf{e}:\mathbf{e} + \frac{1}{2}\left(1-\bar{d}_K\right)K_0\varepsilon_v^2 + h(\xi) \\ \mathbf{s} = \frac{\partial \Psi}{\partial \mathbf{e}} = \left(1-\bar{d}_G\right)2G_0\mathbf{e}; \quad p = \frac{\partial \Psi}{\partial \varepsilon_v} = \left(1-\bar{d}_K\right)K_0\varepsilon_v; \quad \chi = \frac{\partial \Psi}{\partial \xi} = \frac{\partial h}{\partial \xi} \\ Y_G = -\frac{\partial \Psi}{\partial \bar{d}_G} = \frac{1}{2}\left(2G_0\mathbf{e}:\mathbf{e}\right); \quad Y_K = -\frac{\partial \Psi}{\partial \bar{d}_K} = \frac{1}{2}K_0\varepsilon_v^2 \end{gathered} \qquad [12]$$

where:

$$\bar{d}_G(\mathbf{x}) = \int_\Omega W(\mathbf{x},\mathbf{s})d_G(\mathbf{s})ds, \quad \bar{d}_K(\mathbf{x}) = \int_\Omega W(\mathbf{x},\mathbf{s})d_K(\mathbf{s})ds \qquad [13]$$

The dissipation rate density takes the expression:

$$D = Y_G\dot{\bar{d}}_G + Y_K\dot{\bar{d}}_K - \chi\dot{\xi} + P \geq 0 \qquad [14]$$

P being the so called nonlocality residual representing the energy exchanged between the considered material point and other points belonging to its interaction domain due to the intrinsic nonlocality of the developing damage mechanism. The fact that the system is thermodynamically isolated implies the following *insulation condition* [ERI 81], [BOR 99]

$$\int_{\Omega} P d\Omega = 0 \qquad [15]$$

The insulation condition allows to eliminate the nonlocality residual and to transfer the nonlocality onto the dual variables of the nonlocal damage variables defined in the model, i.e. the energy release rates Y_G and Y_K. One can write

$$D = Y_G \dot{\bar{d}}_G + Y_K \dot{\bar{d}}_K - \chi\dot{\xi} + P = \bar{Y}_G \dot{d}_G + \bar{Y}_K \dot{d}_K - \chi\dot{\xi} \qquad [16]$$

From the insulation condition [15] it follows that:

$$\int_{\Omega} P d\Omega = \int_{\Omega} \left(\bar{Y}_G \dot{d}_G + \bar{Y}_K \dot{d}_K - Y_G \dot{\bar{d}}_G - Y_K \dot{\bar{d}}_K \right) d\Omega = 0 \qquad [17]$$

Having in mind the definitions [13], from eq. [17 one obtains

$$\bar{Y}_G(\mathbf{x}) = \int_{\Omega} W^*(\mathbf{x},\mathbf{s}) Y_G(\mathbf{s}) ds, \quad \bar{Y}_K(\mathbf{x}) = \int_{\Omega} W^*(\mathbf{x},\mathbf{s}) Y_K(\mathbf{s}) ds \qquad [18]$$

where W^* is the adjoint function of W, i.e.

$$W^*(\mathbf{x},\mathbf{s}) \equiv \frac{W_0(\|\mathbf{x}-\mathbf{s}\|)}{\bar{W}(\mathbf{s})} \quad \text{with} \quad \bar{W}(\mathbf{s}) \equiv \int_{\Omega} W_0(\|\mathbf{x}-\mathbf{s}\|) dx \qquad [19]$$

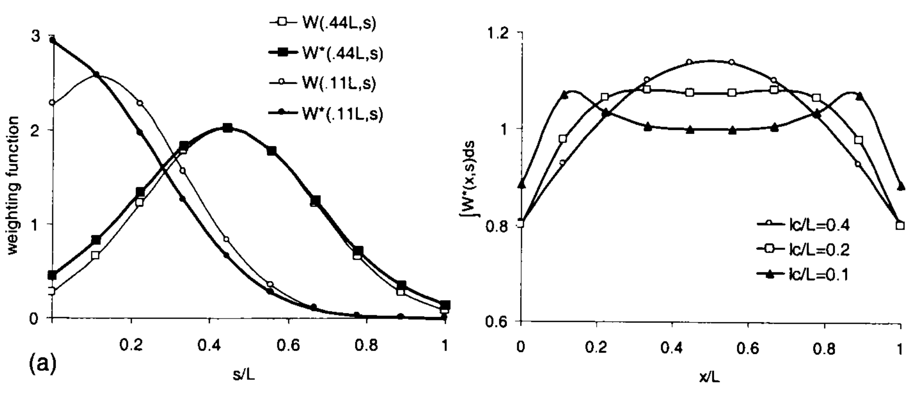


Figure 1. *a) One-dimensional weight functions W(x,s) and W*(x,s) for varying position x over a bar for l_c/L=0.2; (b) function* $\int_0^L W^*(x,s)ds$ *for varying characteristic length* l_c

A plot of the weight functions W and W^* centered at various positions over a bar of length L is shown in figure 1a where the influence of the boundary on the shape of the weight function is also evidenced. The activation function and the evolution equations [3] and [4] are now written in terms of the dual nonlocal variables as in the non-symmetric non-local model [11]

$$\begin{gathered} f\left(\bar{Y}_G, \bar{Y}_K, \chi\right) \leq 0, \quad \dot{\gamma} \geq 0; \quad \dot{\gamma} f = 0 \\ \dot{d}_G = \frac{\partial f}{\partial \bar{Y}_G}\dot{\gamma}; \quad \dot{d}_K = \frac{\partial f}{\partial \bar{Y}_K}\dot{\gamma}; \quad \dot{\xi} = -\frac{\partial f}{\partial \chi}\dot{\gamma} \end{gathered} \qquad [20]$$

REMARK 3.– A dual nonlocal damage model based on the thermodynamic approach has been presented by Benvenuti *et al.* [BEN 00]. In their model, however, only the kinematic internal variable ξ has a nonlocal nature, while the damage variable is local. While the issue of the most appropriate choice seems to be still open from the mechanical point of view, from the computational standpoint the definition of a nonlocal kinematic internal variable leads to a nonlocal constitutive problem in the corrector phase of a standard finite element implementation. On the contrary, the integration of eqs. [20] leads to the same local, and therefore computationally convenient, problem as in the non-symmetric model [11] [COM 00]. □

REMARK 4.– The weight function used for the definition of the dual nonlocal variables in [18 does not allow for the reproduction of a uniform field as, in general (see figure 1b),

$$\int_\Omega W^*(\mathbf{x},\mathbf{s})ds = \int_\Omega \frac{W_0\left(\|\mathbf{x}-\mathbf{s}\|\right)}{\bar{W}(\mathbf{s})}ds \neq 1 \qquad [21]$$

However, in the absence of damage, it appears to be an obvious requirement that a uniform strain field generates a uniform field of strain energy release rates $\bar{Y}_G$ and $\bar{Y}_K$. Therefore, in the applications, the weight function W in eq. $[10]_1$ will be used for $\bar{Y}_G$ and $\bar{Y}_K$, while the weight function W^* in [19] will be used for $\bar{d}_G$ and $\bar{d}_K$, satisfying in this way condition [17]. □

The algorithmic tangent matrix can be computed for the dual nonlocal model following the procedure proposed by Jirásek [JIR 99]. Let

$$\mathbf{F} = \int_\Omega \mathbf{B}^T \boldsymbol{\sigma} d\Omega \qquad [22]$$

be the vector of internal equivalent nodal forces and let $\mathbf{u}$ be the vector of nodal displacements in a finite element discretization. Let N_g be the total number of Gauss points used to carry out the numerical integration over all the elements in the mesh.

and let N_g^{act} be the number of Gauss points where $f(Y_G, Y_K, \chi) = 0$ and $\dot{\gamma} > 0$ at the end of the correction phase. Let us also define the following quantities at Gauss points q and p

$$A_{GG_q} \equiv A_q \left(\frac{\partial f}{\partial \bar{Y}_G} \right)_q^2, \quad A_{KK_q} \equiv A_q \left(\frac{\partial f}{\partial \bar{Y}_K} \right)_q^2, \quad A_{GK_q} = A_{KG_q} \equiv A_q \left(\frac{\partial f}{\partial \bar{Y}_G} \frac{\partial f}{\partial \bar{Y}_K} \right)_q \qquad [23]$$
$$W_{pq} \equiv W(\mathbf{x}_p, \mathbf{s}_q), \quad W^*_{pq} \equiv W^*(\mathbf{x}_p, \mathbf{s}_q)$$

where A is defined in $[7]_2$. After some algebra one obtains the symmetric elasto-damage tangent matrix $\mathbf{K}^{ed}$

$$\begin{aligned}\mathbf{K}^{ed} \equiv \frac{\partial \mathbf{F}}{\partial \mathbf{u}^{\mathrm{T}}} = & \sum_{p=1}^{N_g} \omega_p \left[\left(1 - \bar{d}_{G_p}\right) \mathbf{K}^e_{G_p} + \left(1 - \bar{d}_K\right) \mathbf{K}^e_{K_p} \right] + \\ & - \sum_{p=1}^{N_g} \sum_{q=1}^{N_g^{\text{act}}} \sum_{l=1}^{N_g} \Big\{ \omega_p \omega_q \omega_l W_{pq} W^*_{ql} \Big[A_{GG_q} \mathbf{K}^e_{G_l} \mathbf{u}\mathbf{u}^{\mathrm{T}} \mathbf{K}^e_{Gp} + \\ & + A_{GK_q} \left(\mathbf{K}^e_{G_l} \mathbf{u}\mathbf{u}^{\mathrm{T}} \mathbf{K}^e_{Kp} + \mathbf{K}^e_{K_l} \mathbf{u}\mathbf{u}^{\mathrm{T}} \mathbf{K}^e_{Gp} \right) + A_{KK_q} \mathbf{K}^e_{K_l} \mathbf{u}\mathbf{u}^{\mathrm{T}} \mathbf{K}^e_{Kp} \Big] \Big\}\end{aligned} \qquad [24]$$

where ω_p denotes the Gauss weight at Gauss point p and

$$\mathbf{K}^e_{G_p} \equiv \mathbf{B}^{\mathrm{T}}_G(\mathbf{x}_p) \mathbf{D}_0 \mathbf{B}_G(\mathbf{x}_p), \quad \mathbf{K}^e_{K_p} \equiv \mathbf{B}^{\mathrm{T}}_K(\mathbf{x}_p) \mathbf{D}_0 \mathbf{B}_K(\mathbf{x}_p) \qquad [25]$$

represent the deviatoric and volumetric contributions of the same Gauss point to the initial undamaged stiffness matrix. In [25], $\mathbf{D}_0$ is the matrix of initial elastic moduli, $\mathbf{B}_G$ and $\mathbf{B}_K$ are compatibility matrices such that:

$$\mathbf{e}(\mathbf{x}) = \mathbf{B}_G(\mathbf{x})\mathbf{u}, \quad \mathbf{P}\varepsilon_v(\mathbf{x}) = \mathbf{B}_K(\mathbf{x})\mathbf{u} \qquad [26]$$

and $\mathbf{P}^{\mathrm{T}} \equiv \{1/3 \quad 1/3 \quad 1/3 \quad 0 \quad 0 \quad 0\}$. It should be noted that in [24 the index p runs over the whole set of Gauss points. This is because the global damage variable at a point varies as a consequence of the variation of the local damage at any point in the body. Thus, even though at a point one has $\dot{\gamma} = 0$ and the material point unloads elastically, at the same point one has $\dot{\bar{d}} \neq 0$ if there is at least one active point in the structure. By contrast, the index q runs only over the active Gauss points since it concerns the dependence of the nonlocal damage variables at point p on their corresponding local variables which are zero at inactive Gauss points. On the basis of these considerations and noting that, while $W_{pq} \neq W_{qp}$ and $W^*_{pq} \neq W^*_{qp}$, one has $W_{pq} W^*_{ql} = W_{lq} W^*_{qp}$, the symmetry of $\mathbf{K}^{ed}$ can be easily assessed.

4. A simple nonlocal damage model

To study the effects of the dual nonlocal regularization described in the previous Section a simple model, with only one damage variable d, is considered. Applications of the two damage variables model to concrete problems will be presented in a forthcoming paper. The simplified model is based on the following free energy density:

$$\begin{aligned}\Psi &= \frac{1}{2}\left(1-\bar{d}\right)\varepsilon : \mathrm{D}_0 : \varepsilon \\ &-k\left(1-\xi\right)\left[\ln^n \frac{c}{1-\xi} + n\ln^{n-1}\frac{c}{1-\xi} + n\left(n-1\right)\ln^{n-2}\frac{c}{1-\xi} + \ldots + n!\ln\frac{c}{1-\xi} + n!\right]\end{aligned} \quad [27]$$

where D_0 is the undamaged elastic tensor and k, c and n are material parameters. The state equations are given by:

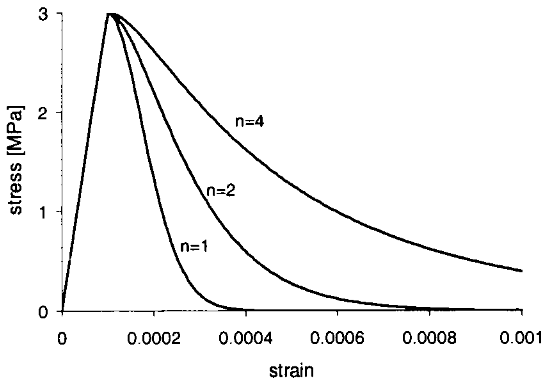


Figure 2. *Stress-strain behavior for the simple damage model for varying n*

$$\sigma = \frac{\partial \Psi}{\partial \varepsilon} = \left(1-\bar{d}\right)\mathrm{D}_0 : \varepsilon,\ \ \chi = \frac{\partial \Psi}{\partial \xi} = k\ln^n\left(\frac{c}{1-\xi}\right),\ \ Y = -\frac{\partial \Psi}{\partial \bar{d}} = \frac{1}{2}\varepsilon : \mathrm{D}_0 : \varepsilon \quad [28]$$

The activation function, loading-unloading conditions and evolution equations are defined as:

$$\begin{aligned} &f\left(\bar{Y},\chi\right) = \bar{Y} - \chi \le 0,\ \ \dot{\gamma} \ge 0,\ \ f\dot{\gamma} = 0 \\ &\dot{d} = \frac{\partial f}{\partial \bar{Y}}\dot{\gamma} = \dot{\gamma},\ \ \dot{\xi} = -\frac{\partial f}{\partial \chi}\dot{\gamma} = \dot{\gamma}\end{aligned} \quad [29]$$

and therefore the kinematic internal variable ξ coincides with the damage variable d.

The local model is such that, in one dimension, the stress vanishes only asymptotically, for $\varepsilon \to \infty$, but with a bounded fracture energy density. This can be seen by specializing the model to one dimension. For $\varepsilon > \varepsilon_0$, ε_0 being the strain at the linear elastic limit, from the condition $f = 0$, one has (figure 2)

$$\sigma = c\exp\left[-\left(\frac{E\varepsilon^2}{2k}\right)^{\frac{1}{n}}\right]E\varepsilon \qquad [30]$$

E denoting the Young's modulus. The fracture energy density is defined as

$$g_f = \frac{1}{2}E\varepsilon_0^2 + \int_{\varepsilon_0}^{\infty}\sigma d\varepsilon = \frac{1}{2}E\varepsilon_0^2 + \int_{\varepsilon_0}^{\infty}c\exp\left[-\left(\frac{E\varepsilon^2}{2k}\right)^{\frac{1}{n}}\right]E\varepsilon d\varepsilon \qquad [31]$$

If I denotes the integrand in $[31]_2$, the boundedness of g_f can be established noting that

$$\lim_{\varepsilon\to\infty}\frac{I}{1/\varepsilon} = 0 \qquad [32]$$

The nonlocal variables $\bar{d}$ and $\bar{Y}$ are defined according to [13] and [18] taking into account Remark 4, i.e.

$$\bar{d}(\mathbf{x}) = \int_{\Omega} W^*(\mathbf{x},\mathbf{s})d(\mathbf{s})ds, \quad \bar{Y}(\mathbf{x}) = \int_{\Omega} W(\mathbf{x},\mathbf{s})Y(\mathbf{s})ds \qquad [33]$$

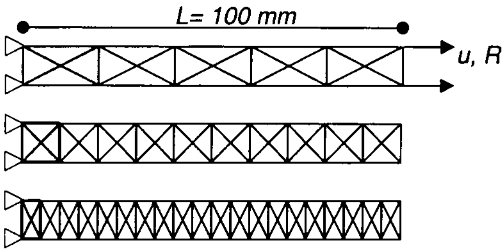
The symmetric consistent tangent matrix can be computed for the simplified model following the same procedure as in the previous Section

$$\begin{aligned}\mathbf{K}^{ed} \equiv \frac{\partial \mathbf{F}}{\partial \mathbf{u}^{\mathrm{T}}} = &\sum_{p=1}^{N_g}\omega_p\left(1-\bar{d}_p\right)\mathbf{K}_p^e + \\ &-\sum_{p=1}^{N_g}\sum_{q=1}^{N_g^{act}}\sum_{l=1}^{N_g}\left(\frac{k}{1-d_q}n\ln^{n-1}\frac{c}{1-d_q}\right)^{-1}\omega_p\omega_q\omega_l W_{pq}^* W_{ql}\mathbf{K}_p^e\mathbf{u}\mathbf{u}^{\mathrm{T}}\mathbf{K}_l^e\end{aligned} \qquad [34]$$

5. One-dimensional numerical application

The simplified damage model is used for the simulation of a tensile test on a prismatic bar. The problem data and geometry, together with the adopted meshes are shown in figure 1. To trigger the damage localization, the elements at the left boundary have been slightly weakened. The problem has been solved adopting two nonlocal approaches: (model A) the non-symmetric approach of Comi [COM 00] based on the definition of nonlocal strain invariants (in this simple case coinciding with the energy release rate); (model B) the symmetric dual nonlocal approach of Borino *et al.* [BOR 99] in the form discussed in Section 3. Note that different values of l_c have been adopted for the two models to obtain comparable damage accumulation in the part of the bar where unloading occurs after localization.

The parameter l_c can be identified using a back-analysis technique based on one dimensional tests where the width of the process zone is measured. Alternatively, an analytical approach can be pursued where l_c is related to the length of the stationary harmonic localization wave (see *e.g.* [SLU 93]). This type of study has still to be carried out for the symmetric nonlocal model considered here.



A = 100 mm^2 (cross section)
E = 30000 MPa
ν = 0.
k = .00015 MPa
c = 2.72
n = 2
l_c = 20 mm (model A)
l_c = 15 mm (model B)

Figure 3. *One-dimensional test problem: geometry, adopted meshes and material data*

As shown in figure 4, both approaches provide an effective regularization of the problem as the results in terms of reaction force versus imposed displacement rapidly converge towards a mesh independent solution. From figure 4, it appears that the dual regularization technique produces an initially more ductile response with a subsequent very steep drop of the reaction force. The displacement controlled analysis cannot proceed further due to a global snap-back behavior which is not observed in the analysis with model A regularization. Both behaviors can be observed in uniaxial tension tests depending on the material properties and testing conditions. Since the present numerical test does not simulate a physical experiment, it is not possible to assess which one of the two results is more realistic.

The longitudinal strain evolution obtained by means of the two regularizations is shown in figure 5. While model A regularization gives rise to a sharp strain localization, the model B technique produces a smoother profile with a much lower peak value developing at a significant distance from the boundary, where some elements have been weakened. This is a consequence of the effect of the boundary due to the particular shape of the weight function as already mentioned in Section 3 (see figure 1).

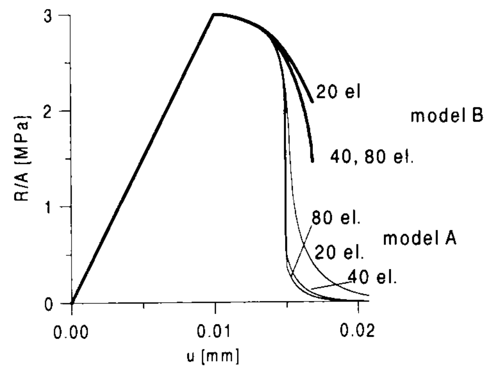


Figure 4. *Reaction per unit cross-section area versus imposed displacement with nonlocal models A and B: convergence with mesh refinement*

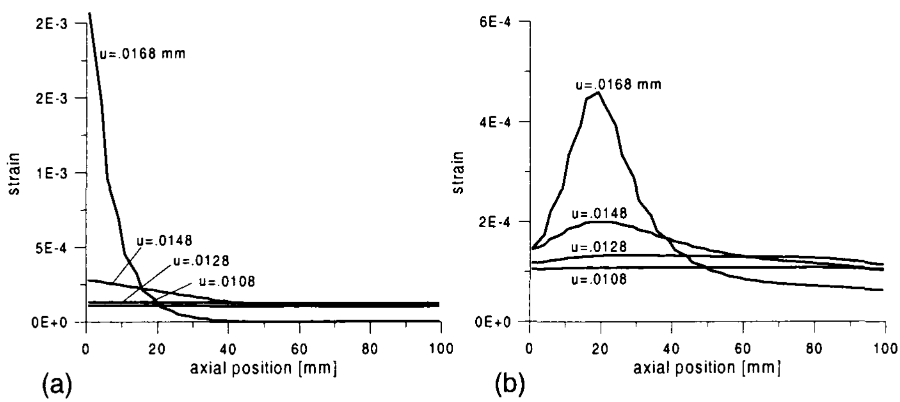


Figure 5. *Strain evolution for imposed displacement u: (a) model A;* (b) *model B*

The difference is less pronounced in terms of local damage profiles, as illustrated in figure 6a for a displacement u=0.0168 mm. Again, with model B regularization the damage peak is offset with respect to the boundary. It should also be noted that for equal imposed displacement u, the model A regularization leads to a higher damage peak. The comparison between the local and nonlocal damages in model B analysis is shown in figure 6b. It can be noted that the nonlocal damage presents a sharper peak though at almost the same value of the local one.

Finally, the stress profiles are shown in figure 7. It turns out that the dual regularization has the beneficial effect to reduce the stress oscillation caused by the weighting process [JIR 99]. Furthermore, in both cases stress oscillation tends to reduce as the mesh is refined.

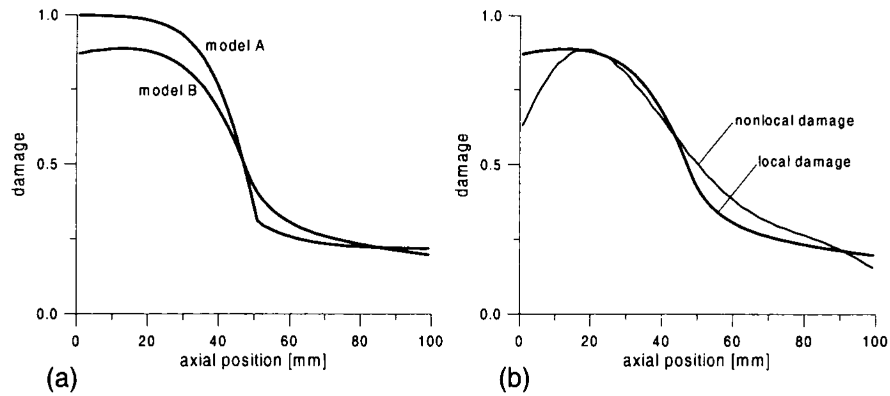


Figure 6. *Imposed displacement u=0.0168 mm: (a) damage profiles* $d(x)$ *for models A and B; (b) local* $d(x)$ *and non local* $\bar{d}(x)$ *damage profiles for model B*

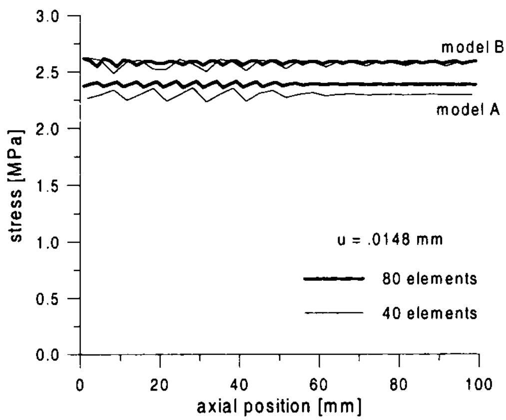


Figure 7. *Stress distributions along the bar for model A and model B upon mesh refinement*

6. Conclusions

The finite element implementation of a family of isotropic nonlocal damage models has been discussed. Attention has been focussed on the issue of the symmetry of the consistent tangent operator.

A rather general isotropic local damage model based on two damage variables affecting separately the shear and bulk moduli has been presented. The explicit expression of the consistent tangent matrix has been derived and it has been shown that symmetry is obtained provided that associative evolutions are postulated for the damage and the internal variables. Then the model has been re-formulated as a nonlocal model following the approach proposed in [PIJ 87] and [COM 00] which consists of assuming as nonlocal variable the elastic energy release rate. This has the advantage that all constitutive calculations can be carried out separately at each Gauss point during the corrector phase of the standard finite element iterative procedure. The consistent tangent matrix for the considered nonlocal model is well known to be non-symmetric [BAZ 88], [JIR 99].

A nonlocal version of the same model, based on the thermodynamically founded nonlocal theory recently put forward by Borino *et al.* [BOR 99] and preserving the symmetry of the underlying local model has also been formulated. In this new version of the model, the nonlocal nature, originally conferred to the damage variables, is transferred to their conjugate variables, the energy release rates, on the basis of an energy equivalence which allows to eliminate the so called nonlocality residual. The explicit expression of the finite element tangent stiffness matrix of the new nonlocal model has been derived and it has been shown that it is symmetric.

A one-dimensional test has been carried out for a simpler nonlocal model based on a single damage variable. The regularization property of the dual nonlocal formulation has been assessed even though the issue of the influence of the boundary conditions with the development of a significant boundary layer seems to deserve further consideration.

Acknowledgements

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